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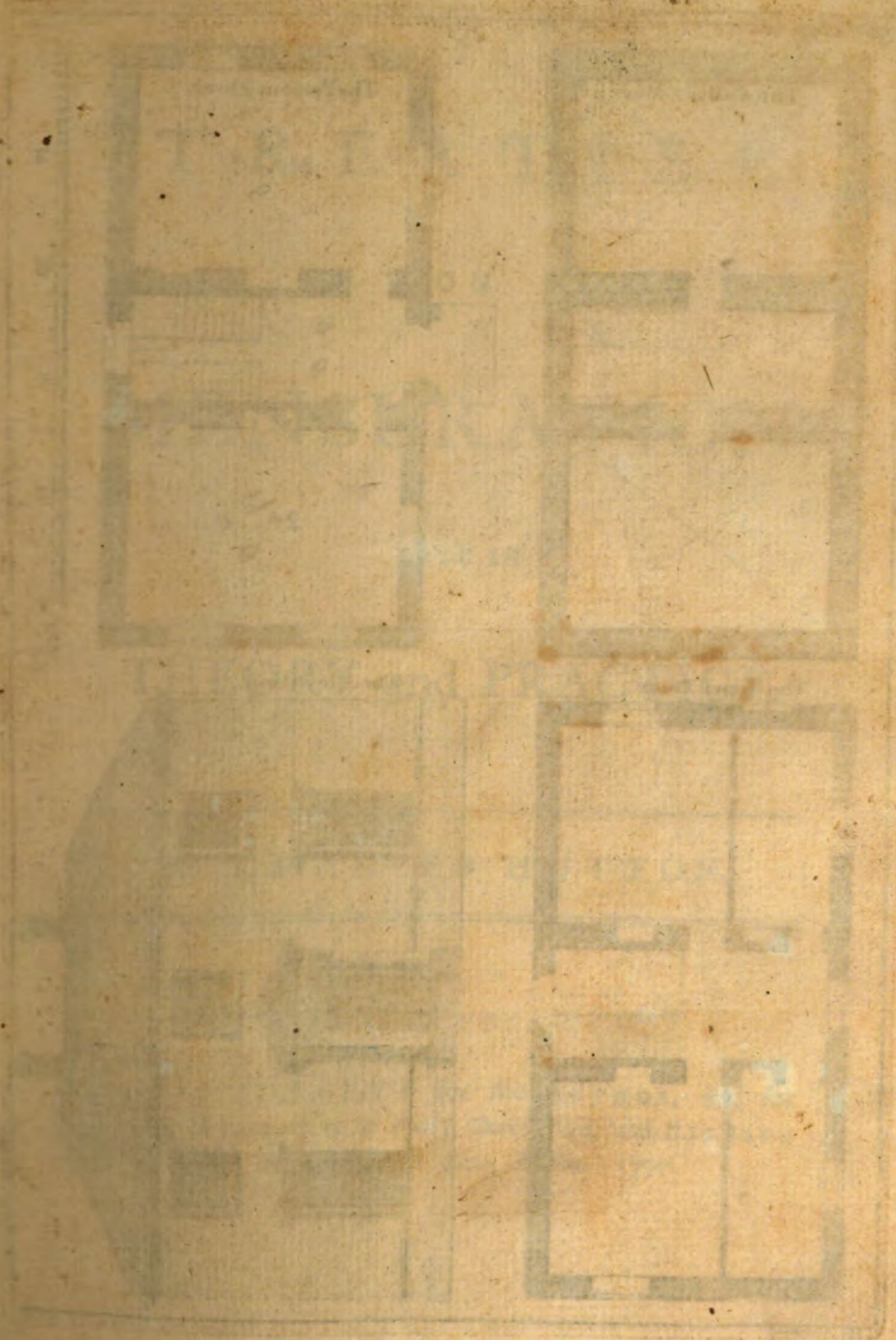


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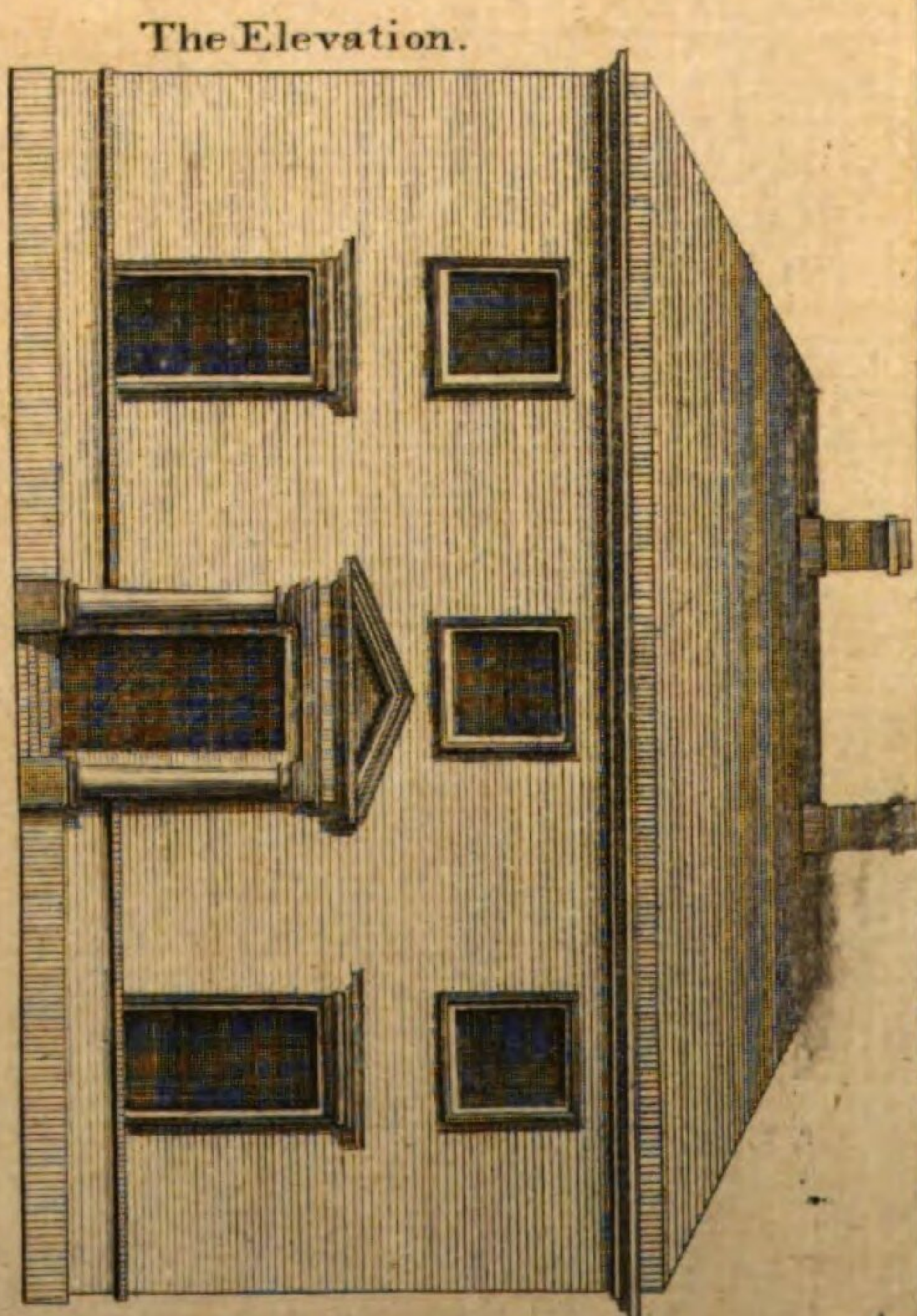
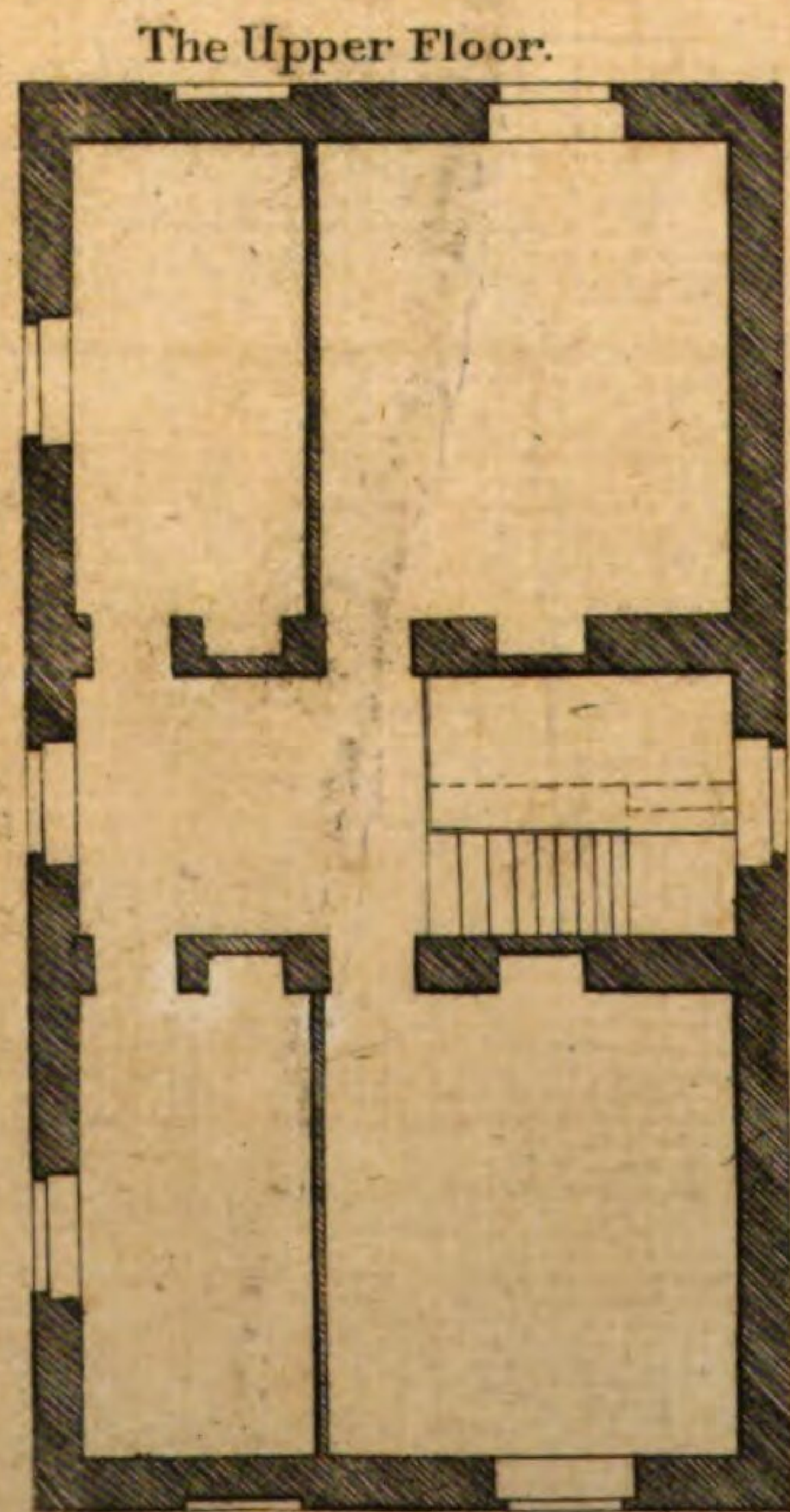
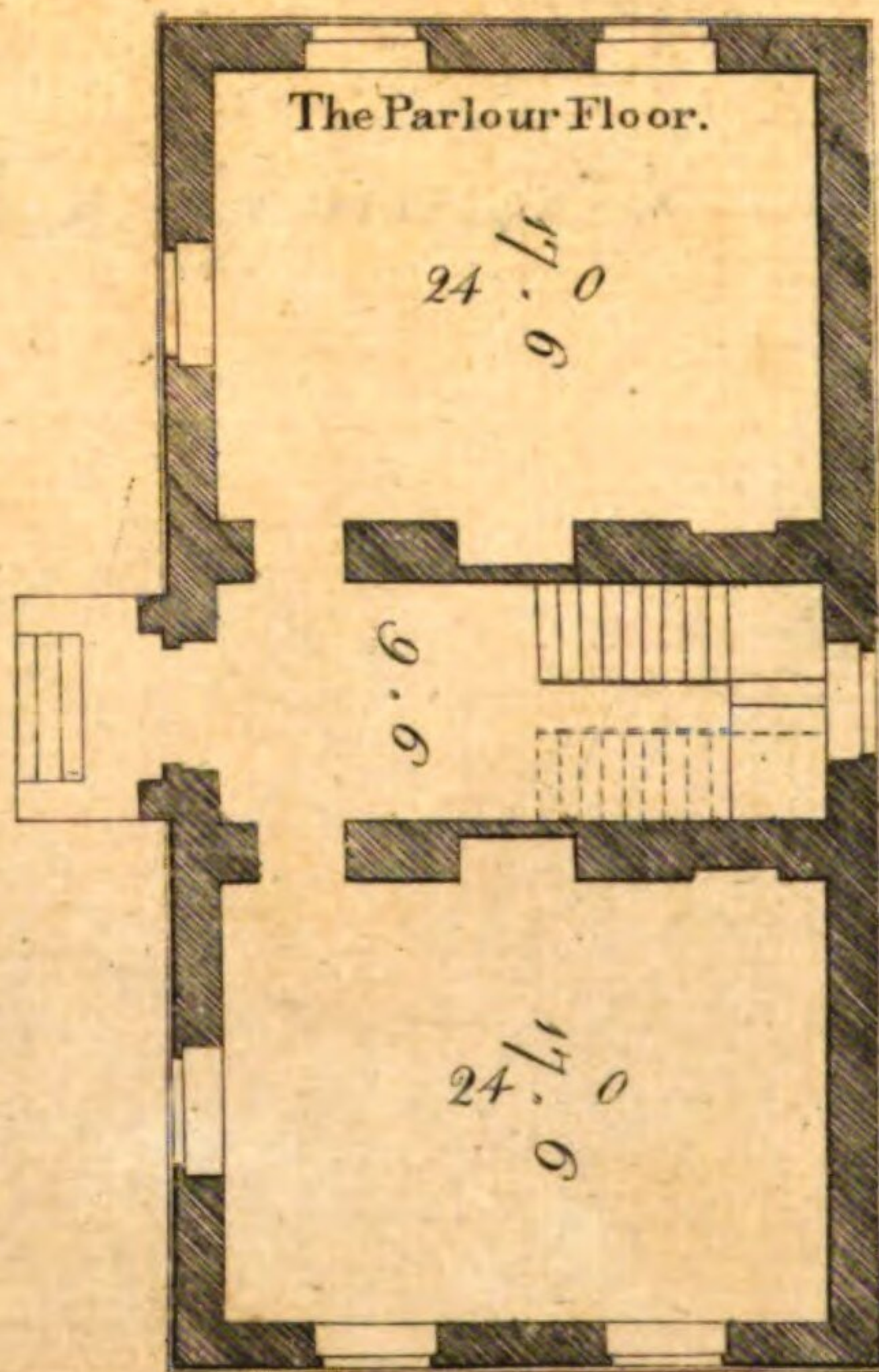
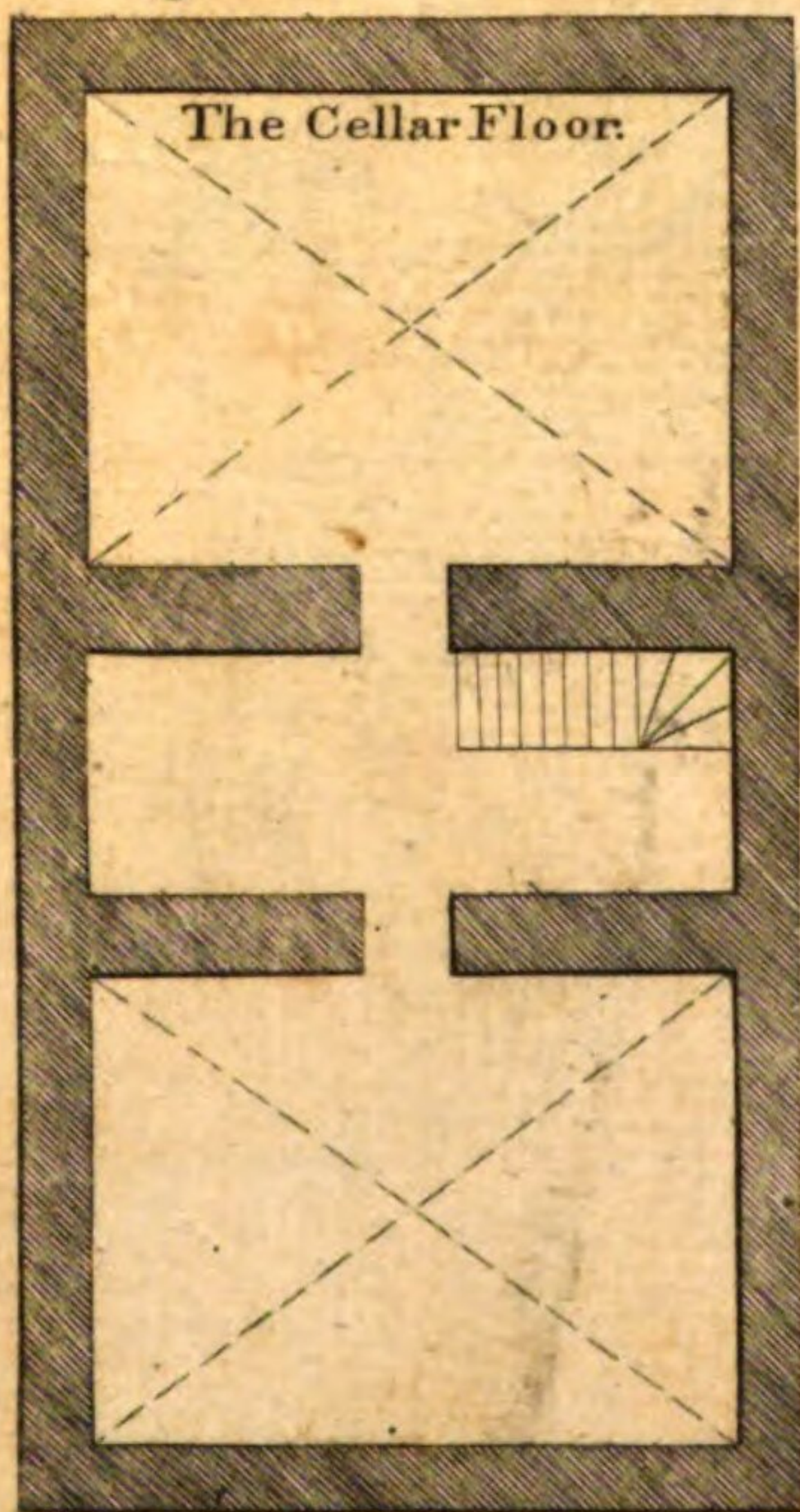
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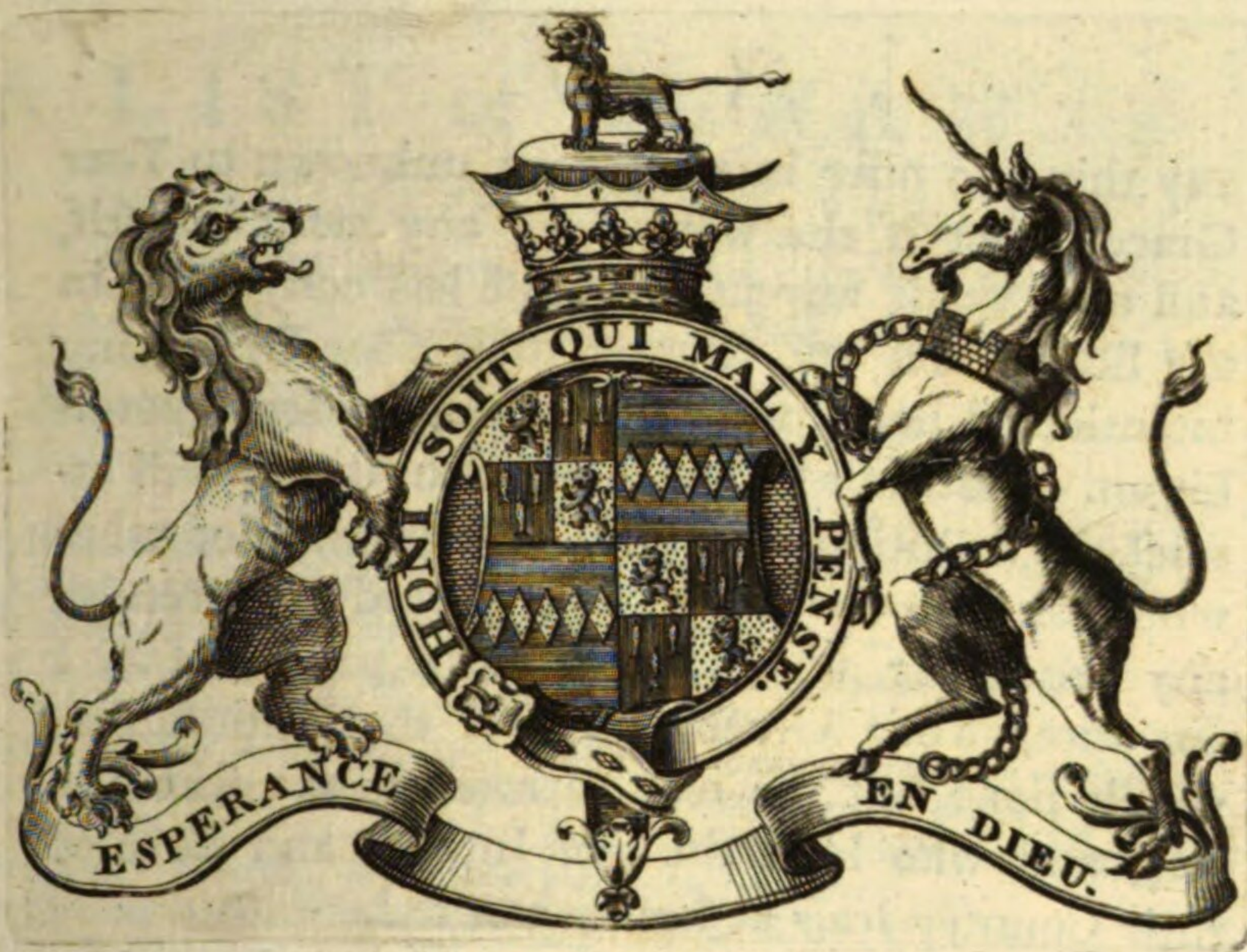
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any



any thing to offer in this work unknown to Your Grace. But if the work have any merit in itself, and may be of any use to those less conversant in the liberal arts and sciences than Your Grace, my intention, I hope, will fully apologize for my ambition. For Your name, I am convinced, will attract the attention of many, who might not otherwise look into the work; and should such receive any benefit or advantage from the perusal of it, I am sure, Your Grace will feel that satisfaction which none but the *real* Patrons of the scientific arts, and who have the *real* Interest and Love of their Country *truly* at heart, can feel.

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— John Thompson, Free School,  
Whitechapel, London

— William Teague, Redruth,  
Cornwall

— John Telifer, Jedburgh

— John Turnbull, Schoolmaster,  
Jedburgh

— Tate, Schoolmaster, S. Shields

— Thurnam, Whitby

Mr Robert Taylor, Berwick, 4 B.

— Sam. Toft, Norwich

— Augustus Tillet, Norwich

— Thompson, Schoolmaster, Hull

— Charles Taylor, Coventry

— Wm Turner, Birmingham

## V

The Hon. Raby Vane

Rev. Dr Vane

Dr Nicola Vilant, Profess. of Math.  
in the University of St An-  
drews, 2 Books

Mr Thomas Vanner, Walton upon  
Thames

— Cha. Vyse, Math. M. Portland  
Street, Cavendish Square

## W

Dr James Williamson, Profess. of  
Math. in the University of  
Glasgow

Dr Wilson, Profess. of Astronomy  
in ditto

Mr Professor Watson of Cambridge

Rev. Mr Watson, Fellow of Tri-  
nity College, Cambridge

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of Christ Church, Oxford

Rev. Mr Wolfe, Howick

Rev. Mr Warkman, Earlsdon

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Major Wilson

Capt. Edward Williams, of the  
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— Thomas Wilkin, Hexham

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— John Warden, London

— Winship, Corbridge

— John Wales, Schoolmaster,  
Tanfield

— Thomas Wibberfly

— Ralph Waters, jun. Newcastle

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Mr



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| — William Wafcoe, Free School,  | — John Wrathall, Sheffield       |
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| — Walter Matthewfon, Wr. and    |                                  |
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| — John Yarrow, Schoolmaster   |
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## P R E F A C E.

**B**Y Mensuration I understand the art and science which is concerned about the measure of extention, or the magnitude of figures; and it is, next to arithmetic, a subject of the greatest use and importance, both in affairs that are absolutely necessary in human life, and in every branch of the mathematics: A subject by which sciences are established, and commerce is conducted; by whose aid we manage our business, and inform ourselves of the wonderful operations in nature; by which we measure the heavens and the earth, estimate the capacities of all vessels and bulks of all bodies, gauge our liquors, build edifices, measure our lands and the works of artificers, buy and sell an infinite variety of things necessary in life, and are supplied with the means of making the calculations which are necessary for the construction of almost all machines.

IT is evident that the close connection of this subject with the affairs of men would very early evince its importance to them; and accordingly the greatest among them have paid the utmost attention to it; and the chief and most essential discoveries in geometry in all ages, have been made in consequence of their efforts in this subject. Socrates thought that the prime use of geometry was to measure the ground, and indeed this business gave name to the subject; and most of the ancients seem to have had no other end besides mensuration in view in all their laboured geometrical disquisitions. Euclid's elements are almost entirely devoted to it; and although there be contained in them many properties of geometrical figures which may be applied to other purposes, and indeed of which the moderns have made the most material uses in various disquisitions of exceedingly different kinds, I say notwithstanding this, yet Euclid himself seems to have adapted them entirely to this purpose. For, if it be considered that his elements contain a continued chain of reasoning and of truths, of which the former are successively applied to the discovery of the latter, one proposition depending on another, and the succeeding propositions still approximating towards some particular object near the end of each book; and when at the last we find that object to be the equality, proportion, or relation between the magnitudes of figures both plane and solid; it is scarcely possible to avoid allowing this to have been Euclid's grand object. And accordingly he determined the chief properties in the mensuration of rectilineal  
plane



plane and solid figures; and squared all such planes, and cubed all such solids. The only curve figures which he attempted besides are the circle and sphere; and when he could not accurately determine their measures, he gave an excellent method of approximating to them, by shewing how in a circle to inscribe a regular polygon which should not touch another circle, concentric with the former, although their circumferences should be ever so near together; and, in like manner, between any two concentric spheres to describe a polyhedron which should not any where touch the inner one: And approximations to their measures are all that have hitherto been given. But although he could not square the circle, nor cube the sphere, he determined the proportion of one circle to another, and of one sphere to another, as well as the proportions of all rectilineal similar figures to one another.

ARCHIMEDES took up mensuration where Euclid left it, and carried it a great length. He was the first who squared a curvilinear space, unless Hypocrates must be excepted on account of his lunes. In his time the conic sections were admitted in geometry, and he applied himself closely to the measuring of them as well as other figures. Accordingly he determined the relations of spheres, spheroids, and conoids, to cylinders and cones; and the relations of parabolas to rectilineal planes whose quadratures had long before been determined by Euclid.—He hath left us also his attempts upon the circle: He proved that a circle is equal to a right-angled triangle whose base is equal to the circumference and its altitude equal to the radius; and consequently that its area is found by drawing the radius into half the circumference; and so reduced the quadrature of the circle to the determination of the ratio of the diameter to the circumference; but which however hath not yet been done. Being disappointed of the exact quadrature of the circle, for want of the rectification of its circumference, which all his methods would not effect, he proceeded to assign an useful approximation to it; this he effected by the numerical calculation of the perimeters of the inscribed and circumscribed polygons; from which calculations it appears, that the perimeter of the circumscribed regular polygon of 192 sides is to the diameter, in a less ratio than that of  $3\frac{1}{7}$  ( $3\frac{10}{70}$ ) to 1, and that the inscribed polygon of 96 sides is to the diameter in a greater ratio than that of  $3\frac{10}{71}$  to 1; and consequently much more that the circumference of the circle is to the diameter in a less ratio than that of  $3\frac{1}{7}$  to 1, but greater than that



that of  $3\frac{10}{71}$  to 1: The first ratio of  $3\frac{1}{7}$  to 1, reduced to whole numbers, gives that of 22 to 7, for  $3\frac{1}{7} : 1 :: 22 : 7$ , which therefore will be nearly the ratio of the circumference to the diameter. From this ratio of the circumference to the diameter he computed the approximate *area* of the circle, and found it to be to the square of the diameter as 11 is to 14.—He likewise determined the relation between the circle and ellipse, with that of their similar parts.—The hyperbola too in all probability he attempted, but it is not to be hoped that he met with any success, since approximations to its area are all that can be given by all the methods that have since been invented.

Besides these figures, he hath left us a treatise on the spiral described by a point moving uniformly along a right line, which at the same time moves with an uniform angular motion; and determined the proportion of its area to that of its circumscribed circle, as also the proportion of their sectors.

Throughout the whole works of this great man, which are chiefly on mensuration, he every where discovers the deepest design and the finest invention; and seems to have been (with Euclid) exceedingly careful of admitting into his demonstrations nothing but principles perfectly geometrical and unexceptionable: And although his most general method of demonstrating the relations of curved figures to streight ones, be by inscribing polygons in them; yet to determine those relations, he does not increase the number and diminish the magnitude of the sides of the polygon *in infinitum*; but from this plain fundamental principle, allowed in Euclid's elements, viz. that any quantity may be so often multiplied, or added to itself, as that the result shall exceed any proposed finite quantity of the same kind, he proves that to deny his figures to have the proposed relations, would involve an absurdity.

He demonstrated also many properties, particularly in the parabola, by means of certain numerical progressions whose terms are similar to the inscribed figures; but without considering such series to be continued *in infinitum*, and then summing up the terms of such infinite series.

He had another very curious and singular contrivance for determining the measures of figures, in which he proceeds as it were mechanically by weighing them.

SEVERAL other eminent men among the ancients wrote on this subject, both before and after Euclid and Archimedes; but their attempts were usually upon particular parts of it, and according



to methods not essentially different from theirs. Among these are to be reckoned Thales, Anaxagoras, Pithagoras, Bryson, Antiphon, Hypocrates of Chios, Plato, Apollonius, Philo, and Ptolomy; most of whom wrote of the quadrature of the circle, and those after Archimedes, by his method, usually extended the approximation to a greater degree of accuracy.

MANY of the moderns also have prosecuted the same problem of the quadrature of the circle, after the same methods, to greater lengths; such are Vieta, and Metius whose proportion between the diameter and circumference is that of 113 to 355, which is within about  $\frac{3}{10000000}$  of the true ratio; but above all, Ludolph van Ceulen, who with an amazing degree of industry and patience, by the same methods extended the ratio to 20 places of figures, making it that of 1 to 3.14159265358979323846+.

THE first material deviation from the principles used by the ancients in geometrical demonstrations was made by Cavalieri: The sides of their inscribed and circumscribed figures they always supposed of a finite and assignable number and length; he introduced the doctrine of indivisibles, a method which was very general and extensive, and which with great ease and expedition served to measure and compare geometrical figures. Very little new matter however was added to geometry by this method, its facility being its chief advantage. But there was great danger in using it, and it soon led the way to infinitely small elements, and infinitesimals of endless orders; methods which were very useful in solving difficult problems, and in investigating or demonstrating theories that are general and extensive; but sometimes led their incautious followers into errors and mistakes, which occasioned disputes and animosities among them. There were now however many excellent things performed in this subject; not only many new things were effected concerning the old figures, but new curves were measured; and for many things which could not be exactly squared or cubed, general and infinite approximating series were assigned, of which the laws of their continuation were manifest, and of some of which the terms were independent on each other. Mr Wallis, Mr Huygens, and Mr James Gregory performed wonders; Huygens in particular must be admired for his solid, accurate, and very masterly works.

DURING the preceding state of things, several men, whose vanity seemed to have overcome their regard for truth, asserted that they had discovered the quadrature of the circle, and published their

attempts



attempts in the form of strict geometrical demonstrations, with such assurance and ambiguity as staggered and misled many who could not so well judge for themselves, and perceive the fallacy of their principles and arguments. Among those were Longomontanus, and our countryman Hobbs, who obstinately refused all conviction of his errors.

THE use of infinites was however disliked by several people, and particularly by Sir Isaac Newton, who among his numerous and great discoveries hath given us that of the method of fluxions; a discovery of the greatest importance both in philosophy and mathematics; it being a method so general and extensive, as to include all investigations concerning magnitude, distance, motion, velocity, time, &c. with wonderful ease and brevity; a method established by its great author upon true and incontestible principles; principles perfectly consistent with those of the ancients, and which were free from the imperfections and absurdities attending some that had lately been introduced by the moderns: he rejected no quantities as infinitely small, nor supposed any parts of curves to coincide with right lines; but proposed it in such a form as admits of a strict geometrical demonstration. Upon the introduction of this method most sciences assumed a different appearance, and the most abstruse problems became easy and familiar to every one; things which before seemed to be insuperable, became easy examples or particular cases of theories still more general and extensive; rectifications, quadratures, cubatures, tangencies, cases *de maximis & minimis*, and many other subjects became general problems, and delivered in the form of general theories which included all particular cases; thus, in quadratures, an expression would be investigated which defined the areas of all possible curves whatever, both known and unknown, and which, by proper substitutions, brought out the area for any particular case, either in finite terms, or in infinite series of which any term or any number of terms could be easily assigned; and the like in other things. And although no curve whose quadrature was unsuccessfully attempted by the ancients, became by this method perfectly quadrable, there were assigned many general methods of approximating to their areas, of which in all probability the ancients had not the least idea or hope, and innumerable curves were squared which were utterly unknown to them.

The excellency of this method revived some hopes of squaring the circle, and its quadrature was attempted with eagerness. The quadrature of a space was now reduced to the finding of the fluent  
of



of a given fluxion, but this problem however was found to be incapable of a general solution in finite terms; the fluxion of every fluent was always assignable, but the reverse of this problem could be effected only in particular cases; among the exceptions, to the great grief of the geometers, was included the case of the circle, with regard to all the forms of fluxions attending it. Another method of obtaining the area was tried: of the quantity expressing the fluxion of any area, in general, could be assigned the fluent in the form of an infinite series, which series therefore defined all areas in general, and which, on substituting for particular cases, was often found to break off and terminate, and so afford an area in finite terms; but here again the case of the circle failed, its area still coming out an infinite series.—All hopes of the quadrature of the circle being now at an end, the geometers employed themselves in discovering and selecting the best forms of infinite series for determining its area, among which it is evident that those were to be preferred which were simple and which would converge quickly; but it generally happened that these two properties were divided, the same series very rarely including them both: the mathematicians in most parts of Europe were now busy, and many series were assigned on all hands, some admired for their simplicity, and others for their rate of convergence; those which converged the quickest, and were at the same time simplest, which therefore were most useful in computing the area of the circle in numbers, were those in which, besides the radius, the tangent of some certain arc of the circle, was the quantity by whose powers the series converged; and from some of these serieses the area hath been computed to a great extent of figures: Mr Edmund Hally gave a remarkable one from the tangent of 30 degrees, which was rendered famous by the very industrious Mr Abraham Sharp, who by means of it extended the area of the circle to 72 places of figures, as may be seen in Sherwin's book of logarithms; but even this was afterwards outdone by Mr John Machin, who, by means hereafter described in this book, composed a series so simple and which converged so quickly, that by it, in a very little time, he extended the quadrature of the circle to 100 places of figures; from which it appears, that if the diameter be 1, the circumference will be 3.1415926535, 8979323846, 2643383279, 5028841971, 6939937510, 5820974944, 5923078164, 0628620899, 8628034825, 3421170679 +, and consequently the area will be .7853981633, 9744830961, 5660845819, 8757210492, 9234984377, 6455243736, 1480769541, 0157155224, 9657008706, 3355292669+.

WHILST



WHILST I have been giving the preceding account of the progress of this subject, I have at the same time unawares been writing its panegyric; for, from hence it appears, that all or most of the material improvements or inventions in the principles or method of treating of geometry, have been made especially for the improvement of this chief part of it, mensuration; which abundantly shews, what I at first undertook to declare, the dignity of this subject; a subject which, as Doctor Barrow says, after mentioning some other things, “deserves to be more curiously weighed, because from hence a name is imposed upon that mother and mistress of the rest of the mathematical sciences, which is employed about magnitudes, and which is wont to be called geometry (a word taken from ancient use, because it was first applied only to measuring the earth and fixing the limits of possessions) though the name seemed very ridiculous to Plato, who substitutes in its place that more extensive name of Metrics or Mensuration; and others after him give it the title of Pantometry, because it teaches the method of measuring all kinds of magnitudes.”

But notwithstanding the dignity and importance of this subject, the books which have lately been offered to the public under the title of mensuration, have treated it in a very unworthy manner, and have brought it into much contempt. My design in this work therefore is to place the subject in a more favourable light, and in some measure to retrieve its reputation; and of which I shall now give some account.

THIS work is first divided into five principal parts, each part being cut into several sections, each of which sections contains several problems or propositions. The problems and propositions are all demonstrated after the simplest and shortest methods that could be discovered; I having proceeded sometimes by pure geometry, and sometimes by algebra, the method of fluxions, the method of increments, &c. according as the one or the other seemed to be the best adapted to the business then in hand. And thus, by not confining myself to one mode of demonstrating, I was enabled to make my book exceedingly more compleat than it could otherwise be; not only by using that particular method which would perform the business in the shortest or clearest manner, but which also would serve to extend the subject to the greatest length. However I had not *always* my choice of my method of demonstrating, the subject sometimes confining to one method, and sometimes to another; for although the method of fluxions be generally the most concise,



cise, and, in other respects, the most proper method of investigating the measure of extensions, there are some things in this business that are too simple for it, as well as some others that arise above its reach : Thus, the method of fluxions cannot determine the measure of a rectangle, that determination being below its root, by being prior to the fluxionary method of determining areas ; for in this method, that determination is supposed, and its result assumed in the very notation ; for, the rectangle which is considered as the fluxion of a surface, is denoted by its length drawn into its breadth : And on this account those fluxionists who, in their chapter of quadratures, attempt to determine the measure of a rectangle, argue in a circle, or deduce as a conclusion what was necessarily supposed in the premises. The same may be said with regard to prisms. And, on the other hand, the same method alone would not extend to the determination of the equality of a triangle and hyperbola of the same base and of an infinite length, I being there obliged to call in the assistance of the method of increments.

As extensions are of three kinds, longitudinal, superficial, and solid ; accordingly in this work the science is treated of as distinguished by nature into these three principal parts ; that is, so far as the nature of geometrical figures would conveniently admit. With regard to right lines, plane surfaces, and solids, the distinction is general ; but it does not obtain with regard to curved lines and surfaces. For, by preserving this distinction so entire, as to determine the measure of all kinds of lines in the first part, and of all kinds of surfaces in the second ; I immediately perceived that I could not so well adapt my book to the convenient study of the generality of readers : on which account, in the first part I have treated only of the measure of right lines ; and in the second part, only of the measure of planes which are bounded by right and circular lines, without any curve surfaces excepting that of the sphere. In the third part, which treats of solids, I have generally placed the problems which relate to the measures of the lines, surfaces, and solidities of each particular figure, immediately after each other, because that generally the knowledge of the one of them led to *that* of the others. The latter parts of the book are employed chiefly about the applications of the general problems to several interesting practical subjects in life.

So much for the distribution and order of the parts in general. I shall now proceed to describe the contents of the parts themselves more particularly.

THE



THE whole work consists of 5 parts.

PART I. Contains the mensuration of right lines and right-lined angles, and is divided into 3 sections.

*Sect. 1.* Contains some geometrical definitions and problems. Some of the definitions are new, and, it is presumed, more compleat and less exceptionable than those instead of which they have been substituted. The problems will prepare the learner for making the several figures which afterwards are treated of.

*Sect. 2.* Contains plane Trigonometry, or the measuring of lines and angles. All the cases of trigonometry, both right and oblique angled are here reduced to 3 only; by which means it is easier to be remembered, and more clearly understood. Besides these cases, which perform the business in the common way by means of the sines, &c. of angles, I have given a new and extensive method, by which all the cases of trigonometry are performed independent of all sines, tangents, &c. and without any kind of tables.

*Sect. 3.* Contains the application of trigonometry to the determination of heights and distances; in which a great variety of cases and methods concerning this curious subject are explained.

PART II. Treats on superficial mensuration, or the mensuration of plane figures, and is divided into 2 sections.

*Sect. 1.* Treats on the areas, &c. of right-lined and circular figures. Here, besides many things that are new and curious, are given an explanation of Professor Machin's famous quadrature of the circle, and the demonstrations of some useful approximations to the measures of circular arcs and areas, which had been given by Mr Huygens and Sir Isaac Newton without demonstrations.

*Sect. 2.* Contains a curious and useful collection of questions concerning areas, promiscuously placed, and solved by the rules in the former sections.

PART III. Contains the measuring of solids, and is divided into 8 sections.

*Sect. 1.* Treats on bodies that are bounded by right or by circular lines, viz. prisms, pyramids, the sphere, and the circular spindle.

*Sect. 2.* Treats on the conic sections in general; and, though it be short, contains several things that are new and of great importance.

*Sect. 3.* Treats on the ellipse and the figures generated by it, viz. spheroids and elliptic spindles.

*Sect. 4.* In like manner treats on parabolic lines, areas, surfaces, and solidities. And

*Sect. 5.* On hyperbolic lines, areas, surfaces, and solidities.

In



In these sections the several figures and bodies are very extensively and particularly handled, many of the rules, &c. being new (which indeed is the case throughout the whole book) and interesting; and I have given throughout many pretty approximations of the values of several things which cannot be truly expressed otherwise than by an infinite series, which approximations are generally new, excepting two or three formerly given by Sir I. Newton, and which I have demonstrated.

*Sect. 6.* Treats on the five regular solids or bodies. And

*Sect. 7.* On solid rings. And then

*Sect. 8.* Or the last of this part, contains a promiscuous collection of questions concerning solids, to exercise the learner in the foregoing rules.

PART IV. Contains, in 3 sections, several subjects relating to mensuration in general.

*Sect. 1.* Contains a treatise on the true quadrature and cubature of curves in general. In which are contained some of the most universal and most important propositions that can be made in the subject.

*Sect. 2.* Contains the equidistant-ordinate method; or, the approximate quadrature and cubature of curves in general, by means of equidistant ordinates or sections. A subject by which general and finite rules are discovered for all figures; for some of which they are accurately true, and in the others they are exceedingly near approximations; which are often the most useful rules that can be applied to many things in real practice.

*Sect. 3.* Contains, in a very concise but copious treatise, the relations between the areas and solidities of figures, and the centers of gravity of their generating lines and planes.

Then the

Vth and last PART, in 4 sections, contains the application of the general rules to the most useful subjects of measuring that happen in life. In these subjects very material improvements are almost every where made, both with respect to the matters and the disposition of them.

*Sect. 1.* Contains a treatise of land surveying; explaining the use of the instruments, the methods of surveying, of planning, of computing the contents, of reducing plans, and of division of ground.

*Sect. 2.* Contains a very curious and compleat treatise on gauging. As in like manner doth

*Sect. 3.* On the measuring of artificers works; viz. Bricklayers, Masons,



Masons, Carpenters and Joiners, Slaters and Tilers, Plasterers, Painters, Glaziers, Pavers, and Plumbers. Containing the description of the carpenters rule, the several measures used by each, with the methods of taking the dimensions, and of squaring and summing them up. The whole illustrated by a real case of a building, in which are shewn the methods of entering the dimensions and contents in the pocket book, of drawing out the abstracts, and from them drawing out the forms of the bills.

*Sett.* 4. Contains a curious treatise on timber measuring; in which, among several other things, is given a new rule for measuring round timber, which not only brings out the true content sufficiently exact, but is full as easy in the operation as the common false one either by the pen or the sliding rule; as also some curious rules for cutting timber to the most advantage.

The book then concludes with a large table of the areas of circular segments, extended to ten times the usual length. Other tables are suppressed on account of the great size to which the book hath increased.

It may may be necessary to remark that, in this book, where a curve or a space is said to be non-quadrable, or it is said that the value of a thing cannot be expressed but by an infinite series, or any such-like expression is used, the meaning is that it is not geometrically quadrable, or that its area or value cannot be expressed in a finite number of terms by any method yet known, or by the method there used; and not that it is a thing any way impossible in itself. For, although a space be not quadrable by the methods yet known, it does not therefore follow that its quadrature is an impossible thing, or that some method may not hereafter be discovered by which it may be squared. All the methods used by the geometers before Archimedes were insufficient for the quadrature of any curve space whatever; but were they therefore to infer that no curve could by any means be squared? What have since been done abundantly shew the imprudence and falshood of the assertion. Archimedes discovered a method by which he squared the parabola; and, by the lately-discovered method of fluxions, we can find as many quadrable curves as we please.—But whilst I am urging the possibility of the quadrature of *any* space, I am not ignorant of the pretensions of several people to prove the impossibility of *that* of the *circle* in particular. There are attempts to demonstrate this impossibility in the *Leipfic Acts*, as well as in our own *Philosophical Transactions*; but those



demonstrations are far from being sufficiently general to afford me any conviction of it. On the other hand, several considerations may induce us to believe the possibility of its quadrature: This quadrature is known to depend upon the rectification of the circumference, and that a right line may be of the exact length of the circumference, may be thus conceived; In the description of a circle by a pair of compasses, supposing the describing point or leg to move uniformly, the description will be performed in a certain time and with a certain velocity, which time drawn into the velocity will produce a certain length for the circumference: Again, by conceiving the circle to roll upon a right line at the same time when it turns round its own center, after the manner of the motion of a wheel, the right line and circumference will be congruous and equal by successive and continual application; But without these illustrations, the circumference is evidently of some certain length (for it is not infinitely great) and every finite length in nature must have a measure, it being only the fault of our methods of notation or expression that we have not determined it.—It may be farther added, that a space which by some methods hath a finite expression, is by others expressed by an infinite series, as appears in page 318 and throughout sect. 1 part 4. Even a number is the same way affected, and what is a finite expression in one method or scale, is, in another, expressed by an infinite series; thus  $\frac{1}{3} = .333 \text{ \&c.}$   $\frac{1}{4} = .166 \text{ \&c.}$  and these infinite decimals, and many others in the present decimal scale, would be finite expressions in a duodecimal, and innumerable other scales; and, again, some numbers that are finite in the present scale, would be infinite in them. And who can prove that even the root of 2, or of any other number whose root in the present scale is an infinite series, may not be a terminate quantity in some scale whose root is a square number (for such an one it must be). It is true we have not yet found the area of the circle, nor the root of 2, of 3, &c. in finite terms, yet for each of these we can assign infinite series whose laws of progression are visible; which is more than the ancients could do, or perhaps ever expected could be done, if they even at all thought of such things. And I have no doubt but that hereafter will be discovered a method of squaring any figure whatever. Which is the chief problem in geometry.



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A  
T R E A T I S E  
O F  
M E N S U R A T I O N.

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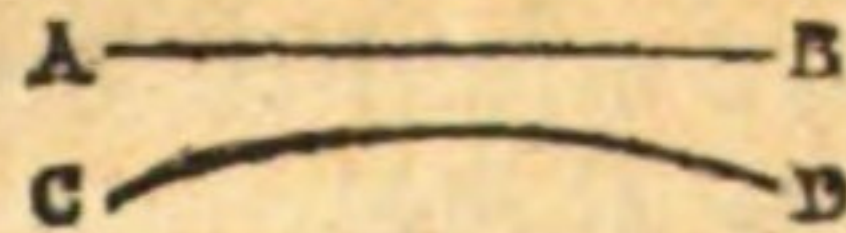
P A R T I.

*Of lineal and angular Mensuration, or the Mensuration of Lines and Angles.*

S E C T I O N I.

*Geometrical Definitions and Problems.*

A Line is a length conceived without breadth, and is denoted by a stroke, as  $AB$  or  $CD$ .



If every part of a line be directed towards the same place, it is called a right line, as  $AB$ ; and if the parts continually vary their directions, it is a curve line, as  $CD$ .

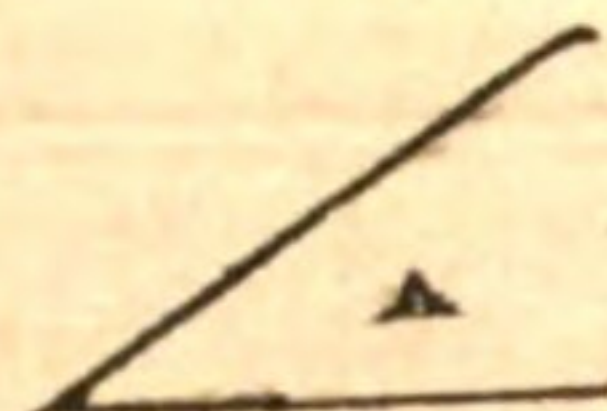
An angle is the mutual inclination of two lines which meet.

A

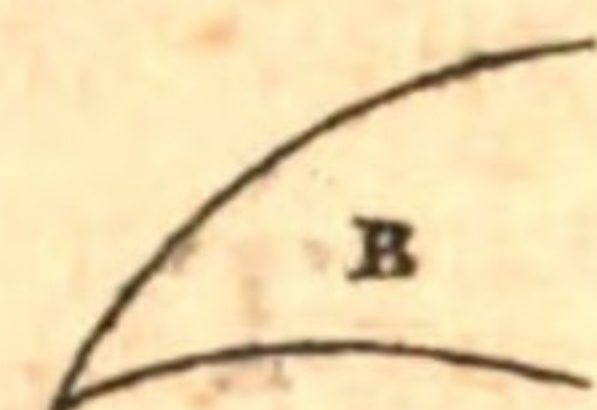
If



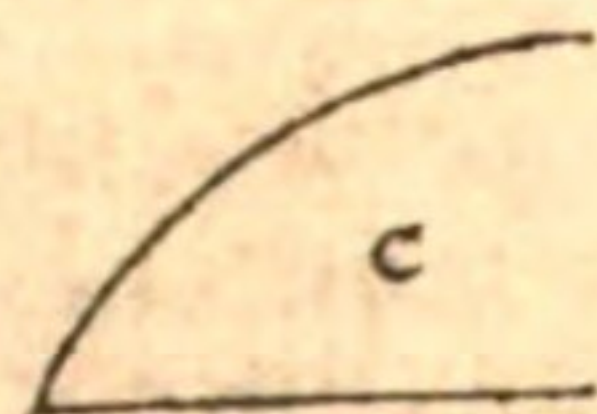
If they be right lines, it is called a rectilinear or a plane angle, as *A*.



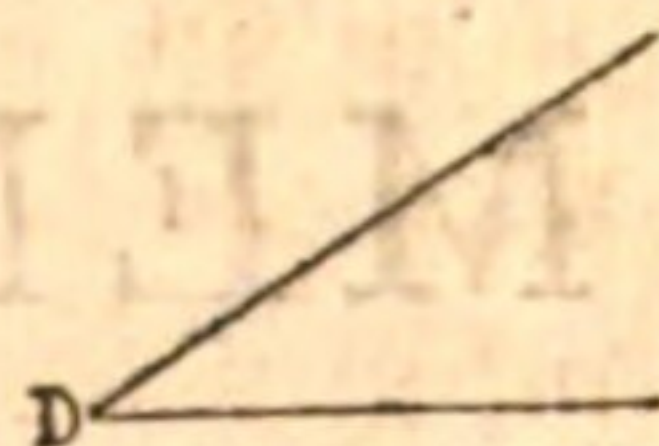
If they be curves, it is a curvilinear angle, as *B*.



If the angle be made by a right line and a curve, it is a mixt angle, as *C*.

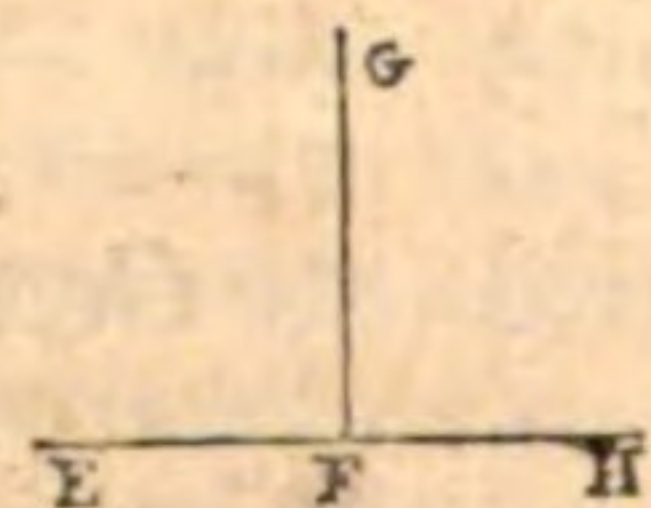


The point where the two lines, or legs of the angle, meet, is called the angular point, as *D*.

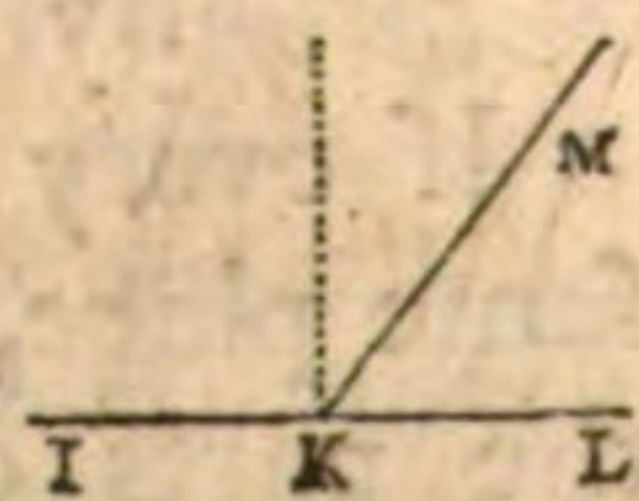


To express an angle, it is sufficient to name the letter denoting the angular point, if no other angle have the same angular point, as the angle *D*; but if there be more than one angle formed at the same point, it is common to express the intended angle by the two letters which are at the ends of its legs with the letter denoting the angular point between them, as the angle *GFE*, or *GFH*.

A right angle is that which is made by two lines meeting so, that either of them being produced, the two angles made by it on each side of the other are equal to one another, as the angle *GFH*, or angle *GFE*.



An acute angle is that which is less than a right angle, as *MKL*; and an obtuse angle is that which is greater than a right angle, as *MKI*.—These two are, also, called oblique angles.



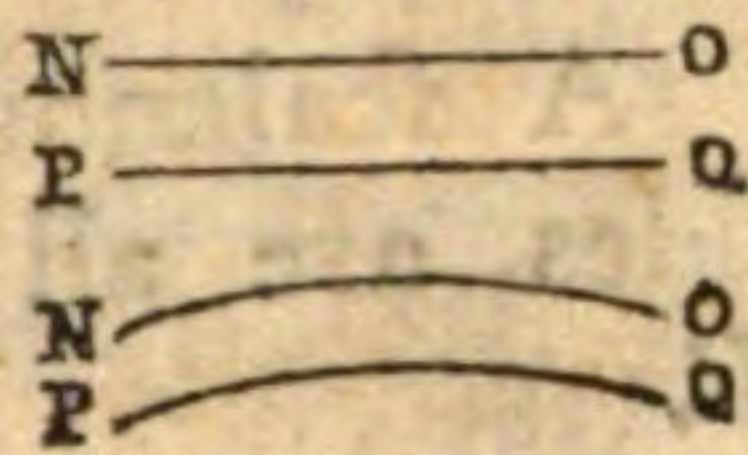
Perpen-



Perpendicular lines are those which make a right angle, as  $GF$  and  $EFH$ .

The distance between any point and a line is a line drawn from that point perpendicularly to the line.

Parallel lines are those of which every point of the one is at the same distance from the other, as  $NO$ ,  $PQ$ .



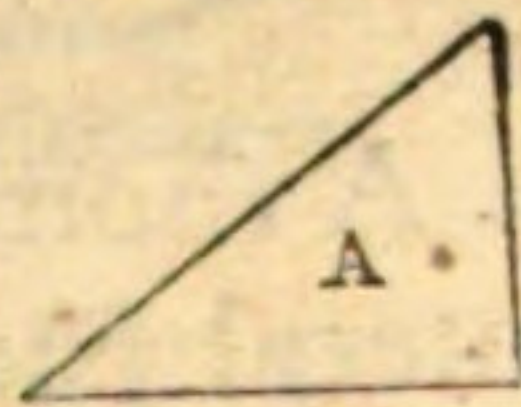
A superficies, or surface is an extension of two dimensions, *viz.* length and breadth.

A plane, or plane superficies, is that with which a right line may, every way, coincide.

A plane superficies receives several denominations according to the number and positions of the lines by which it is terminated, as follows.

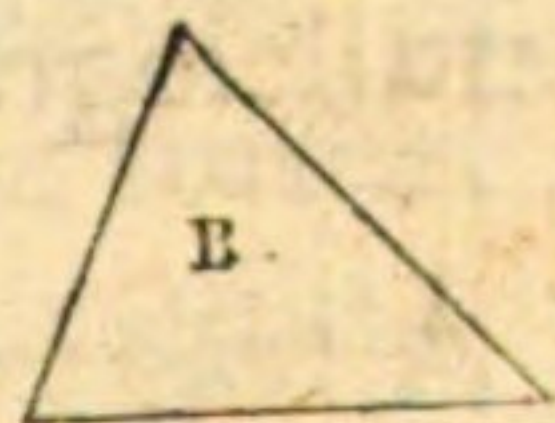
A triangle is a space included by three lines, and of consequence hath three angles; for every rectilineal plane figure hath as many angles as sides.

A right-angled triangle is that which has one right angle, as  $A$ .

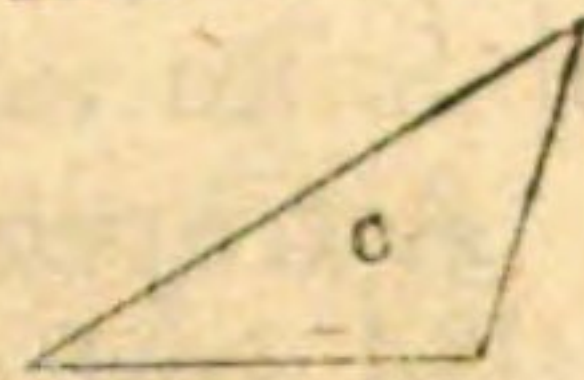


An oblique-angled triangle, is that which hath oblique angles only, as  $B$ , or  $C$ .

An acute-angled triangle is that which hath acute angles only, as  $B$ .



An obtuse-angled triangle is that which hath an obtuse angle, as  $C$ .



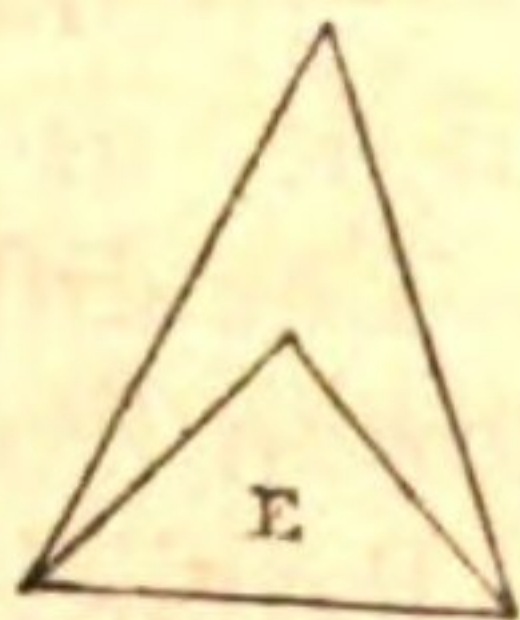
An equilateral, or equal-sided, triangle, is that whose three sides are all equal to each other, as  $D$ .



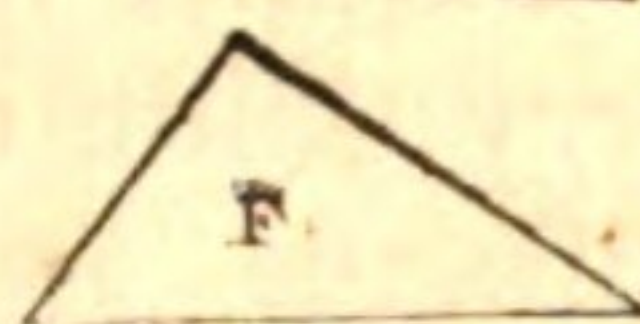
An



An ifosceles triangle is that which hath only two fides equal to each other, as *E*.



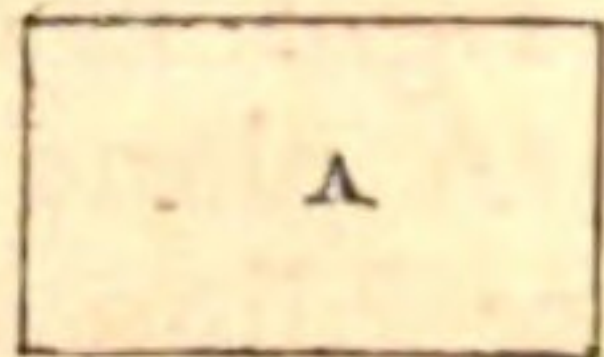
A scalenous triangle is that whose fides are all of different lengths, as *F*.



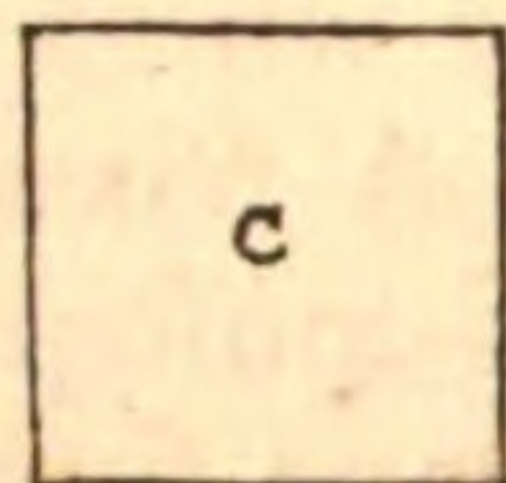
A quadrangle, or quadrilateral, is a plane figure of four fides.

A parallelogram is a quadrangle whose opposite fides are parallel.

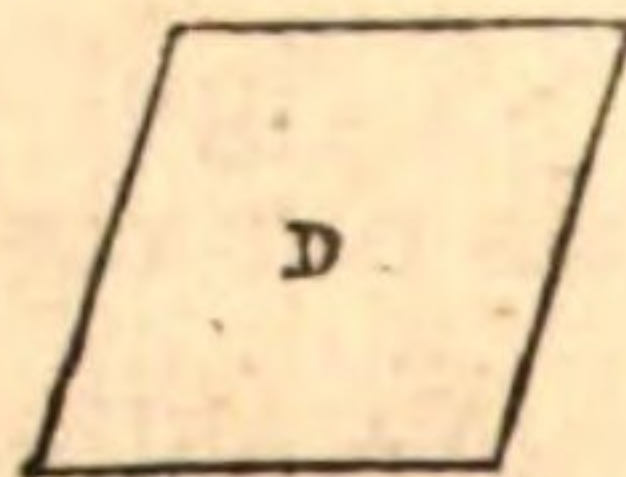
A rectangle is a right-angled parallelogram, as *A*.



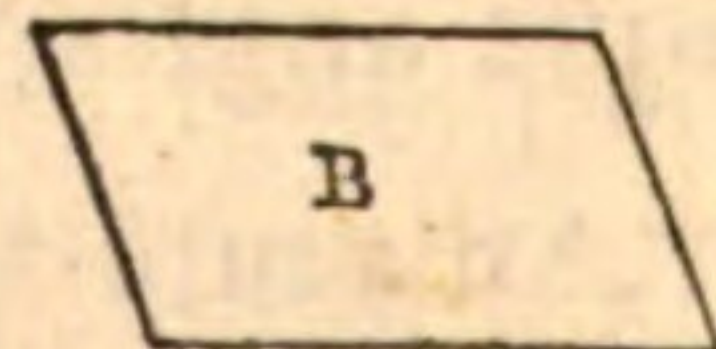
A square is a right-angled equilateral parallelogram, as *C*; or it is an equilateral rectangle.



A rhombus is an oblique-angled equilateral parallelogram, as *D*.



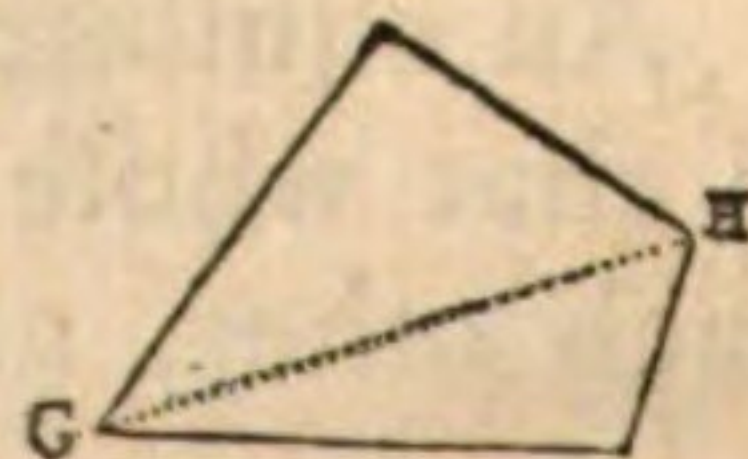
A rhomboides is an oblique-angled parallelogram, as *B*.



A trapezium is a quadrangle both whose pairs of opposite fides are not parallel.

A trapezoid is a trapezium having two fides parallel and the other two inclined.

A line drawn between the two opposite angles of a quadrilateral, is called a diagonal, as *GH*.

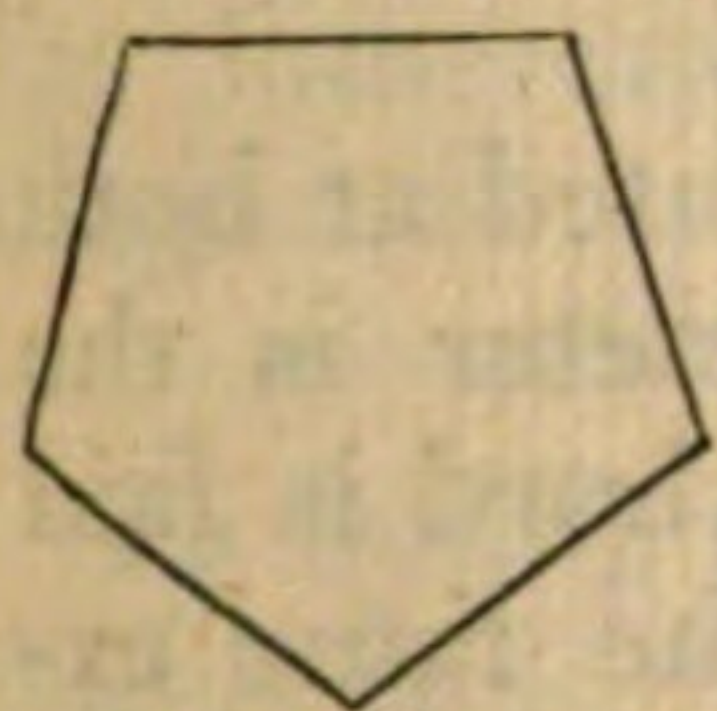


Plain

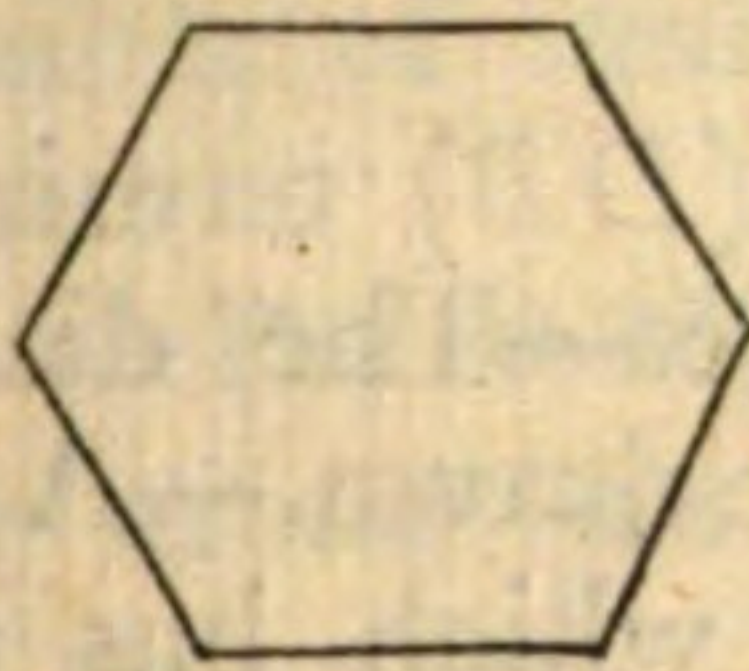


Sect. I. GEOMETRICAL DEFINITIONS. 5

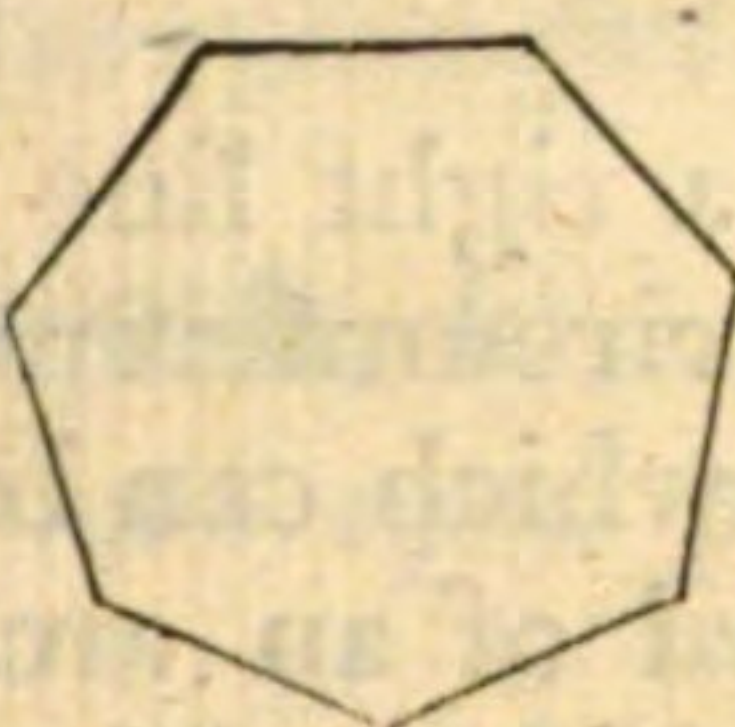
Plane figures of more than four sides are, in general, termed polygons, and receive their particular denominations from the number of their sides; so a pentagon is a polygon consisting of five sides, a hexagon of six, a heptagon of seven, an octagon of eight, a nonagon of nine, a decagon of ten, an undecagon of eleven, and a duodecagon of twelve.



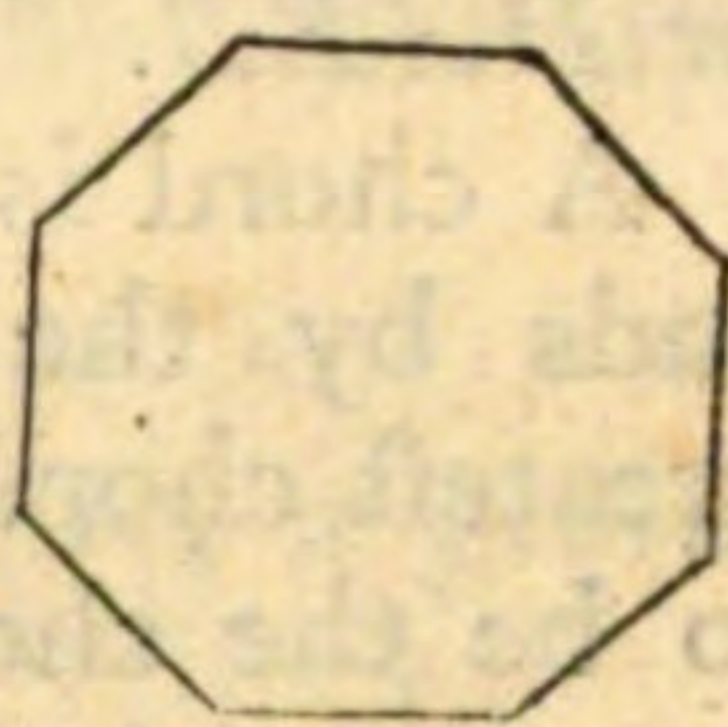
Pentagon



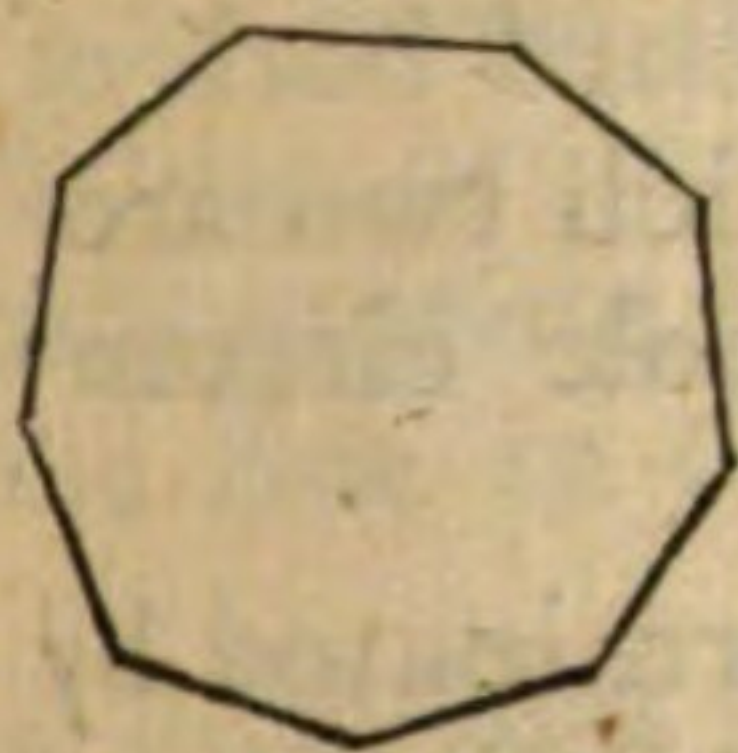
Hexagon



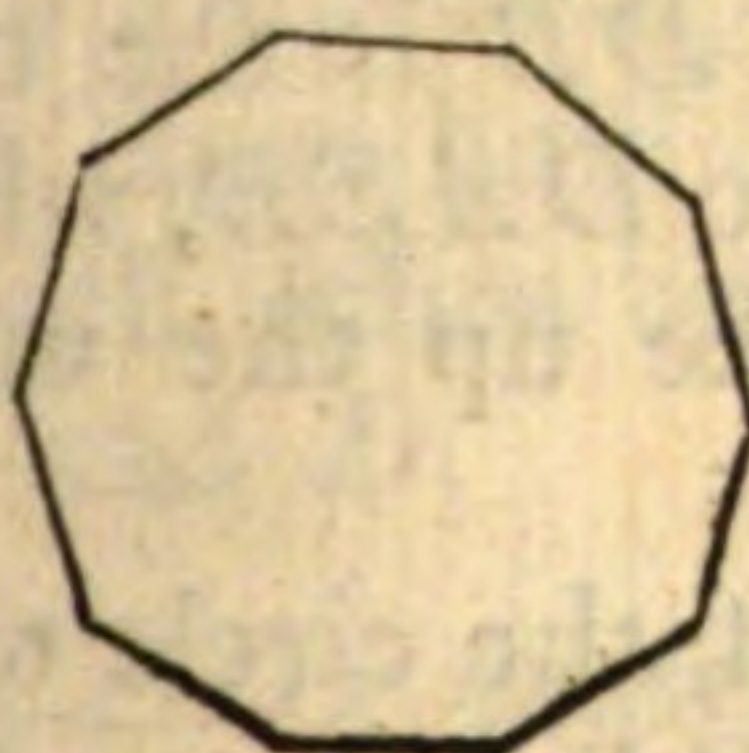
Heptagon



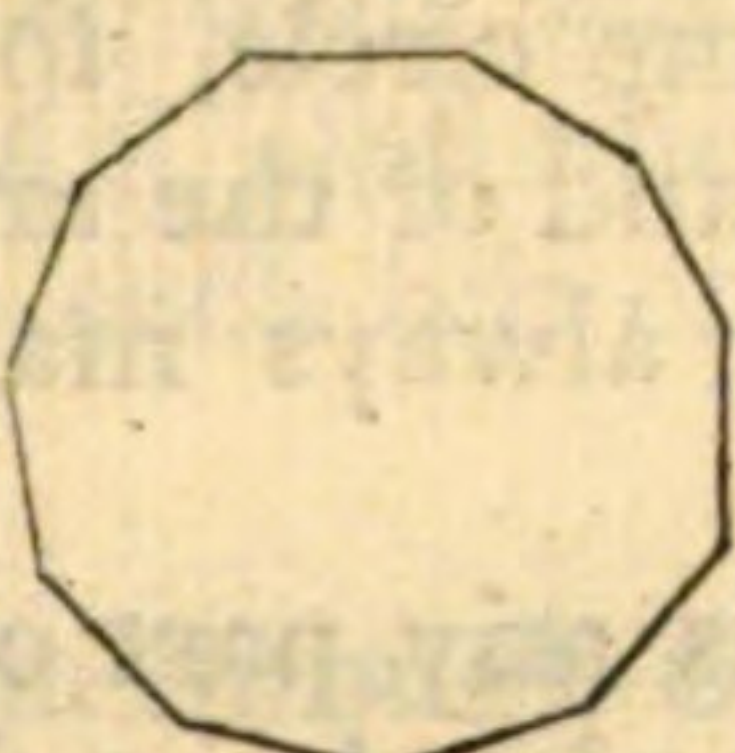
Octagon



Nonagon



Decagon



Undecagon

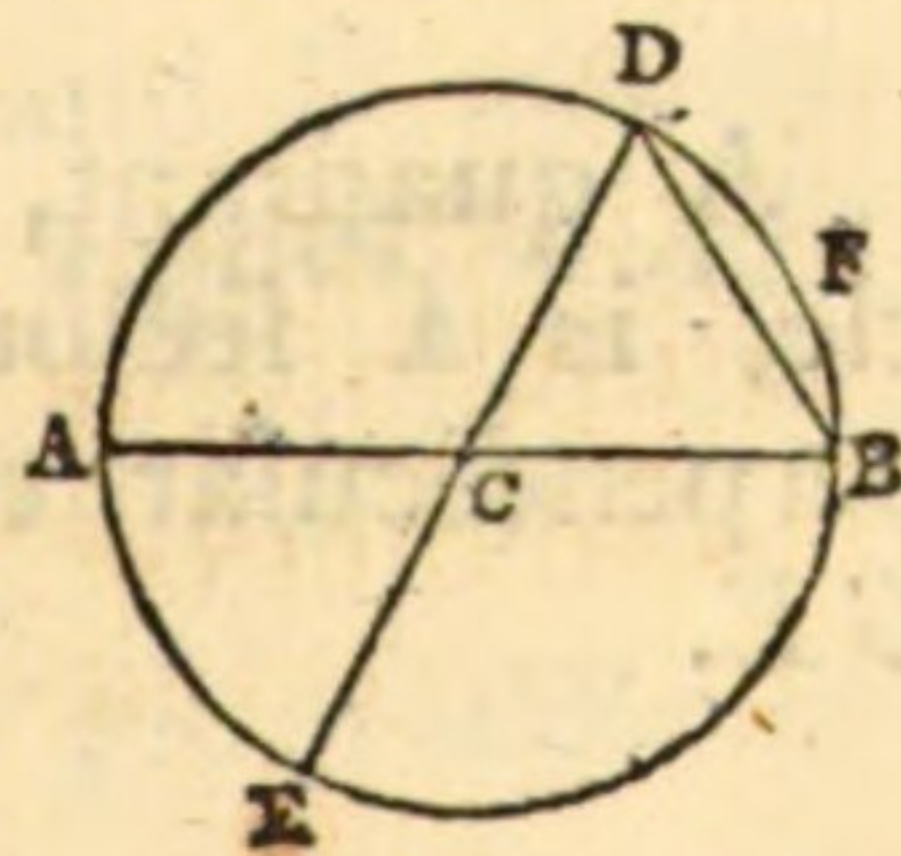


Duodecagon

If the angles, and of consequence the sides of a polygon, be all equal to each other, it is called a regular polygon: if not, it is irregular.

An equilateral triangle is also a regular figure of three sides, and a square is one of four; the former being also called a trigon, and the latter a tetragon.

A circle is a plane figure bounded by a curve (*ADBEA*) called the circumference, every point of which is at the same distance from a certain point (*C*) within, called the center.—The circumference itself is very often called the circle.



B

The



6 GEOMETRICAL DEFINITIONS. Part I.

The radius is the distance between the circumference and the center, or a right line drawn from the circumference to the center, as  $CA$ ,  $CD$ ,  $CB$ , &c.

The diameter is double the radius, or a right line drawn through the center and terminated at the circumference on each side, as  $ACB$ , or  $DCE$ .

An arc is any part of the circumference, as  $DFB$ , or  $DAEB$ .

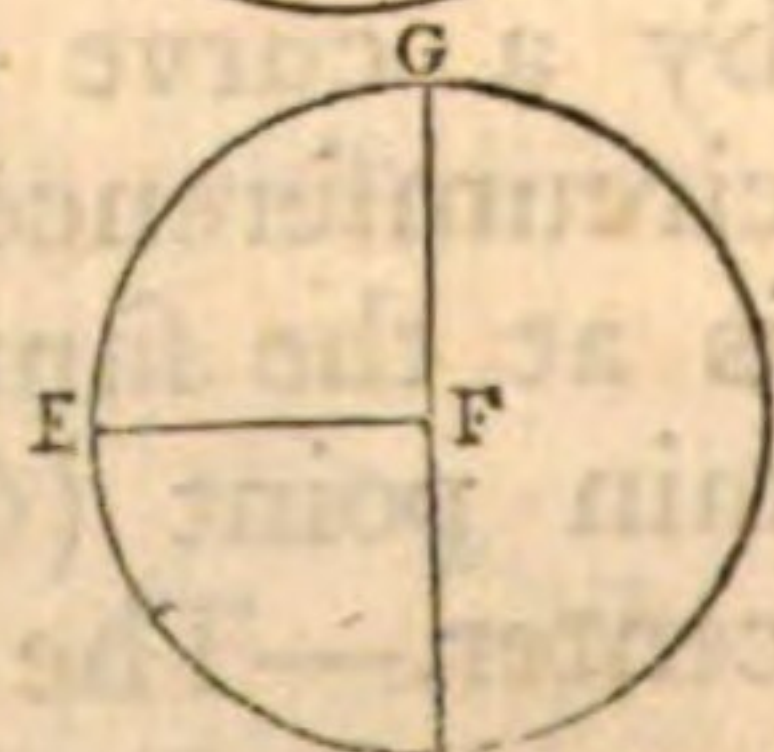
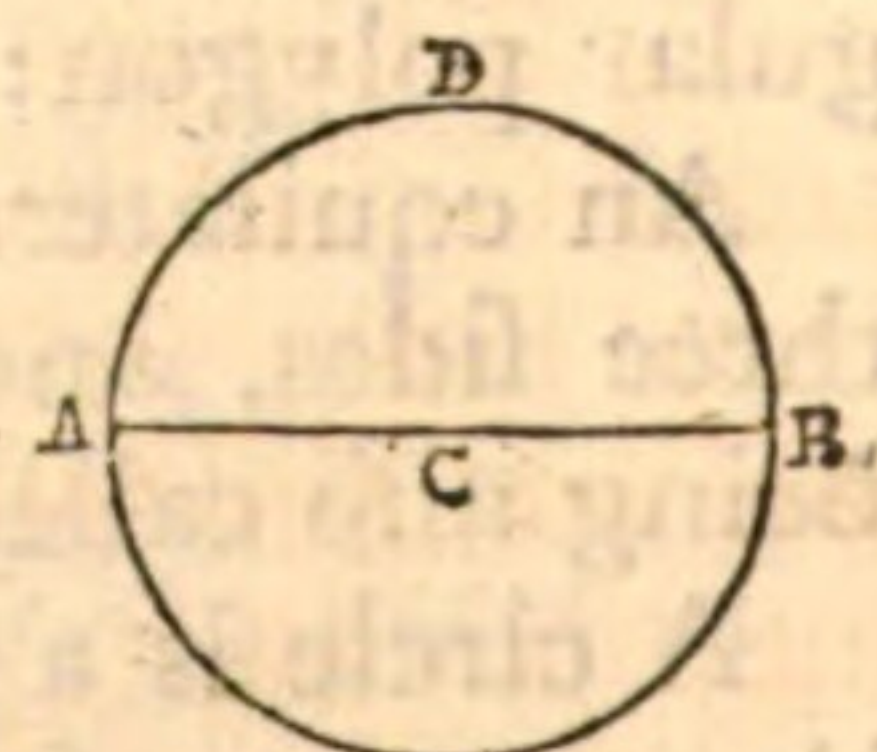
A chord is a right line ( $DB$ ) terminated at both ends by the circumference.—The diameter is the greatest chord which can be drawn.—A chord is said to be the chord of an arc which has the same extremes with itself, and is therefore common to two arcs in the same circle, so  $DB$  is the chord both of the arc  $DFB$ , and of the arc  $DAEB$ , which two arcs taken together always make up the whole circumference.

A segment is any part of the circle terminated by an arc and its chord, as  $DBFD$ , or  $DBEAD$ .

A sector is a part of a circle contained within two radiuses and their included arc, as  $CBFDC$ , or  $CBEADC$ .

A semicircle, or half circle, is a segment whose chord is the diameter, as  $ACBDA$ .

A quadrant, or quarter of a circle, is a sector whose radiuses are perpendicular to each other, as  $EFGE$ .



The



The altitude or height of a figure is a perpendicular let fall from its vertex, or top, to its base.

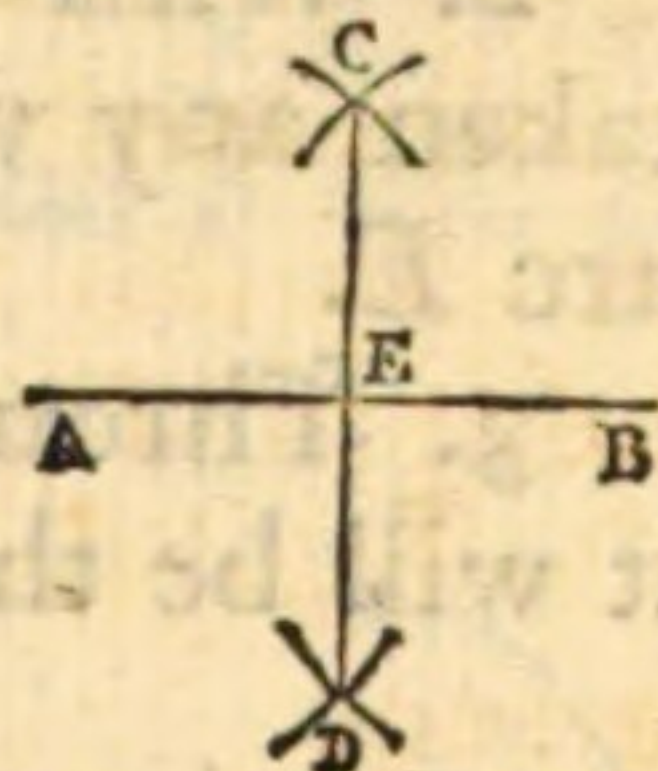
Similar triangles are those whose angles are equal, each to each.

Similar plane figures are those which consist of the same number of angles equal, each to each, and placed in the same order; and may be resolved into the same number of similar triangles by drawing diagonals.

#### PROBLEM I.

*To bisect, or divide into two equal parts, a given line  $AB$ .*

1. From the centers  $A, B$ , with any radius greater than half  $AB$ , describe arcs cutting each other on each side of the line in  $C$  and  $D$ .



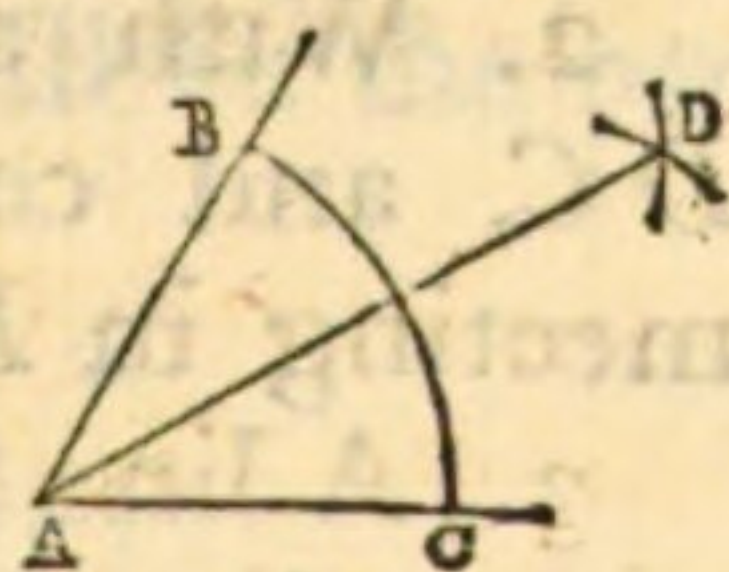
2. A line ( $CED$ ) drawn through  $C$  and  $D$  gives  $E$  the middle of  $AB$ .

#### PROBLEM II.

*To bisect a given angle  $BAC$ .*

1. With the center  $A$ , describe an arc  $BC$ .

2. From the centers  $B$  and  $C$ , with one and the same radius, describe arcs intersecting in  $D$ .



3. A line ( $AD$ ) drawn through  $A$  and  $D$  will divide the angle  $BAC$  into two equal angles,  $BAD$  and  $DAC$ .

#### PROBLEM III.

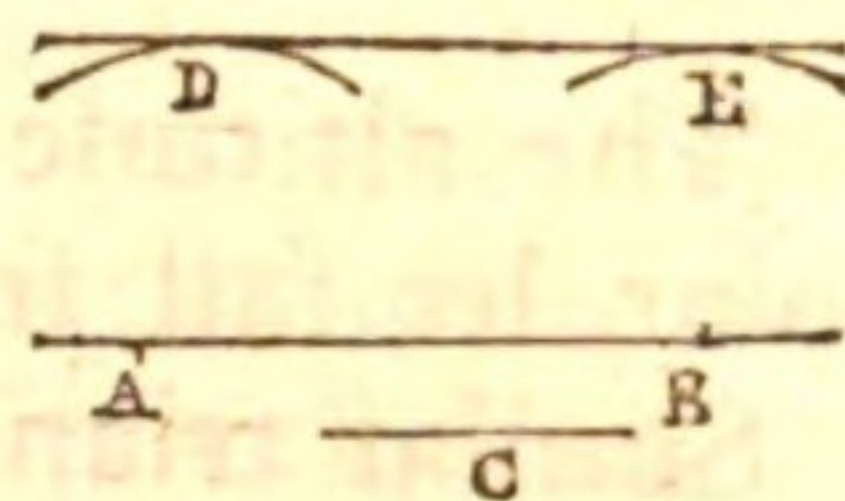
*To draw a line parallel to and at a given distance ( $C$ ) from a given line  $AB$ .*

1. With



1. With the radius  $C$  and centers  $A, B$ , describe the arcs  $D, E$ .

2. Draw a line ( $DE$ ) touching these arcs, and it will be the parallel required.



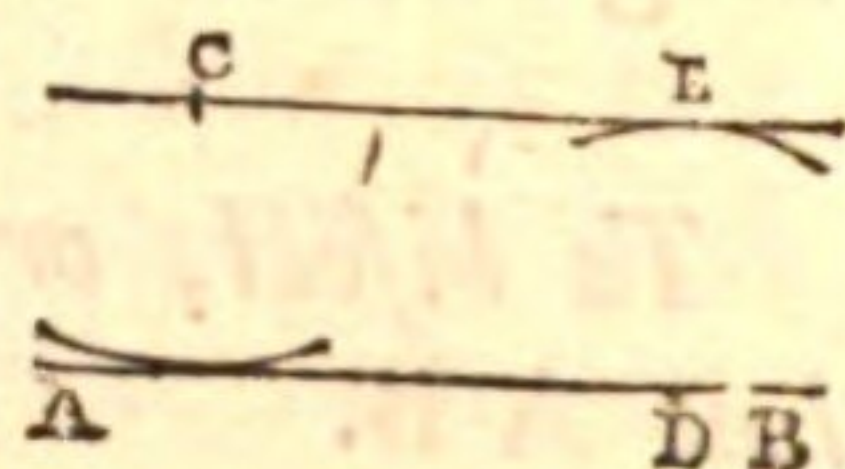
#### PROBLEM IV.

*To draw a line parallel to a given line  $AB$  and passing through a given point  $C$  taken without that line.*

1. Take the nearest distance of the point  $C$  from  $AB$ .

2. With that radius, and center  $D$ , taken any where in  $AB$ , describe an arc  $E$ .

3. Through  $C$  draw  $CE$  to touch the arc  $E$ , and it will be the parallel required.



#### PROBLEM V.

*To raise a perpendicular at a given point  $C$  in a given line  $AB$ .*

1. On each side of  $C$  take the equal distances  $CD, CE$ .

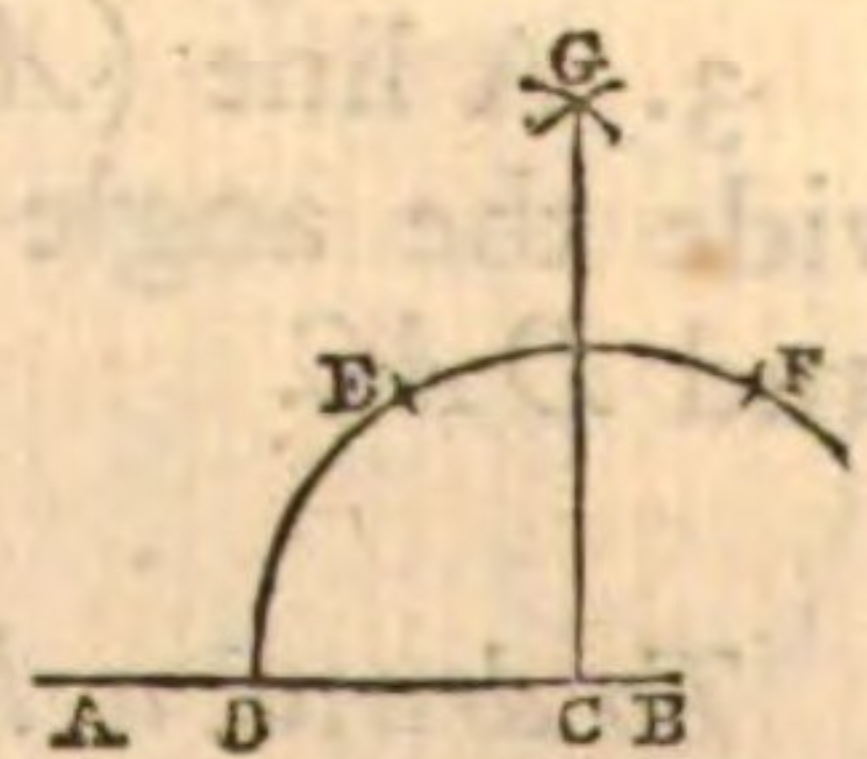
2. With any radius greater than  $DC$ , and centers  $D, E$ , draw arcs meeting in  $F$ .

3. A line ( $CF$ ) drawn through  $C$  and  $F$  will be the perpendicular required.

*Otherwise.*

1. With the center  $C$ , and any radius, describe an arc  $DEF$ .

2. With the same radius, and the center  $D$ , cross the arc in  $E$ , and with the center  $E$  cross it in  $F$ ; and lastly,



with



with the centers  $E$  and  $F$  draw arcs meeting in  $G$ , all with the same radius.

3. A line ( $CG$ ) drawn through  $C$  and  $G$  will be the perpendicular required.

## PROBLEM VI.

*To draw a perpendicular to a line ( $AB$ ) from a point ( $C$ ) taken without the line.*

1. With the center  $C$  describe an arc cutting  $AB$  in  $D$  and  $E$ .

2. With any radius, and the centers  $D$  and  $E$ , describe arcs meeting each other in  $F$ .

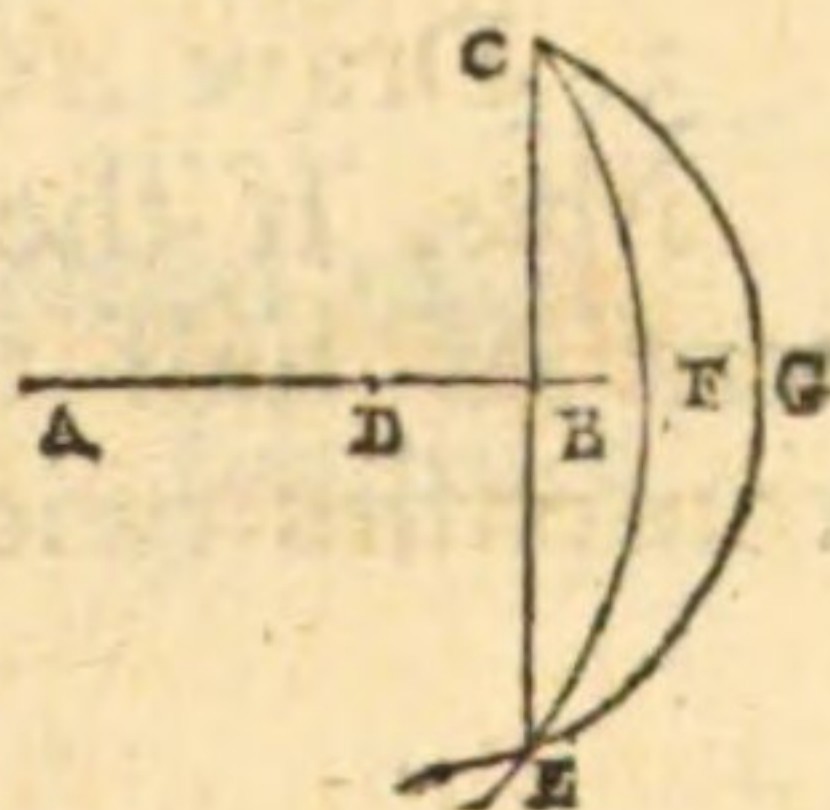
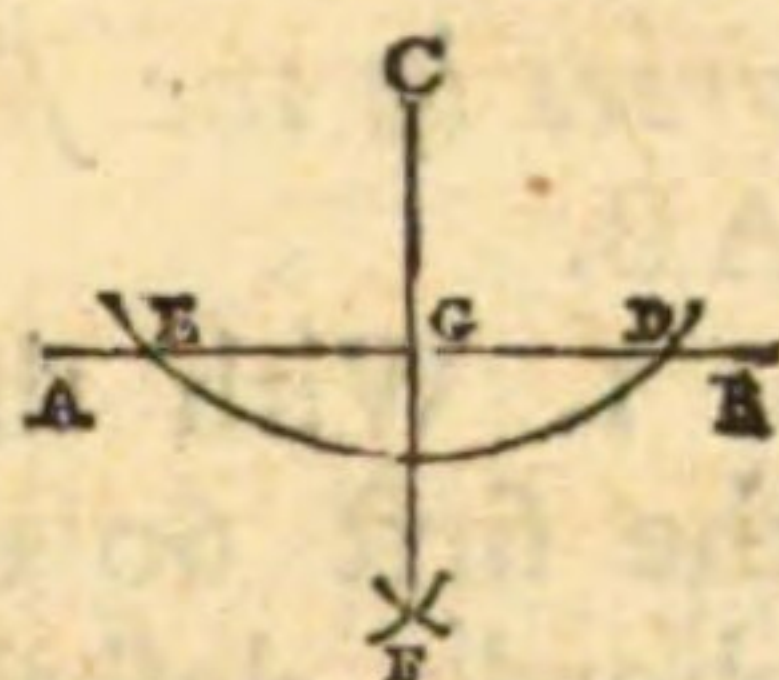
3. A line ( $CGF$ ) drawn through  $C$  and  $F$  will be the perpendicular required.

*Otherwise.*

1. With any center ( $A$ ) in  $AB$ , and radius  $AC$ , describe an arc  $CFE$ .

2. With any other center ( $D$ ) in  $AB$ , and radius  $DC$ , describe another arc  $CGE$ , meeting with the former in  $E$ .

3. A line drawn through  $C$  and  $E$  will be the perpendicular required.



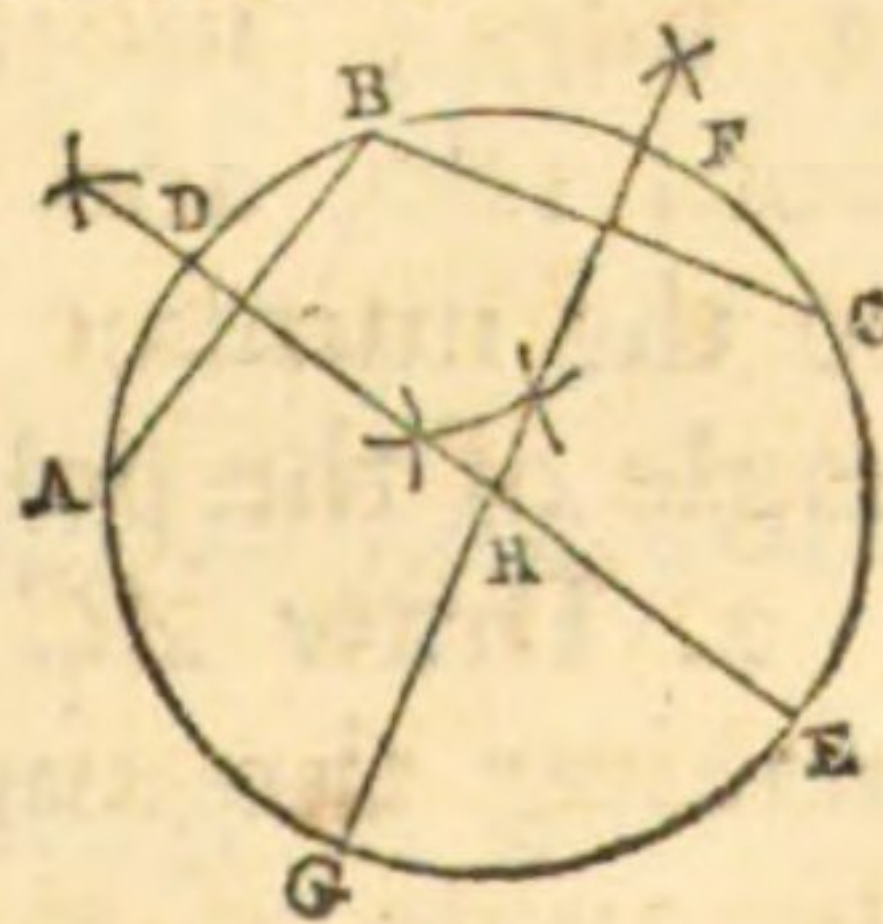
## PROBLEM VII.

*To find the center of a circle ECFDE.*

1. Draw any two chords  $AB$ ,  $BC$ , meeting in the same point  $B$ .

2. Bisect these chords with the chords  $DE$  and  $FG$ , which will be diameters.

Then will the intersection ( $H$ ) of these diameters be the center required.



C

*Note,*



*Note,* It is evident that the same operation will serve when only a part of the circumference is given, as also to find the center of a circle which shall pass through three given points,  $A, B, C$ .

### PROBLEM VIII.

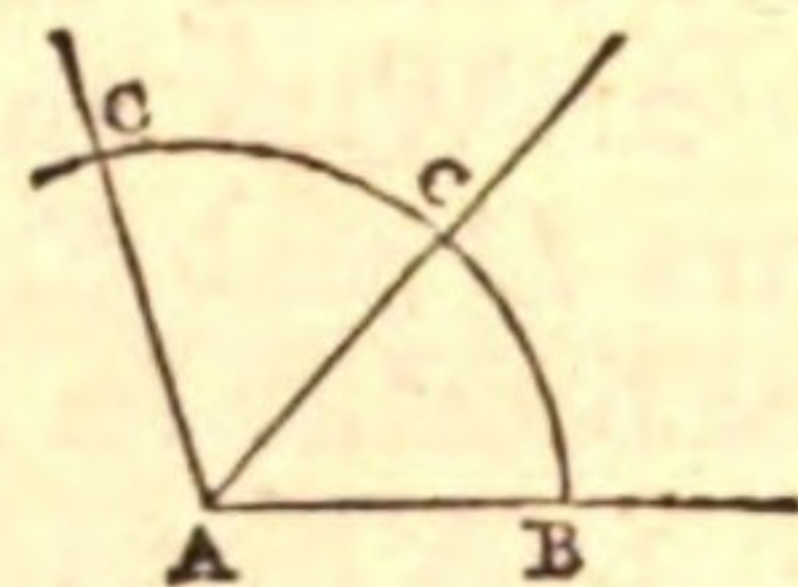
*To make an angle of any proposed number of degrees by the scale of chords, one leg of the angle being  $AB$ .*

1. With the center  $A$  and radius the first 60 degrees on the line of chords, describe an arc  $BC$ .

2. Extend the compasses from the beginning of the chords to the proposed degrees, and set that extent upon the arc from  $B$  to  $C$ .

3. Draw  $AC$ : so will  $BAC$  be the angle required.

*Note,* If the angle exceed 90 degrees take the extent of half the number of degrees and turn the compasses twice over upon the arc from  $B$  to  $C$ .

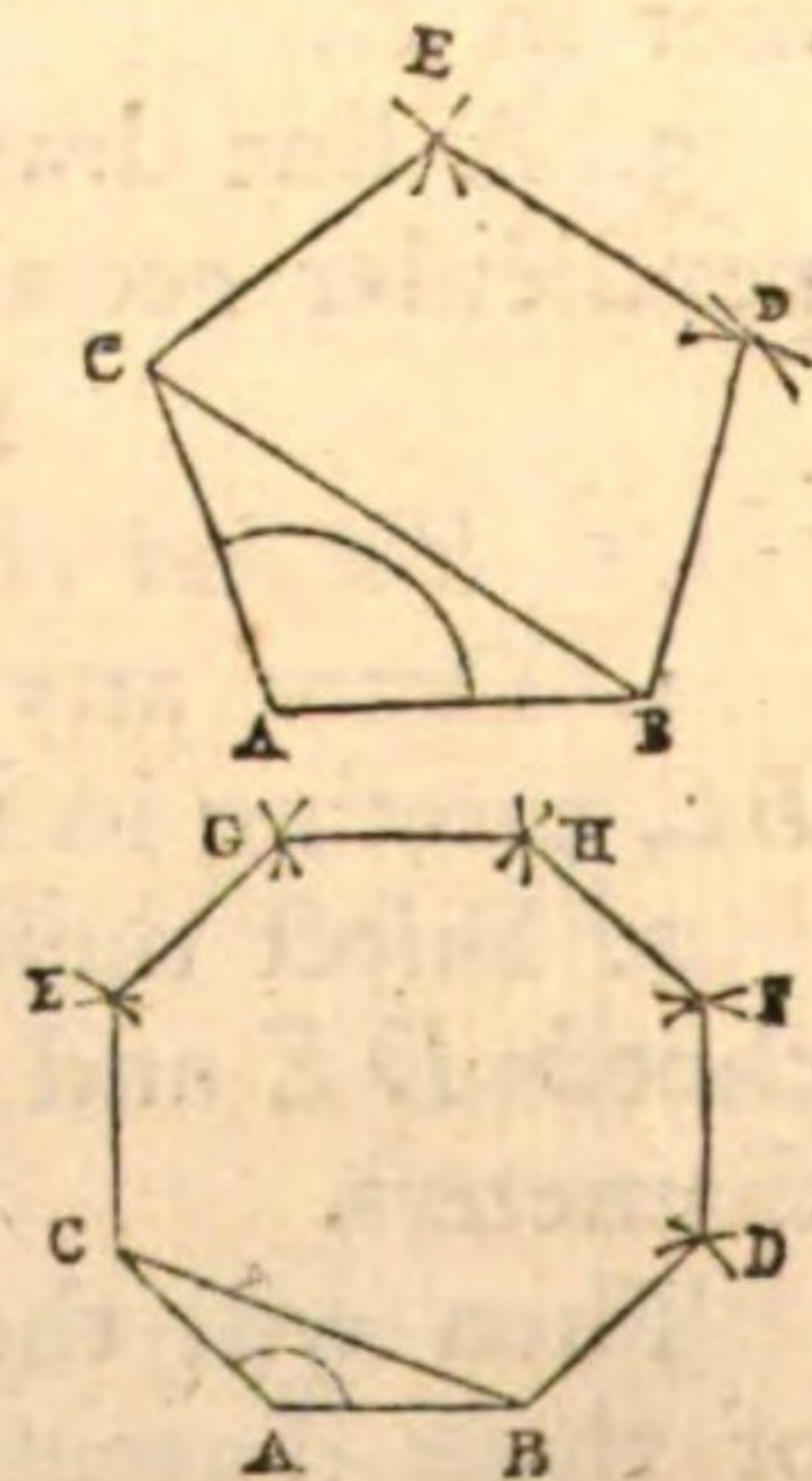


### PROBLEM IX.

*To make a regular figure of any given number of sides, each of which shall be equal to a given line  $AB$ .*

1. Divide 360 by the number of sides; subtract the quotient from 180, and the remainder will be the number of degrees in each angle of the polygon.

2. Draw  $AC$  equal to  $AB$  and making the angle  $BAC$  equal to the angle of the polygon.



3. With

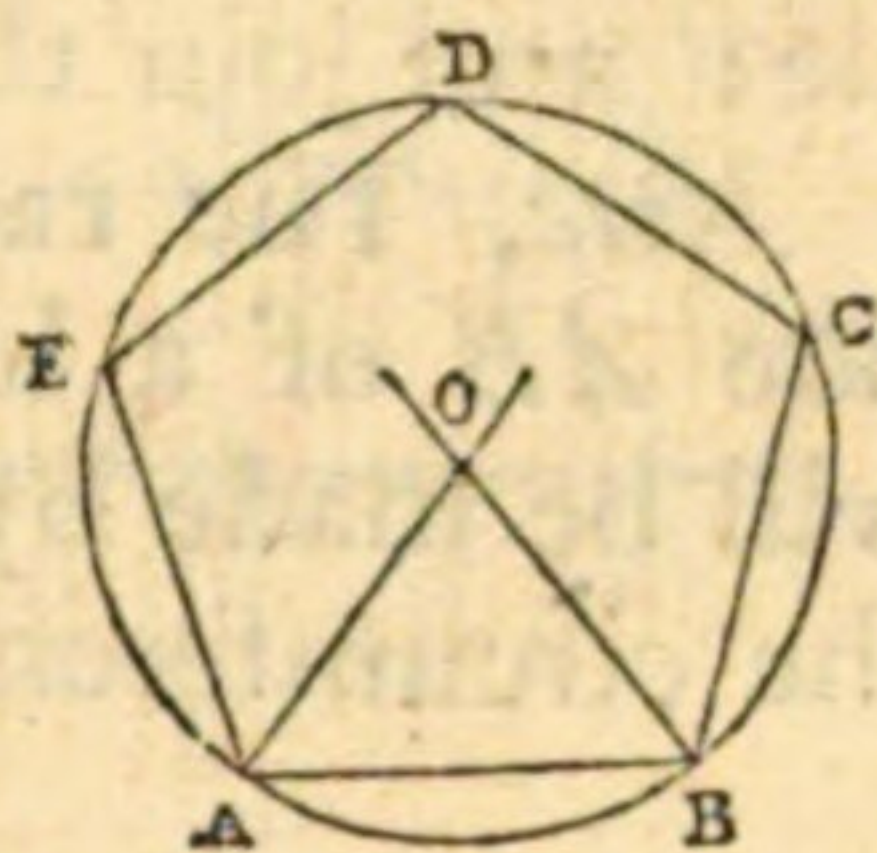


3. With the radius  $AB$  and centers  $B$  and  $C$ , describe arcs; and with the radius  $BC$  and center  $A$ , cross them in  $E$  and  $D$ , two other angular points of the polygon.

4. With the same radius, and in the same manner, find all the other angular points, and then connect them with lines.

*Otherwise.*

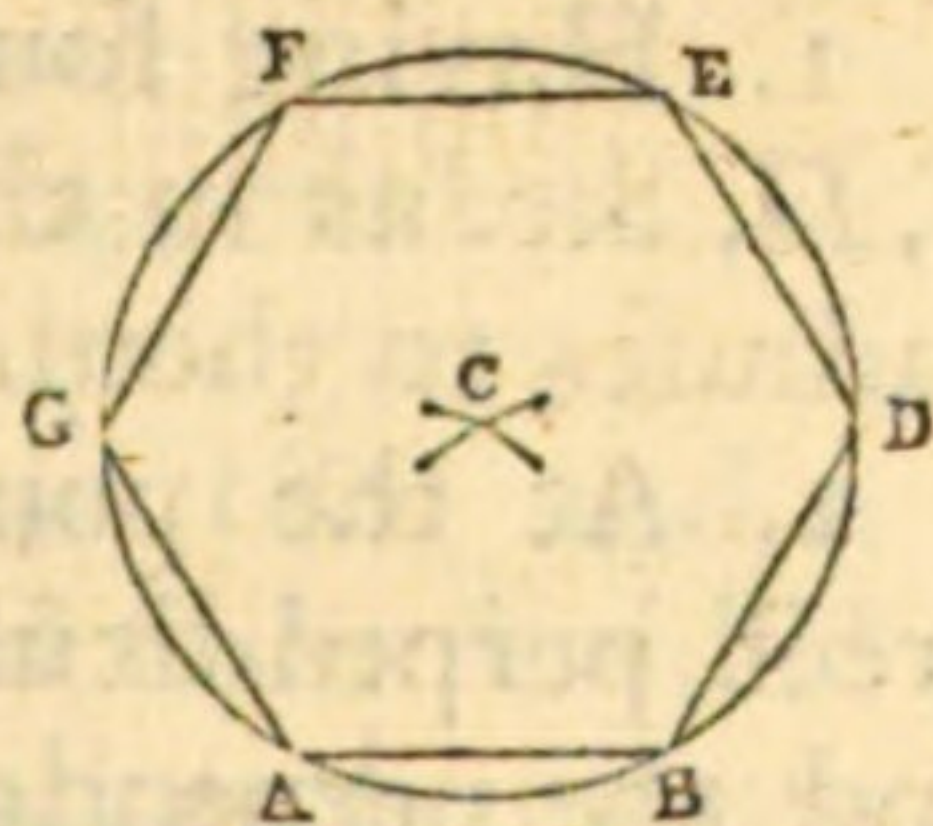
Having found the angle of the polygon, from each end of  $AB$  draw lines  $AO$ ,  $BO$ , making angles with the given line equal to half the angle of the polygon.



Then with the center  $O$  and radius  $OA$  describe a circle, in the circumference of which apply continually the given side  $AB$ .

*Note,* These methods will serve for all polygons; but a hexagon may be much easier formed thus.

1. With the centers  $A$ ,  $B$ , and radius  $AB$ , describe arcs intersecting in  $C$ .



2. With the same radius, and center  $C$ , describe a circle  $ABD EFG$ .

3. Apply  $AB$  continually to the circumference, from  $B$  to  $D$ , from  $D$  to  $E$ , from  $E$  to  $F$ , and from  $F$  to  $G$ . And connect these points with right lines.

#### PROBLEM X.

*In a given circle to inscribe a regular polygon of any proposed number of sides.*

*Note,* A figure is said to be inscribed in a circle, and a circle is said to be circumscribed about a figure when



when all the angular points of the figure touch the circumference of the circle.

1. Divide 360 by the number of sides.
2. At the center  $O$  [*last figure but one*] make the angle  $BOA$  of so many degrees as are expressed by the quotient.
3. Lay the extent  $AB$  from  $B$  to  $C$ , from  $C$  to  $D$ , from  $D$  to  $E$ , &c. till you have gone round the circle; and join the points  $A, B, C, D$ , &c.

*Note*, The radius  $AC$  [*last figure*] is equal to the side  $AB$  of a hexagon, and therefore the hexagon will be made by applying the radius continually to the circumference.

#### PROBLEM XI.

*About a given circle to circumscribe a polygon.*

*Note*, A figure is said to circumscribe a circle, and a circle is said to be inscribed in a figure, when the circumference touches all the sides of the figure.

1. Having found the points  $A, B, C, D$ , &c. as in the last problem, draw radiuses to them.

2. At the points  $A, B, C, D$ , &c. erect perpendiculars to the radiuses, and the perpendiculars will form the polygon required.

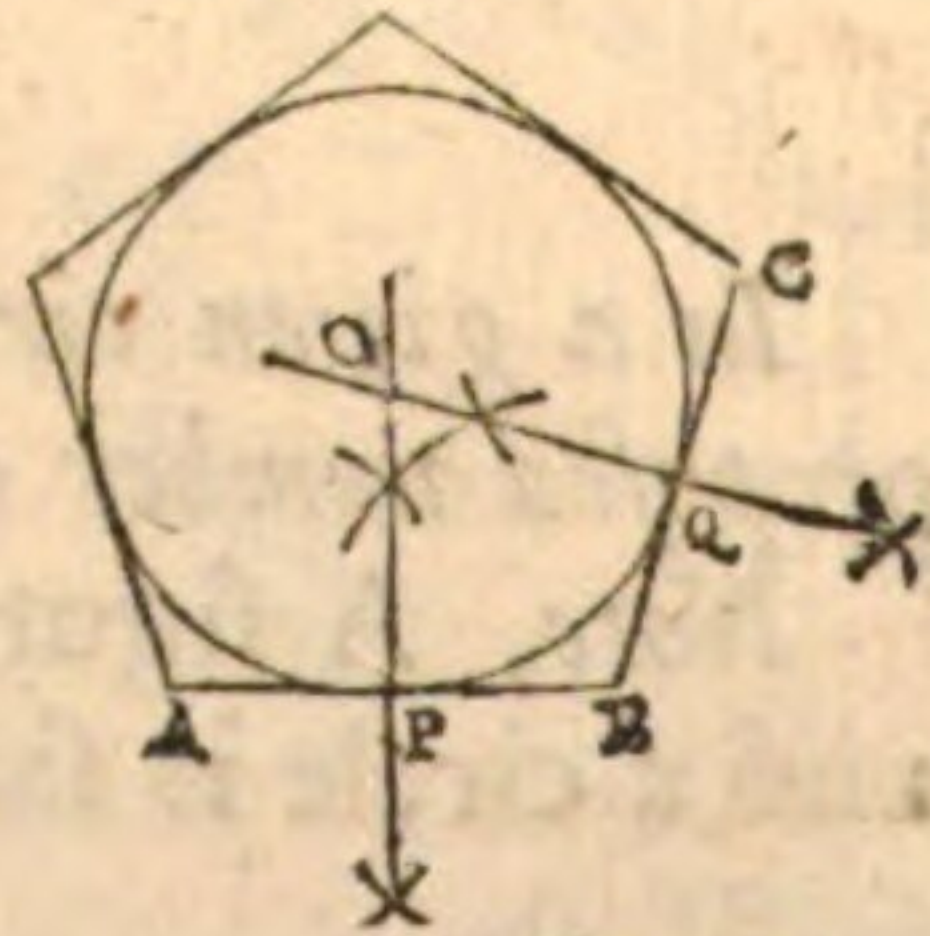


#### PROBLEM XII.

*Within a given polygon to inscribe a circle.*

1. Bisect any two sides  $AB, BC$ , of the polygon by the lines  $PO, QO$ .

2. With the center  $O$  and radius  $OP$ , describe the circle required.



P R O-

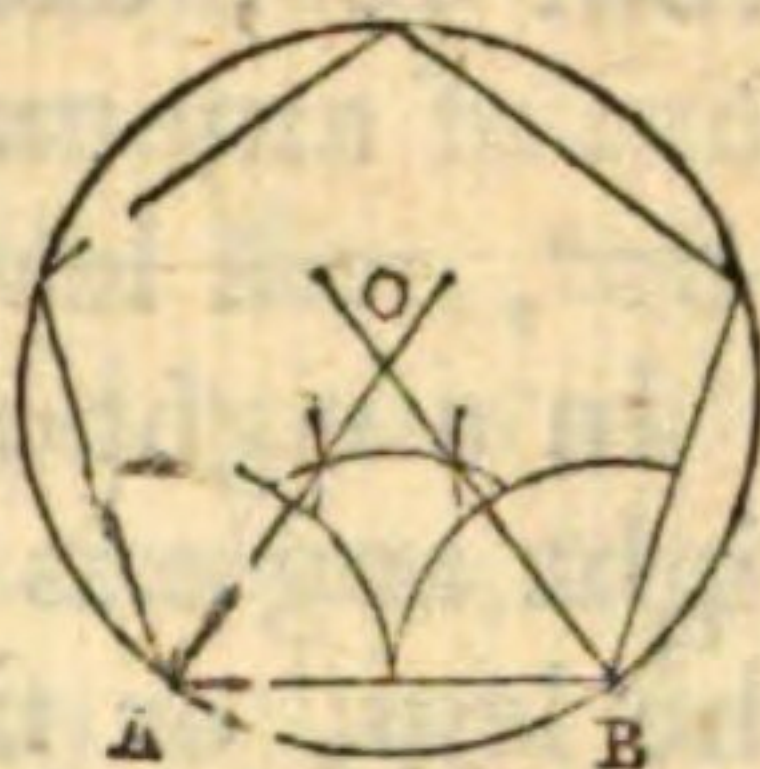


## PROBLEM XIII.

*About any given polygon to circumscribe a circle.*

1. Bisect any two angles,  $A$ ,  $B$ , of the polygon, by the lines  $AO$ ,  $BO$ .

2. With the center  $O$ , and radius  $OA$ , describe the circle.



## S E C T. II.

## PLANE TRIGONOMETRY.

PLANE Trigonometry teaches the relations and calculations of the sides and angles of plane triangles.

The circumference of a circle is supposed to be divided into 360 equal parts, called degrees; of which, therefore, a semicircle contains 180, and a quadrant 90. Each degree is supposed to be divided into 60 equal parts, called minutes, and each minute into 60 seconds, &c.—Degrees are marked with a small  $^{\circ}$  near the top of the figure, minutes with a stroke, seconds with two strokes, &c. in the same place: so 57 degrees, 30 minutes, 12 seconds, are wrote thus,  $57^{\circ} 30' 12''$ .

The angles of triangles are measured by the number of degrees contained in the arc cut off by the legs of the angle, and whose center is the angular point.—A right angle is an angle of 90 degrees.—

D

The







The cosine ( $BH$ ), cotangent ( $CI$ ), and cosecant ( $FBI$ ) of an arc ( $AB$ ) is the sine, tangent, and secant of ( $CB$ ) the complement of that arc.

From the definitions it is evident that the sine, tangent, and secant are common to two arcs which are the supplements of each other: so the sine, tangent, &c. of  $50^\circ$  is also the sine, tangent, &c. of  $130^\circ$ .

The sine, tangent, &c. of an angle, is the sine, tangent, &c. of the arc, or its degrees, by which the angle is measured.

The sines, tangents, &c. of every degree and minute in a quadrant are calculated to an assumed radius, and ranged in tables for use; but because that trigonometrical operations with these natural sines, tangents, &c. require tedious multiplications and divisions, the logarithms of these sines, tangents, &c. are taken, and ranged in tables; and the logarithmic sines, tangents, &c. are generally used, as they require only additions and subtractions instead of the multiplications and divisions.

In a triangle there must be given three parts, one of which, at least, must be a side, because the same angles are common to an infinite number of triangles.

In plane trigonometry there are three cases or varieties only, *viz.*

1. When two of the three parts given are a side and its opposite angle.
2. When there are given two sides and their included angle.
3. When the three sides are given.

When the terms are stated according to rule, then

*To work by the logarithms.*

Take, from the tables, the logarithms answering  
to



to the three given terms; add those of the second and third together, subtract the first from the sum, and the remainder will be the logarithm of the fourth term; then find, in the tables, the number answering to this logarithm, and it will be the fourth term required.

*To work a stating by the lines on the back of the common two-foot scales.*

Extend the compasses from the first term to the second, or third, which happens to be of the same kind with it, then that extent will reach from the other term to the fourth, taking both extents towards the same side.

*Note 1.* For the sides of triangles you use the line of numbers (marked Num.) and for the angles, the lines of sines or tangents (marked Sin. and Tan.) according as the proportion respects sines or tangents. —If the extent upon the tangents reach beyond the line, set it so far back as it reaches over.

2. If the figure be constructed, and the unknown terms measured, it will confirm the other methods of solution.

3. In the figure it is common to mark the given terms with a dash ('), and the required terms with a cypher (°). —The word angle is sometimes expressed by this mark <.

#### PROBLEM I.

*Given three parts, such that an angle and its opposite side are two of them, to find the rest.*

RULE.



## R U L E.

\* In any plane triangle, the sides are proportional to the sines of their opposite angles. That is,

As one side:

Is to the sine of its opposite angle::

So is any other side:

To the sine of its opposite angle.

And the contrary.

*Note 1.* To find an angle, begin the proportion with a side opposite to a given angle; and to find a side, begin with an angle opposite to a given side.

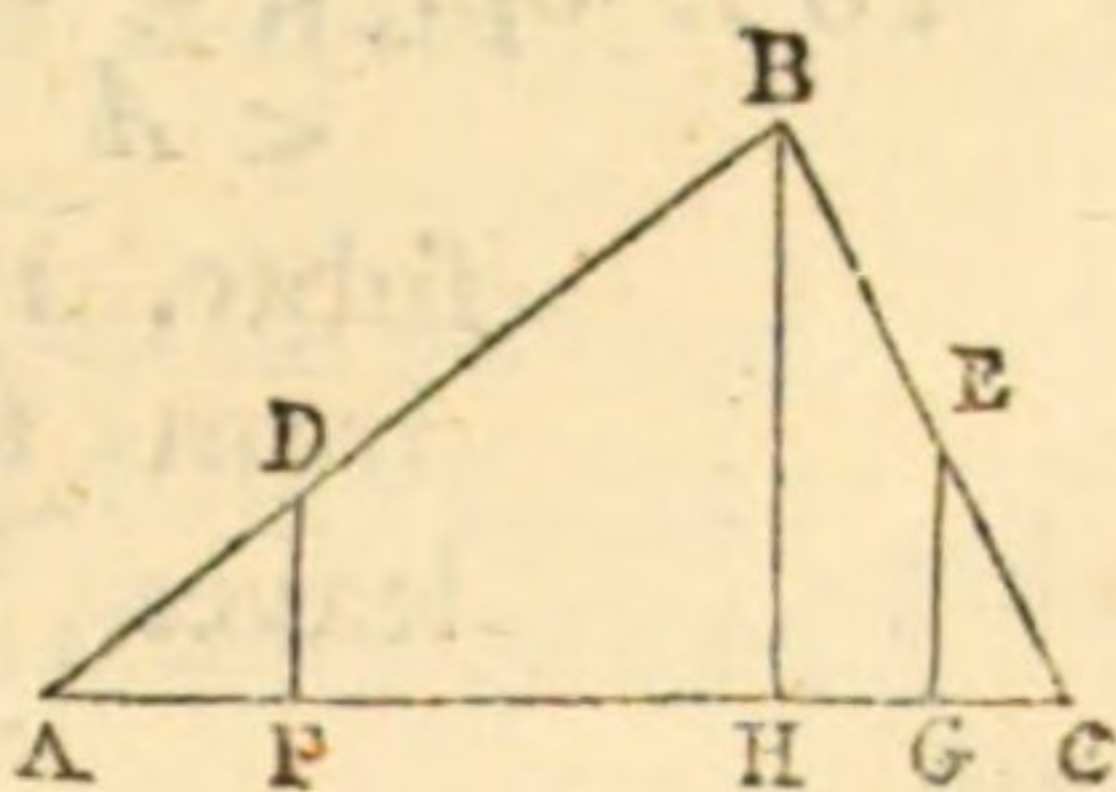
2. An angle found by this rule is always ambiguous, except it be a right angle, or except that the magnitude of the given angle prevent the ambiguity, because that the sine answers to two angles which are the supplements of each other; and accordingly the construction gives two triangles with the same given parts; and when there is no restriction or limitation included in the proposition, either of them may be taken. The degrees, in the table, answering to the sine, is the acute angle; and if the angle be obtuse, take those degrees from  $180^\circ$ , and the remainder will be the obtuse angle required.

E

When

## \* DEMONSTRATION.

Let  $ABC$  be any triangle: in  $AB$  assume any point  $D$ , take  $CE = AD$ , and upon  $AC$  demit the perpendiculars  $DF$ ,  $EG$ ,  $BH$ ; then will  $DF$  and  $EG$  be the sines of the angles  $A$ ,  $C$ , to the general radius  $AD$  (or  $CE$ ).—Now, from similar triangles, we shall have  $\left\{ \begin{array}{l} AB : BH :: AD : DF \\ CB : BH :: AD (=CE) : EG \end{array} \right\}$  and hence, of equality,  $AB : BC :: DF : EG$ .





When the given angle is obtuse, or right, there can be no ambiguity; for then neither of the other angles can be obtuse, and then the construction will produce but one triangle.

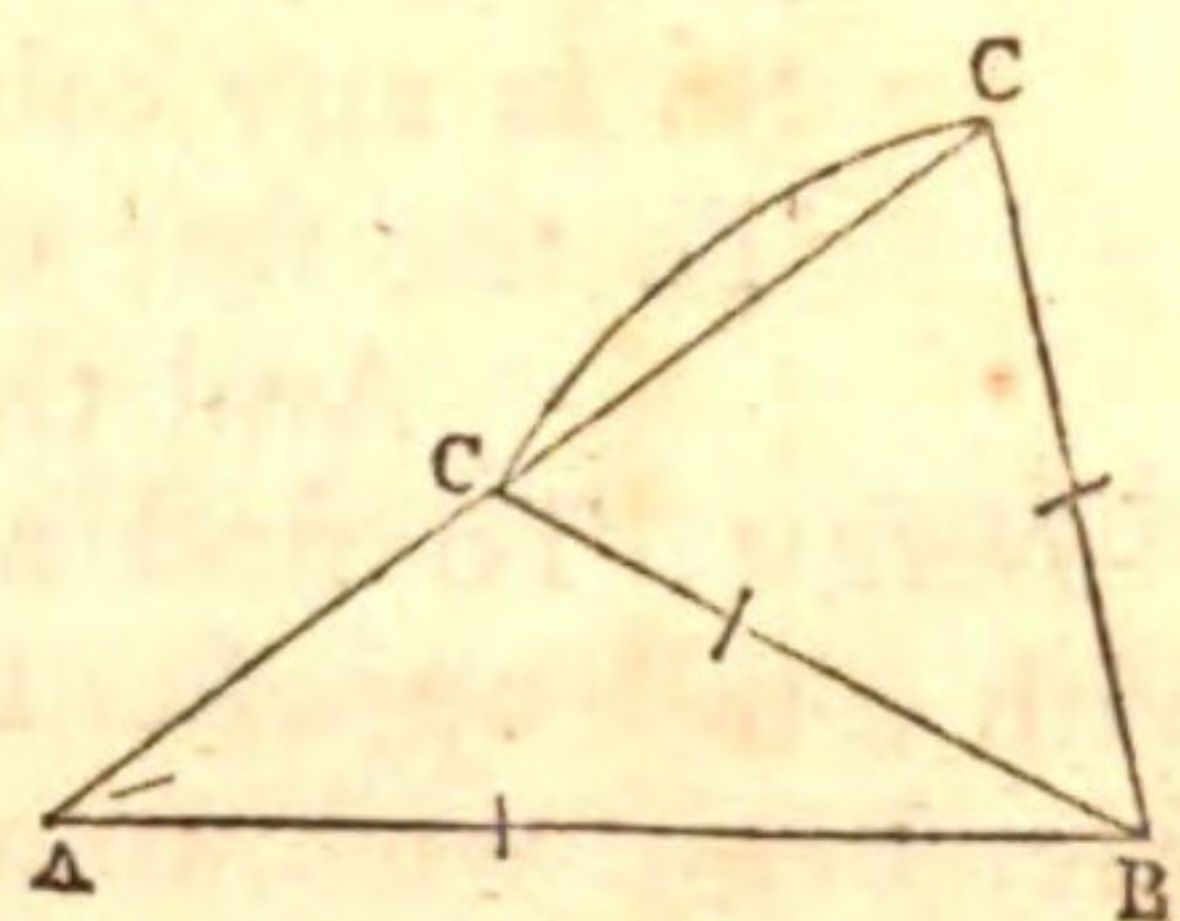
## EXAMPLE I.

In the plane triangle  $ABC$ ,

Given  $\begin{cases} AB=345 \\ BC=232 \end{cases}$  yards  
 $\angle A=37^\circ 20'$

Required the other parts.

*Construction.*



1. Draw the line  $AB=345$  from some convenient scale of equal parts.
2. Make the angle  $A=37^\circ 20'$ .
3. With the center  $B$  and radius 232, taken from the same scale of equal parts, cross  $AC$  in  $C$ .
4. Draw  $BC$ , and the triangle is constructed.

Then the angles  $B$  and  $C$ , measured by the scale of chords, will be found to be

$\begin{cases} 27^\circ 04' \\ 78 \ 16 \end{cases}$  and  $\begin{cases} 115^\circ 36' \\ 64 \ 24 \end{cases}$ , and  $AC$ , measured by the scale of equal parts, will be found to be  $\begin{cases} 174.07 \\ 374.56 \end{cases}$ .

*Calculation by the logarithms.*

|                               |                     |       |        |                  |           |
|-------------------------------|---------------------|-------|--------|------------------|-----------|
| As $BC$                       | —                   | —     | 232    | —                | 2.3654880 |
| To sine of its op. $\angle A$ | 37° 20'             | —     |        | —                | 9.7827958 |
| So $AB$                       | —                   | —     | 345    | —                | 2.5378191 |
| To s. op. $\angle C$          | 115° 36' or 64° 24' | —     |        | —                | 9.9551269 |
|                               | $\angle A$          | 37 20 | 37 20  |                  |           |
| subtr.                        | 152 56 or 101 44    |       |        | sub.             |           |
| from                          | 180 00              |       | 180 00 |                  |           |
| leaves                        | 27 04 or 78 16      |       |        | the $\angle B$ . |           |

Then



Then

$$\begin{array}{rcl}
 \text{As } s. < A & \text{---} & 37^\circ 20' & \text{---} & 9.7827958 \\
 \text{To op. side } BC & \text{---} & 232 & & 2.3654880 \\
 \text{So } s. < B & \text{---} & \left\{ \begin{array}{l} 27^\circ 04' \\ 78 \quad 16 \end{array} \right\} & \text{---} & \left\{ \begin{array}{l} 9.6580371 \\ 9.9908291 \end{array} \right\} \\
 \text{To op. side } AC & \text{---} & \left\{ \begin{array}{l} 174.072 \\ 374.56 \end{array} \right\} & \text{---} & \left\{ \begin{array}{l} 2.2407293 \\ 2.5735213 \end{array} \right\}
 \end{array}$$

*By the scale.*

In the first proportion, Extend from 232 to 345 upon the line of numbers, that extent will reach, upon the fines, from  $37^\circ 20'$  to  $64^\circ 24'$  the angle  $C$ .

In the second proportion, Extend from  $37^\circ 20'$  to  $\left\{ \begin{array}{l} 27^\circ 04' \\ 78 \quad 16 \end{array} \right\}$  upon the fines; that extent will reach,

upon the numbers, from 232 to  $\left\{ \begin{array}{l} 174.072 \\ 374.56 \end{array} \right\} = AC$ .

## EXAMPLE II.

In the plane triangle  $ABC$ 

$$\begin{array}{rcl}
 \text{Given } \left\{ \begin{array}{l} AB = 162 \\ AC = 270 \end{array} \right\} \text{ chains.} & & \text{Anf. } \left\{ \begin{array}{l} BC = 216 \text{ chains} \\ \angle C = 36^\circ 52' 12'' \\ \angle A = 53 \quad 07 \quad 48 \end{array} \right. \\
 \text{Required the other parts.} & &
 \end{array}$$

## EXAMPLE III.

In the plane triangle  $ABC$ 

$$\begin{array}{rcl}
 \text{Given } \left\{ \begin{array}{l} AB = 365 \text{ poles} \\ \angle B = 24^\circ 45' \\ \angle A = 57 \quad 12 \end{array} \right\} & & \text{Anf. } \left\{ \begin{array}{l} \angle C = 98^\circ 03' \\ AC = 154.332 \\ BC = 309.86 \end{array} \right\} \text{ pol.} \\
 \text{Required the other parts.} & &
 \end{array}$$

Ex-



## EXAMPLE IV.

In the plane triangle  $ABC$ 

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AB = 53 \text{ miles} \\ \angle A = 121^\circ 14' \\ \angle C = 29^\circ 23' \end{array} \right\} \quad \text{Anf. } \left\{ \begin{array}{l} \angle B = 29^\circ 23' \\ AC = 53 \text{ miles} \\ BC = 92.36 \text{ miles} \end{array} \right. \\ \text{Required the other parts.} \end{array}$$

## EXAMPLE V.

In the plane triangle  $ABC$ 

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AB = 162 \\ \angle C = 36^\circ 52' 12'' \\ \angle A = 53^\circ 07' 48'' \end{array} \right\} \quad \text{Anf. } \left\{ \begin{array}{l} \angle B = 90^\circ \\ AC = 270 \text{ chains} \\ BC = 216 \text{ chains} \end{array} \right. \\ \text{Required the other parts.} \end{array}$$

## EXAMPLE VI.

In the plane triangle  $ABC$ 

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AB = 365 \text{ poles} \\ AC = 154.33 \text{ poles} \\ \angle C = 98^\circ 03' \end{array} \right\} \quad \text{Anf. } \left\{ \begin{array}{l} \angle B = 24^\circ 45' \\ \angle A = 57^\circ 12' \\ BC = 309.86 \text{ pol.} \end{array} \right. \\ \text{Required the other parts.} \end{array}$$

## EXAMPLE VII.

In the plane triangle  $ABC$ 

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AC = 120 \text{ feet} \\ BC = 112 \text{ feet} \\ \angle A = 57^\circ 27' \end{array} \right\} \quad \text{Anf. } \left\{ \begin{array}{l} \angle B = \left\{ \begin{array}{l} 64^\circ 34' 21'' \\ 115^\circ 25' 39'' \end{array} \right. \\ \angle C = \left\{ \begin{array}{l} 57^\circ 58' 39'' \\ 7^\circ 07' 21'' \end{array} \right. \\ AB = \left\{ \begin{array}{l} 112.65 \\ 16.47 \end{array} \right\} \text{ feet} \end{array} \right. \\ \text{Required the other par.} \end{array}$$

P R O-



## PROBLEM II.

*Given two sides and the angle included by them, to find the rest.*

## \* RULE.

In a plane triangle, As the sum of any two sides is to their difference, so is the tangent of half the sum of their opposite angles to the tangent of half their difference.—Then the half sum augmented and diminished by the half difference of the angles, gives the greater and less angle respectively.

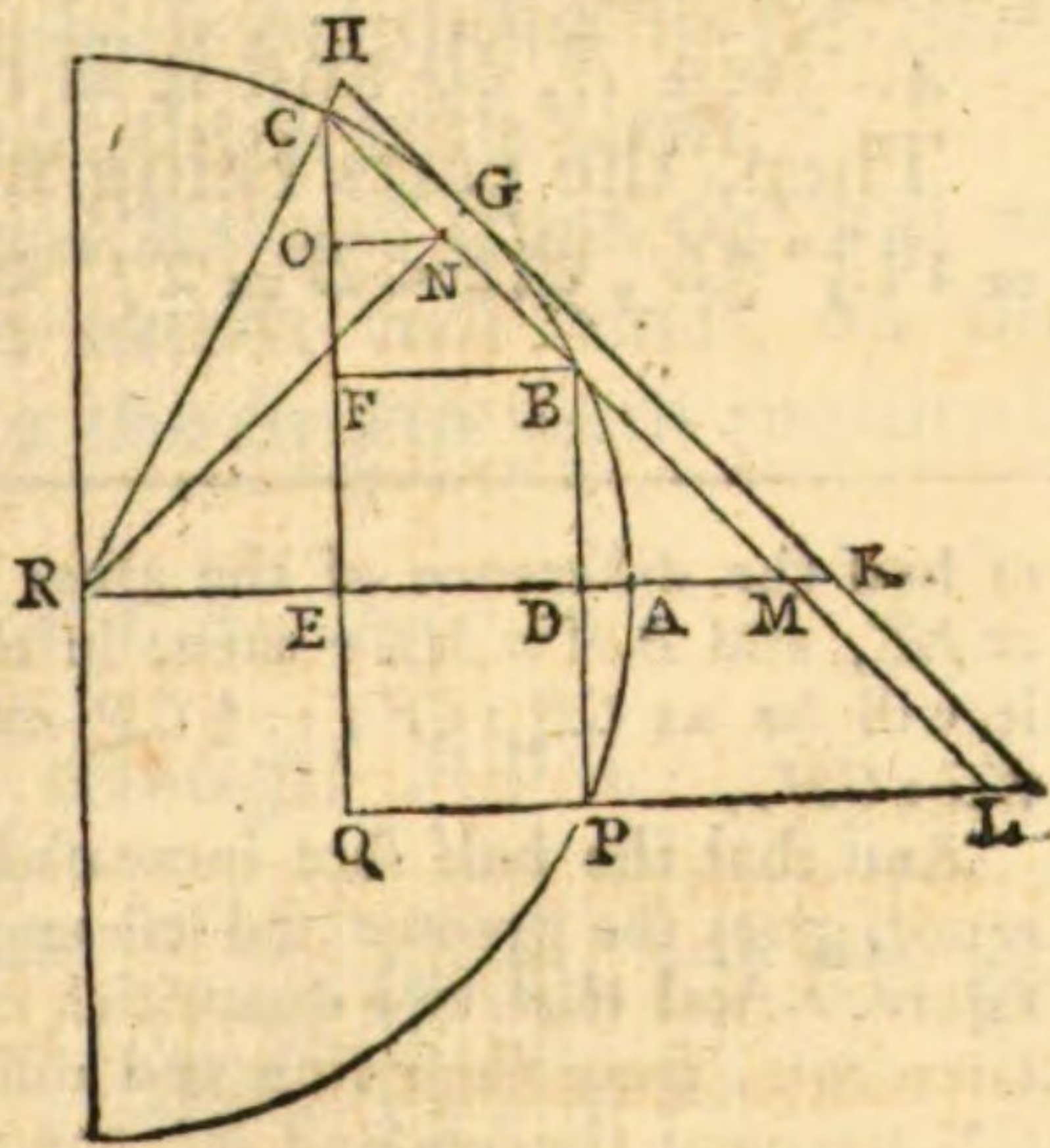
E

Then

## \* DEMONSTRATION.

By the first problem, the sides are as the sines of their opposite angles, and consequently the sum of the sides will be to the difference of the sides, as the sum of the sines to the difference of the sines of the said opposite angles.

Wherefore we have only to prove that the sum of the sines is to the difference of the sines of two arcs, as the tangent of half the sum of those arcs is to the tangent of half their difference: In order to which, let  $BD$ ,  $CE$ , be the sines of the arcs  $AB$ ,  $AC$ ; produce  $BD$  to the circumference at  $P$ , and produce  $CE$  till  $E Q$  be  $= DP$ ; to the middle ( $G$ ) of the arc  $BC$  draw the tangent  $HGK$ , also draw  $CNBML$  parallel to it, and the rest of the lines as are evident from the figure.



Now it is evident that  $CQ$  is the sum and  $CF$  the difference of the sines, and that  $GK$  is the tangent of half the sum and  $GH$  the tangent of



Then, all the angles being known, find the unknown side by the first problem.

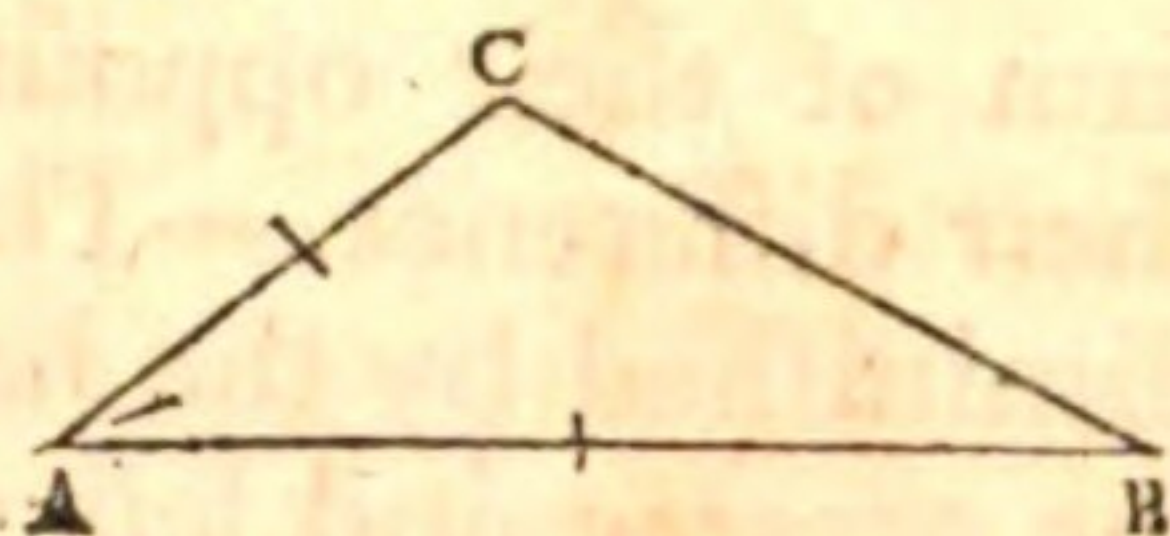
*Note.* When, in this case, the triangle is right-angled, it will be of use to observe that the square of the hypotenuse is equal to the sum of the squares of the legs.

### EXAMPLE I.

In the plane triangle ABC

$$\text{Given } \begin{cases} AB = 345 \text{ yards} \\ AC = 174.07 \text{ yards} \\ \angle A = 37^\circ 20' \end{cases}$$

Required the other parts.



### Construction.

1. Draw  $AB$  equal to 345, from a scale of equal parts.
2. Make the angle  $A$  equal to  $37^\circ 20'$ .
3. Make  $AC$  equal to 174.07, by the scale of equal parts.
4. Join  $B, C$ , and it is done.

Then, the parts being measured, we have the  $\angle C = 115^\circ 36'$ , the  $\angle B = 27^\circ 04'$ , and  $BC = 264$  yards.

### Calculation.

of half the difference of the arcs; as also that  $NM = \frac{1}{2} CL$ , for  $BN = NC$ , and  $BM = ML$ : then, in the similar  $\Delta$ s  $CQL$ ,  $RNM$ ,  $RGK$ , it will be as  $CQ : CF$  ( $:: \frac{1}{2} CQ$  or  $EO : \frac{1}{2} CF$  or  $CO :: MN : NC$ )  $:: KG : GH$ .

And that the half sum increased and diminished by the half difference, gives the greater and less angle respectively, is evident from the figure.—And that two quantities of any kind may be found, by the same rule, from their sum and difference, may be proved thus: Let  $CN$  represent the less and  $NL$  the greater of any two quantities, and let  $B$  be the middle of the right line  $CL$ ; then it is evident that  $BL = BC$  is the half sum, and  $BN$  the half difference, as also that  $LB + BN = NL$  the greater quantity, and  $CB - BN = NC$  the less.



*Calculation by the logarithms.*

As sum of the sides  $AB+AC = 519.07$  —  $2.7152259$   
 To their dif. —  $AB-AC = 170.93$  —  $2.2328183$   
 So tan. of  $\frac{LC+LB}{2}$  — —  $71^{\circ}20'$  —  $10.4712979$   
 So tan. of  $\frac{LC-LB}{2}$  — —  $44\ 16$  —  $9.9888903$   
                     Their sum  $115\ 36\ \angle C$   
                     Their dif.  $27\ 04\ \angle B$ .  
 Then, As s.  $\angle C = 115^{\circ}36'$  or  $64^{\circ}24'$  —  $9.9551259$   
       To its op. side  $AB = 345$  — —  $2.5378191$   
       So s.  $\angle A$  — —  $37^{\circ}20'$  —  $9.7827958$   
       To its op. side  $BC = 232$  — —  $2.3654890$

*By the scale.*

In the first proportion. Extend from  $519.07$  to  $170.93$  on the line of numbers; that extent will reach, upon the tangents, from  $71^{\circ}20'$  (the contrary way, because the tangents are set back again from  $45^{\circ}$ ) a little beyond  $45$ , which being set so far back from  $45$ , falls upon  $44^{\circ}16'$  the fourth term.

In the second proportion. Extend from  $64^{\circ}24'$  to  $27^{\circ}20'$  on the sines, that extent will reach, on the numbers, from  $345$  to  $232$  the fourth term required.

## EXAMPLE II.

In the plane triangle ABC

Given  $\begin{cases} AB = 162 \\ BC = 216 \end{cases}$  chains  
 $\angle B = 90^{\circ}$

Required the other parts.

Ans.  $\begin{cases} \angle A = 53^{\circ}07'48'' \\ \angle C = 36\ 52\ 12 \\ AC = 270 \text{ chains} \end{cases}$

Ex-



## EXAMPLE III.

In the plane triangle ABC

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AB = 53 \\ BC = 92.36 \end{array} \right\} \text{ miles} \\ \left\{ \begin{array}{l} \angle B = 29^\circ 23' \\ \text{Required the other parts.} \end{array} \right\} \end{array} \quad \text{Ans. } \left\{ \begin{array}{l} \angle A = 121^\circ 14' \\ \angle C = 29^\circ 23' \\ AC = 53 \text{ miles} \end{array} \right.$$

## EXAMPLE IV.

In the plane triangle ABC

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AC = 120 \\ BC = 112 \end{array} \right\} \text{ poles} \\ \left\{ \begin{array}{l} \angle C = 57^\circ 58' 39'' \\ \text{Required the other parts.} \end{array} \right\} \end{array} \quad \text{Ans. } \left\{ \begin{array}{l} \angle A = 57^\circ 27' \\ \angle B = 64^\circ 34' 21'' \\ AB = 112.65 \text{ pol.} \end{array} \right.$$

## PROBLEM III.

*Given the three sides of a triangle, to find the angles.*

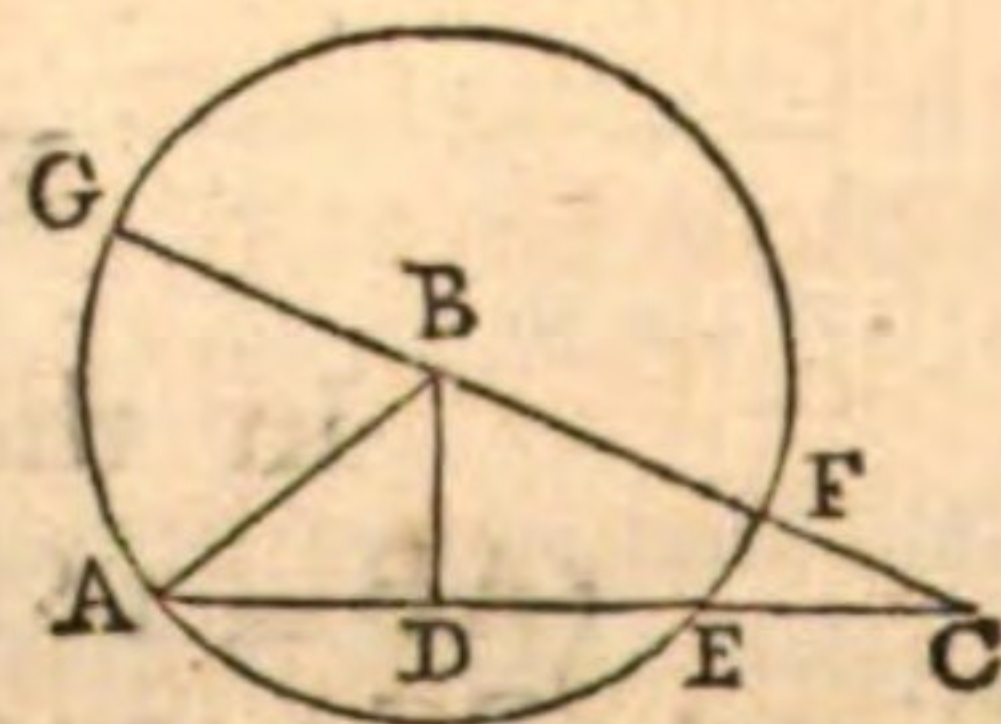
## \* RULE.

In any plane triangle, having let fall a perpendicular from the greatest angle upon the opposite side  
Or

## \* DEMONSTRATION.

From one end  $B$  of the least side  $AB$  of the triangle  $ABC$  as a center, and radius  $BA$ , describe a circle cutting the other two sides in  $E$  and  $F$ ; and let fall the perpendicular  $BD$ .—Then it is evident that  $GB = BF = AB$ , and (by 3. III. Eucl.)  $AD = DE$ , and consequently  $EC = CD - DA$ ,  $FC = CB - BA$ , and  $GC = CB + BA$ . But (by Cor. to 36. III. Eucl.)  $AC \times CE = GC \times CF$ , or  $AC : CG :: FC : CE$ , that is  $AC : CB + BA :: CB - BA : CD - DA$ .

And that the half sum of two quantities increased and diminished by their half difference, gives the greater and less quantities respectively, was proved in the last problem.





or base, dividing it into two segments, and the whole triangle into two right-angled triangles; it will be

As the base, or sum of the segments:

Is to the sum of the other two sides::

So is the difference of those sides:

To the difference of the segments of the base.

Then half the difference being added to and subtracted from half their sum, will give the greater and less segment.

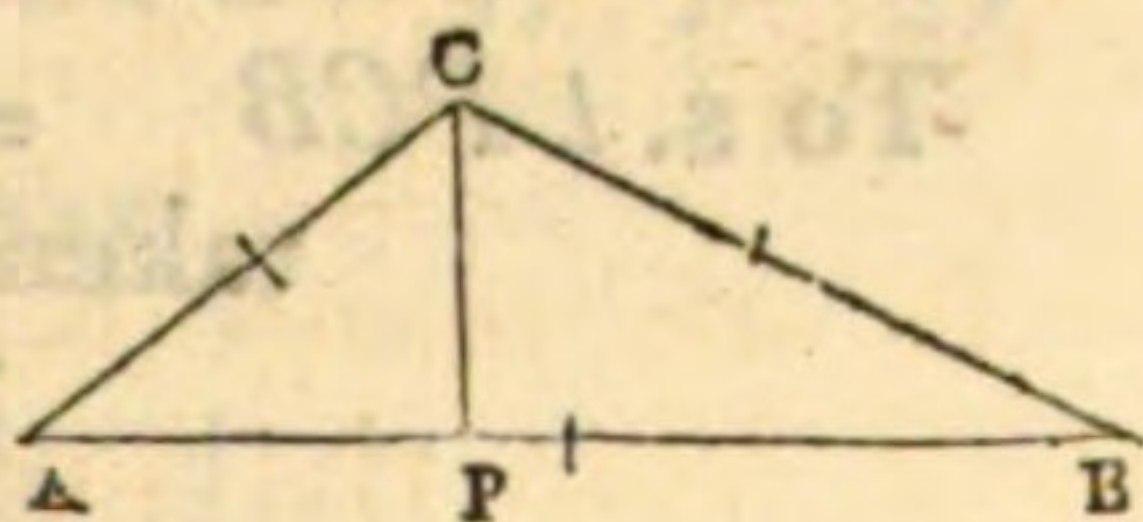
Hence, in each of the right-angled triangles, will be known two sides and the angle opposite to one of them; and consequently the other angles will be found by the first problem.

## EXAMPLE I.

In the plane triangle  $ABC$ .

Given  $\begin{cases} AB=345 \\ AC=174.07 \\ BC=232 \end{cases}$  yards

Required the angles.

*Construction.*

1. Draw  $AB$  equal to 345 by a scale of equal parts.
2. With the centers  $A$  and  $B$ , and radiuses 174.07 and 232, taken from the same scale, describe arcs intersecting in  $C$ .
3. Draw  $AC$  and  $BC$ , and it is done.

Then by measuring the angles, they appear to be nearly of the following dimensions, viz.  $\angle A = 37^\circ 20'$ ,  $\angle B = 27^\circ 04'$ , and  $\angle C = 115^\circ 36'$ .

*Calculation.*

Having let fall the perpendicular  $CP$ , it will be

G

As



As  $AB = 345 : BC + CA = 406.07 :: BC - CA = 57.93 :$

$$\frac{406.07 \times 57.93}{345} = 68.18 = BP - PA.$$

Hence,  $\frac{345 + 68.18}{2} = 206.59 = BP$ , and  $\frac{345 - 68.18}{2} = 138.41 = AP$ .

Then in the triangle  $APC$ , right-angled at  $P$ ,

|                             |   |          |   |                    |
|-----------------------------|---|----------|---|--------------------|
| As $AC$                     | = | 174° 07' | — | 2.2407239          |
| To s. $\angle P$            | = | 90°      | — | 10.0000000         |
| So $AP$                     | = | 138.41   | — | 2.1411675          |
| To s. $\angle ACP$          | = | 52° 40'  | — | 9.9004436          |
| which being taken from 90 — |   |          |   |                    |
| leaves                      |   |          |   | 37° 20' $\angle A$ |

And in the triangle  $BPC$ ,

|                    |   |         |   |                    |
|--------------------|---|---------|---|--------------------|
| As $BC$            | = | 232     | — | 2.3654880          |
| To s. $\angle P$   | = | 90°     | — | 10.0000000         |
| So $BP$            | = | 206.59  | — | 2.3151093          |
| To s. $\angle PCB$ | = | 62° 56' | — | 9.9496213          |
| taken from 90 —    |   |         |   |                    |
| leaves             |   |         |   | 27° 04' $\angle B$ |

Also 52° 40'  $\angle ACP$   
 added to 62 56  $\angle BCP$   
 makes 115 36  $\angle ACB$

Whence the  $\angle A = 37^\circ 20'$ , the  $\angle B = 27^\circ 04'$ , and the  $\angle C = 115^\circ 36'$ .

*By the scale.*

In the first proportion.—Extend from 345 to 406.07, on the line of numbers, that extent will reach, upon the same line, from 57.93 to 68.18, the difference of the segments of the base.

In the second proportion.—Extend from 174° 07' to 138.41, on the numbers, that will reach, on the fines, from 90° to 52° 40'. In



In the third proportion.—Extend from 232 to 206.59, and that extent will reach from 90 to  $62^{\circ} 56'$ .

## EXAMPLE II.

In the plane triangle  $ABC$

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AB=162 \\ AC=270 \\ BC=216 \end{array} \right\} \\ \text{Required the angles.} \end{array} \quad \text{Ans. } \left\{ \begin{array}{l} \angle A = 53^{\circ} 07' 48'' \\ \angle B = 90 \\ \angle C = 36 \ 52 \ 12 \end{array} \right.$$

## EXAMPLE III.

In the plane triangle  $ABC$

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AB=112.65 \\ AC=120 \\ BC=112 \end{array} \right\} \\ \text{Required the angles.} \end{array} \quad \text{Ans. } \left\{ \begin{array}{l} \angle A = 57^{\circ} 27' \\ \angle B = 64 \ 34 \ 21'' \\ \angle C = 57 \ 58 \ 39 \end{array} \right.$$

## EXAMPLE IV.

In the plane triangle  $ABC$

$$\begin{array}{l} \text{Given } \left\{ \begin{array}{l} AB=53 \\ AC=53 \\ BC=92.36 \end{array} \right\} \\ \text{Required the angles.} \end{array} \quad \text{Ans. } \left\{ \begin{array}{l} \angle A = 121^{\circ} 14' \\ \angle B = 29 \ 23 \\ \angle C = 29 \ 23 \end{array} \right.$$

These three problems include all the cases of plane triangles, as well right-angled as oblique; besides which there are some other theorems suited to some particular forms of triangles, which are often more expeditious in use than the general ones, one of which, as the case for which it serves so often occurs, I shall here insert, as follows.

PRO-



## PROBLEM IV.

*Given the angles and a leg of a right-angled triangle, to find the other leg and the hypotenuse.*

## \* RULE.

As the radius (that is, sine of  $90^\circ$  or tangent of  $45^\circ$ ):

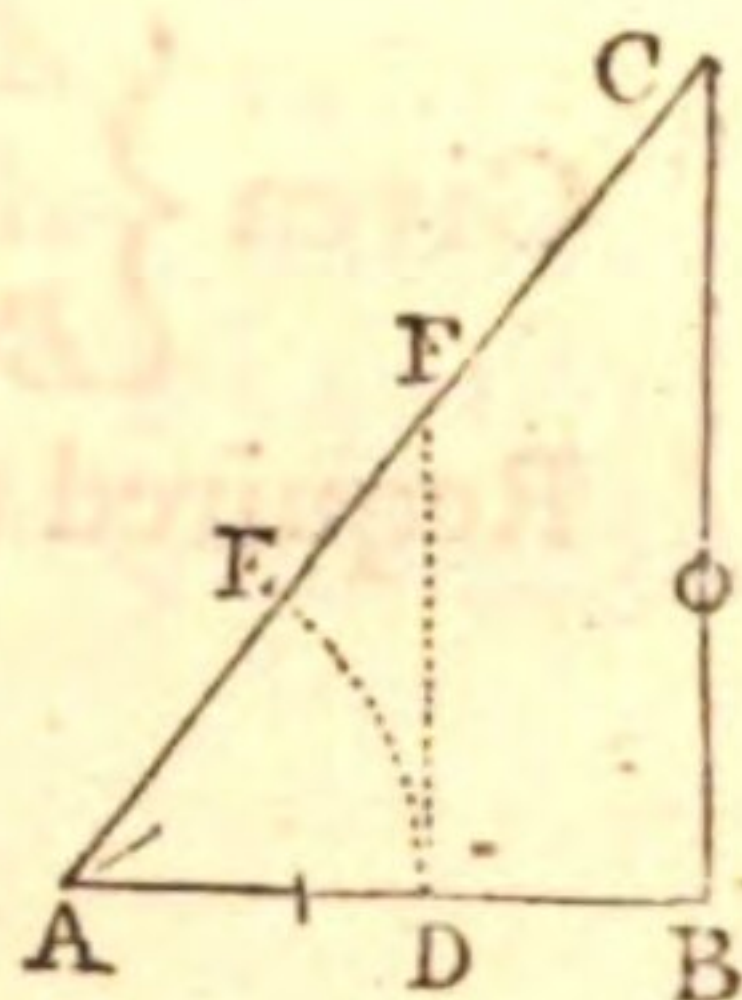
Is to the given leg  $AB$ :

So is the tangent of the angle  $A$  opposite to the required side  $BC$ :

To the required leg  $BC$ :

And, so is the secant of the same  $\angle A$ :

To the hypotenuse  $AC$ .



Ex-

## \* DEMONSTRATION.

With the center  $A$  and any radius  $AD$ , describe an arc  $DE$ , and erect the perpendicular  $DF$ , which, it is evident, will be the tangent, and  $AF$  the secant of the arc  $DE$  or angle  $A$  to the radius  $AD$ .—And in the similar triangles  $ADF$ ,  $ABC$ , it will be  $AD : AB :: DF : BC :: AF : AC$ . Q.E.D.

## GENERAL SCHOLIUM.

Besides these rules I shall here put down some new theorems concerning the relations of the sides and angles of triangles, independent of any tables.

## PROPOSITION.

If  $2a$  denote any side of a triangle,  $A$  the number of degrees contained in its opposite angle, and  $r$  the radius of the circle circumscribing the triangle: Then I say that  $A$  is equal to

$$57.2957795 \times \left( \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9} \right) \&c.$$

For since  $2a$  is the chord of the arc upon which the angle, whose measure is  $a$ , insists;  $a$  will be the sine of half that arc, or the sine of the angle to the radius  $r$ , since an angle in the circumference of a circle, is measured by half the arc upon which it stands: now it appears, from rule 2. prob. 7. sect. 1. part 2. of this work, that the said



## EXAMPLE.

In the plane triangle  $ABC$ , right-angled at  $B$ ,  
 Given  $\left\{ \begin{array}{l} AB = 162 \\ \angle A = 53^\circ 07' 48'' \end{array} \right\}$  Required  $AC$  and  $CB$ .  
 H Con-

faid half arc  $z$  is equal to  $a + \frac{a^3}{2 \cdot 3 r^2} + \frac{3a^5}{2 \cdot 4 \cdot 5 r^4} + \frac{3 \cdot 5 a^7}{2 \cdot 4 \cdot 6 \cdot 7 r^6} \&c.$  and,  $3 \cdot 14159 r$  denoting half the circumference of the same circle, or the arc of 180 degrees, we shall have  $3 \cdot 14159 r : 180^\circ :: z : \frac{180z}{3 \cdot 14159 r} = \frac{57 \cdot 2957795 z}{r} = 57 \cdot 2957795 \times : \frac{a}{r} + \frac{a^3}{2 \cdot 3 r^3} + \frac{3a^5}{2 \cdot 4 \cdot 5 r^5} + \frac{3 \cdot 5 a^7}{2 \cdot 4 \cdot 6 \cdot 7 r^7} \&c.$   
 $=$  the degrees in the angle or half arc.

*Corollary 1.* By reverting the above series we obtain  $\frac{a}{r} = \frac{A}{n} - \frac{A^3}{2 \cdot 3 n^3} + \frac{A^5}{2 \cdot 3 \cdot 4 \cdot 5 n^5} - \frac{A^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 n^7} \&c.$  putting  $n = 57 \cdot 2957795 = \frac{180}{3 \cdot 14159} \&c.$

*Corollary 2.* If  $2a$  be the hypotenuse of a right-angled triangle,  $a$  will be  $= r$ , and then the general series will become  $n \times : a + \frac{1}{2 \cdot 3} + \frac{3}{2 \cdot 4 \cdot 5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} \&c. = 90$ , or  $\frac{90}{n} = \frac{90 \times 3 \cdot 14159 \&c.}{180} = \frac{3 \cdot 14159 \&c.}{2}$   
 $= 1 + \frac{1}{2 \cdot 3} + \frac{3}{2 \cdot 4 \cdot 5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} \&c.$

*Corollary 3.* Since the chord of 60 degrees is  $=$  the radius, or the sine of 30 degrees  $=$  half the radius, putting  $a$  for  $\frac{1}{2} r$  in the general series will give  $n \times : \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 2^7} \&c. = 30$ ;

and hence the sum of the infinite series  $\frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 2^7} \&c.$  is  $= \frac{30}{n} = \frac{30 \times 3 \cdot 14159 \&c.}{180} = \frac{3 \cdot 14159 \&c.}{6} =$  one-sixth of the circumference of the circle whose diameter is 1.

*Corollary 4.* It might easily be shewn, from the principles of common geometry, that the sine of 60 degrees is to the radius as  $\frac{1}{2} \sqrt{3}$  is to 1; substituting, then,  $\frac{1}{2} r \sqrt{3}$  for  $a$  in the general series, we shall have  $n \sqrt{3} \times : \frac{1}{2} + \frac{3}{2 \cdot 3 \cdot 2^3} + \frac{3 \cdot 3^2}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{3 \cdot 5 \cdot 3^3}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 2^7} \&c. = 60$ ; and hence  
 the



*Construction.*

Make  $AB = 162$ , make the angle  $A = 53^\circ 07' 48''$ , and raise the perpendicular  $BC$  meeting  $AC$  in  $C$ .

*Calcu-*

the sum of the infinite series  $\frac{1}{2} + \frac{3}{2 \cdot 3 \cdot 2^3} + \frac{3 \cdot 3^2}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{3 \cdot 5 \cdot 3^3}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 2^7} \&c.$  will be  $= \frac{60}{n\sqrt{3}} = \frac{60 \times 3 \cdot 14159 \&c.}{180\sqrt{3}} = \frac{3 \cdot 14159 \&c.}{3\sqrt{3}}$ , and is, therefore, to the infinite series in the third corollary, as 2 is to  $\sqrt{3}$ .

*Corollary 5.* If  $b, c$  be the halves of the other two sides of the triangle, and  $B, C$  the degrees contained in their opposite angles; since  $B = nx : \frac{b}{r} + \frac{b^3}{2 \cdot 3 r^3} + \frac{3b^5}{2 \cdot 4 \cdot 5 r^5} \&c.$  and  $C = nx : \frac{c}{r} + \frac{c^3}{2 \cdot 3 r^3} \&c.$  and the 3 angles of any triangle equal to 180 degrees; we shall have  $180 = A + B + C = nx : \frac{a+b+c}{r} + \frac{a^3+b^3+c^3}{2 \cdot 3 r^3} \&c.$  or the sum of the infinite series  $\frac{a+b+c}{r} + \frac{1}{2 \cdot 3} \cdot \frac{a^3+b^3+c^3}{r^3} + \frac{3}{2 \cdot 4 \cdot 5} \cdot \frac{a^5+b^5+c^5}{r^5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} \cdot \frac{a^7+b^7+c^7}{r^7} \&c.$  will be  $= \frac{180}{n} = \frac{180 \times 3 \cdot 14159 \&c.}{180} = 3 \cdot 14159 \&c. =$  the circumference of a circle whose diameter is 1;  $a, b, c$  being the halves of the three sides of any triangle, and  $r$  the radius of its circumscribing circle.

*Corollary 6.* Since, by prob. 3,  $b : a+c :: a-c : \frac{aa-cc}{b} =$  half the difference of the segments of the base ( $b$ ) made by a perpendicular demitted from its opposite angle, and  $b + \frac{aa-cc}{b} + \frac{aa+bb-cc}{b} =$  the segment adjoining to the side  $2a$ , we shall have  $\sqrt{4a^2 - \frac{aa+bb-cc}{bb}} = \frac{\sqrt{4a^2 b^2 - aa+bb-cc}}{b}$  for the value of the said perpendicular to the base, and hence  $\frac{\sqrt{4a^2 b^2 - aa+bb-cc}}{b} : 2a :: c : \frac{2abc}{\sqrt{4a^2 b^2 - aa+bb-cc}}$   $= r$  the radius of the circumscribing circle.

Having



*Calculation.*

As radius = 1:  $AB = 162 :: 1.3333284 = \text{nat. tangent of } 53^\circ 07' 48'' : 215.9992 = BC.$

And, as 1: 162 ::  $1.6666628 = \text{nat. secant of the same } L A : 269.99937 = AC.$

*By*

Having now found the value of  $r$ , we can calculate all the cases of trigonometry without any tables, and without reducing oblique triangles to right-angled ones; for having any three parts (except the three angles) given, we can find the rest from these four equations:

$$1. r = \frac{2abc}{\sqrt{4a^2b^2 - aa + bb - cc}}.$$

$$2. A = nx : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9} \&c.$$

$$3. B = nx : \frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5} + \frac{3.5b^7}{2.4.6.7r^7} + \frac{3.5.7b^9}{2.4.6.8.9r^9} \&c.$$

$$4. C = nx : \frac{c}{r} + \frac{c^3}{2.3r^3} + \frac{3c^5}{2.4.5r^5} + \frac{3.5c^7}{2.4.6.7r^7} + \frac{3.5.7c^9}{2.4.6.8.9r^9} \&c.$$

And for the more convenience we may add the three following, which are derived from the three above by reverting the series.

$$5. a = rx : \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \&c.$$

$$6. b = rx : \frac{B}{n} - \frac{B^3}{2.3n^3} + \frac{B^5}{2.3.4.5n^5} - \frac{B^7}{2.3.4.5.6.7n^7} \&c.$$

$$7. c = rx : \frac{C}{n} - \frac{C^3}{2.3n^3} + \frac{C^5}{2.3.4.5n^5} - \frac{C^7}{2.3.4.5.6.7n^7} \&c.$$

Where  $n = 57.2957795 \&c.$

*EXAMPLE.*

Suppose we take here the first example in prob. 1, in which are given two sides  $2b = 345$ ,  $2c = 232$ , and the angle opposite to  $2c = 37^\circ 20' = 37\frac{1}{3}$  degrees =  $C$ .

Then  $\left(\frac{C}{n} \text{ being } = \frac{37\frac{1}{3} \times 3.14159 \&c.}{180} = .651589587\right) c = \frac{232}{2} = 116$   
 $= rx : .651589587 - .04610744 + .00097879 - .000009894 + .000000058 \&c. = rx .652568435 - .046117334 = .6064511r.$  Hence  
 $r = \frac{116}{.6064511} = 191.27677.$  And  $\frac{b}{r} = \frac{345 \times .6064511}{2 \times 116} = .9018346.$

Again,



*By the scale.*

The extent from  $45^\circ$  to  $53^\circ 08'$ , upon the tangents, will reach from 162 to 216 upon the numbers.

*Note,*

Again,  $B = 57.2957795 \times 1.12402$  (the sum of the series in the third equation)  $= 64.4016$  degrees  $= 64^\circ 24'$ .

And  $A = 180 - 37\frac{1}{2} - 64.4016 = 180 - 101.735 = 78.265^\circ = 78^\circ 16'$  nearly.

Lastly,  $\left(\frac{A}{n} = \frac{78.265}{57.2957795} = 1.365982, \text{ and } r = 191.27677\right)$  from the fifth equation we have  $a = 191.27677 \times 1.365982 - .4247992 + .0396379 - .0017607 + .0000288 - .0000005 = 191.27677 \times .9790883 = 187.27684$ .—And hence  $2a = 374.55368 =$  the third side of the triangle.

*Corollary 7.* As the series by which an angle is found often converges very slowly, I have inserted the following approximation of it, viz

$$A = n \times \frac{4\sqrt{2-2\sqrt{1-\frac{aa}{rr}}} - \frac{a}{r}}{3}$$
 nearly, where the letters denote the same quantities as in the above series.

For since  $P = \sqrt{2-2\sqrt{1-\frac{aa}{rr}}}$  is  $= \frac{a}{r} + \frac{a^3}{2.4r^3} + \frac{7a^5}{2.4.16r^5} \&c.$   
and  $\frac{A}{n}$  is  $= \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} \&c.$

we shall have, by taking the former of these from the latter,

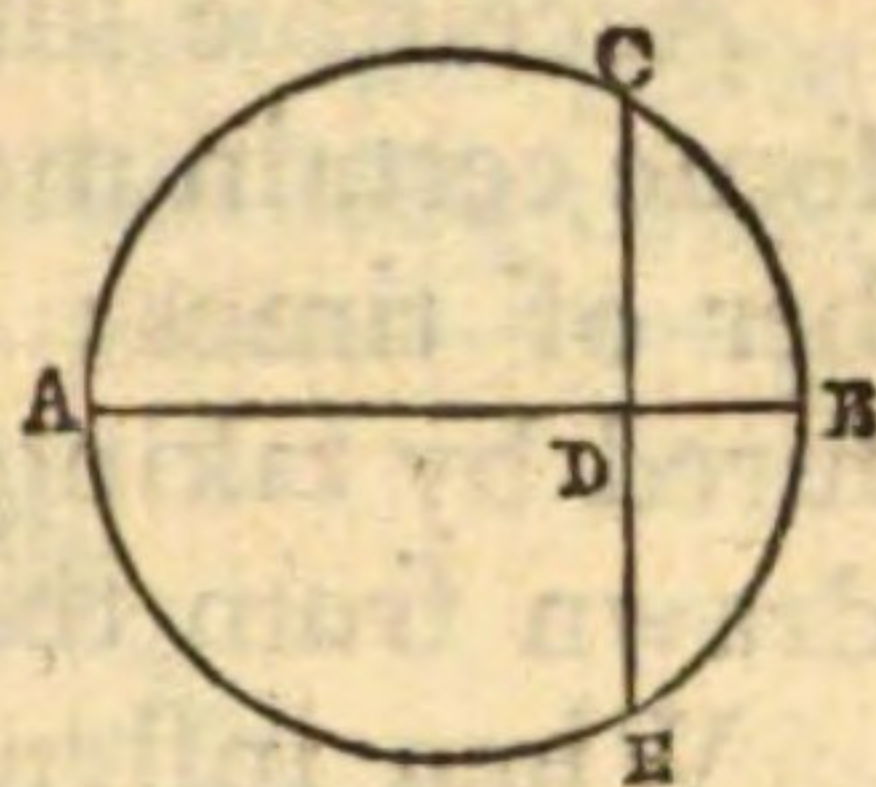
$$\frac{A}{n} - P = \frac{a^3}{24r^3} + \frac{13a^5}{640r^5} \&c. \text{ But, from the first series, } \frac{P - \frac{a}{r}}{3} = \frac{a^3}{24r^3} + \frac{7a^5}{384r^5} \&c. \text{ hence, by subtracting the latter from the former,}$$

$$\frac{A}{n} - P - \frac{P - \frac{a}{r}}{3} = \frac{A}{n} - \frac{4P - \frac{a}{r}}{3} = \frac{a^5}{480r^5} \&c. \text{ and } A = n \times \frac{4P - \frac{a}{r}}{3}$$

$$= n \times \frac{4\sqrt{2-2\sqrt{1-\frac{aa}{rr}}} - \frac{a}{r}}{3}$$
 nearly.



*Note,* That the sine ( $CD$ ) of an arc is a mean proportional between the versed sine ( $BD$ ) and versed sine of the supplement ( $AD$ ). Or, that half the chord is a mean proportional between the segments of the diameter perpendicular to it. That is,  $DC^2$  or  $DC \times DC = AD \times DB$ .



Observe also, that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the two legs; and that the corresponding sides of similar or equiangular triangles are proportional to each other.

These last three theorems I have put down as things necessary to be remembered by every measurer, having omitted their demonstrations as belonging to the theory of geometry.—But I have demonstrated the theorems in trigonometry as being properly a part of mensuration.

## S E C T. III.

*Of HEIGHTS and DISTANCES, &c.*

**B**Y the mensuration and protraction of lines and angles, we determine the lengths, heights, depths, or distances of bodies and objects, and they more immediately respect navigation, surveying, and what is commonly called altimetry and longimetry, or heights and distances, if indeed this must be distinguished from surveying.

I

Accessible



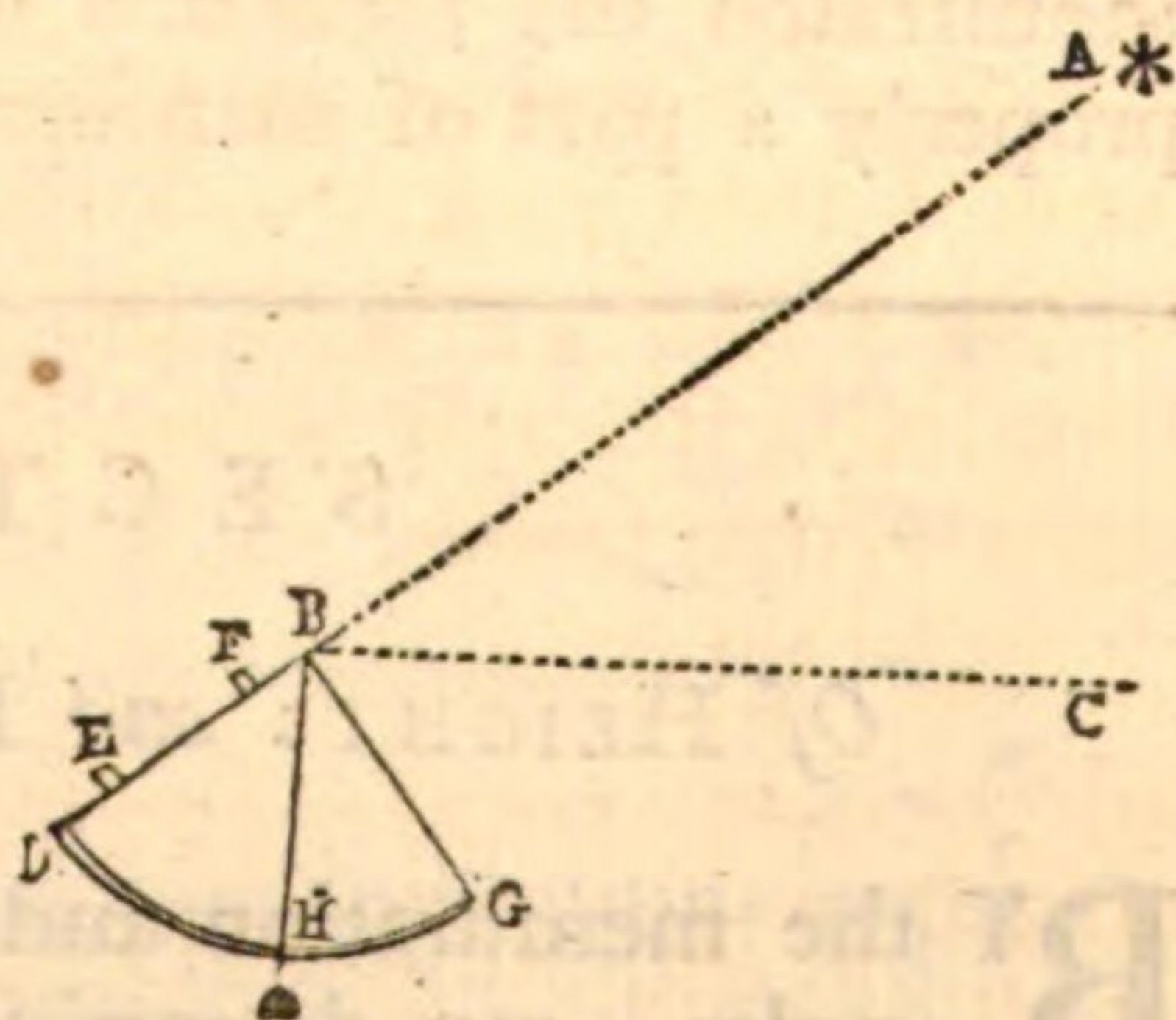
Accessible lines are measured by applying to them some certain measure, as an inch, a foot, &c. a number of times; but inaccessible lines must be measured by taking angles, or by some such-like method drawn from the principles of geometry.

When instruments are used for taking the quantities of the angles in degrees, the lines are then calculated by trigonometry: in the other methods the lines are calculated from the principle of similar triangles without any regard to the quantities of the angles.

Angles of elevation, or of depression, are generally taken either with a theodolite, or with a quadrant divided into degrees and minutes, and furnished with a plummet suspended from the center, and two sights fixed perpendicularly upon one of the radiuses.

*To take an angle of altitude and depression with the quadrant.*

Let  $A$  be any object, as the top of a tower, hill, or other eminence; or the sun, moon, or a star: and let it be required to find the measure of the angle  $ABC$  which a line drawn from the object makes with the horizontal line  $BC$ .

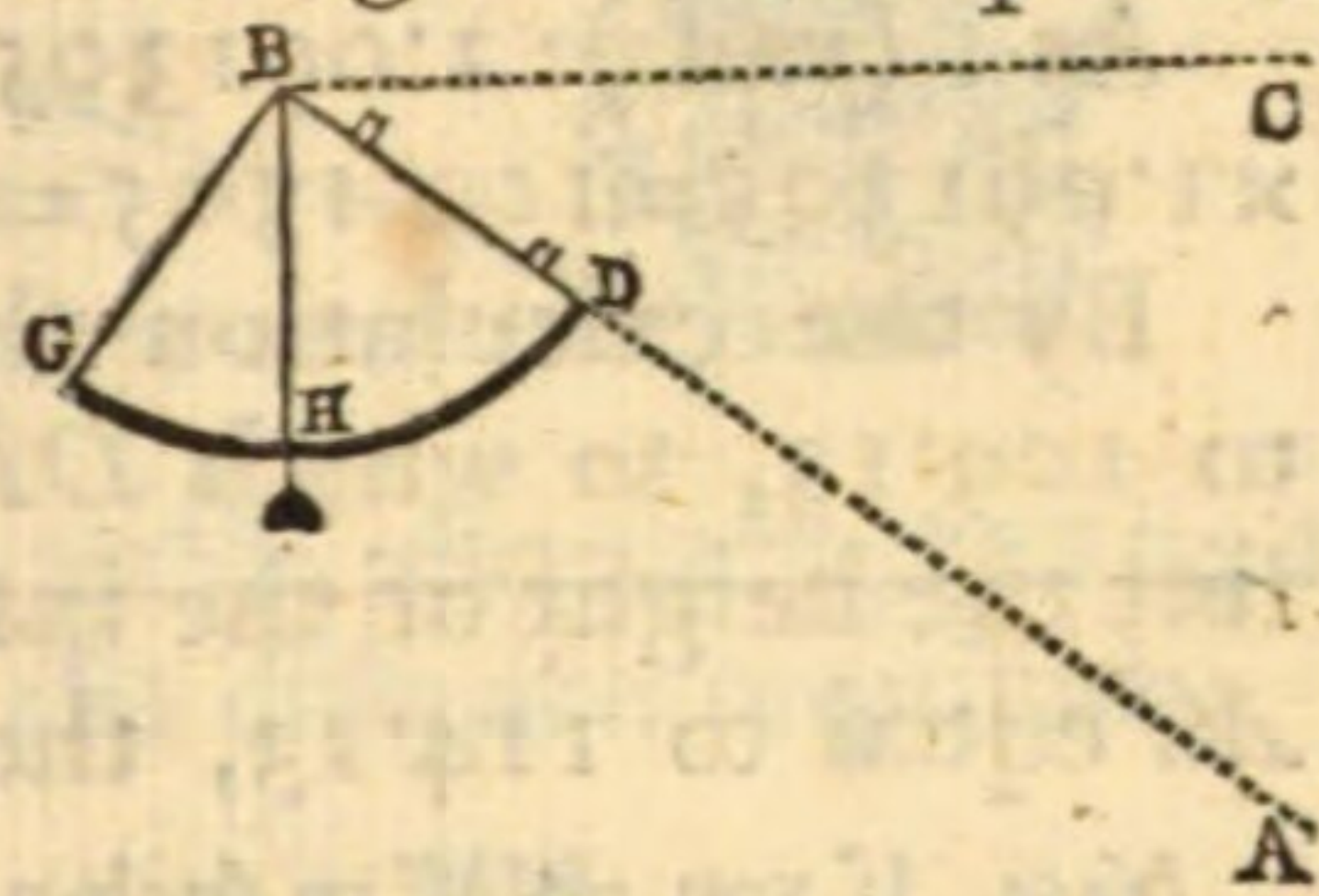


Fix the center of the quadrant in the angular point, and move it round there as a center till with one eye (the other being shut) at  $D$ , you perceive the object  $A$  through the two sights  $F, F$ ; then will the



the arc  $GH$  of the quadrant, cut off by the plum line  $BH$ , be the measure of the angle  $ABC$  required.

The angle  $ABC$  of depression of any object  $A$  is taken in the same manner, except that here the eye is applied to the center and the measure of the angle is the arc  $GH$ .

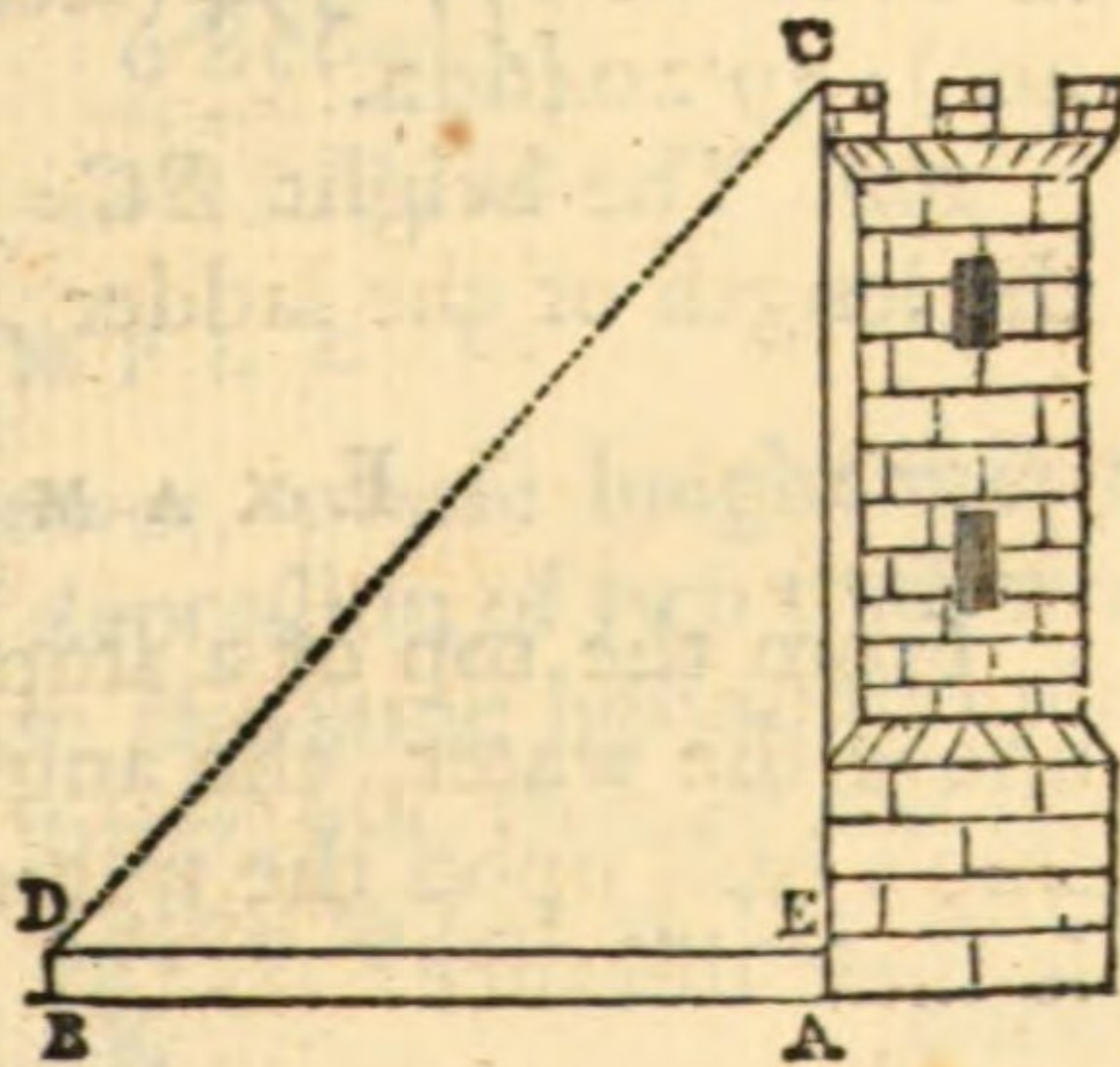


The observations with the quadrant, necessary to determine the heights and distances of objects, will be sufficiently apparent from the manner in which the following examples are proposed, and the solution may easily be given by any one who understands plane trigonometry.

## EXAMPLE I.

Having measured  $AB$  equal to 100 feet from the bottom of a tower, in a direct line on a horizontal plane, I then took the angle  $CDE$  of elevation of the top equal  $47^\circ 30'$ , the center of the quadrant being fixed five feet above the ground; required the height of the tower.

|                        |               |             |
|------------------------|---------------|-------------|
| Rad.                   | —             | 10.00000000 |
| Tan. $LD 47^\circ 30'$ |               | 10.0379475  |
| So $BC 100$            | —             | 2.00000000  |
| To $CE 109.13$         |               | 2.0379475   |
|                        | 5             |             |
|                        | <u>114.13</u> |             |



Or,



*Or, without the logarithms,*

As 1 (rad.): 1.091305 (tan. of  $47^{\circ} 30'$ ) :: 100: 100  
 $\times 1.091305 = 109.1305 = CE$ .

By the calculation the height  $CE$  is found equal to 109.13, to which  $DB$  (equal to  $EA$  equal to five feet the height of the instrument) being added, gives  $AC$  equal to 114.13, the whole height.

*Note.* If you go off to such a distance from the bottom, as that the angle of elevation shall be  $45^{\circ}$ , then will the height be equal to the distance with the height of the eye added.

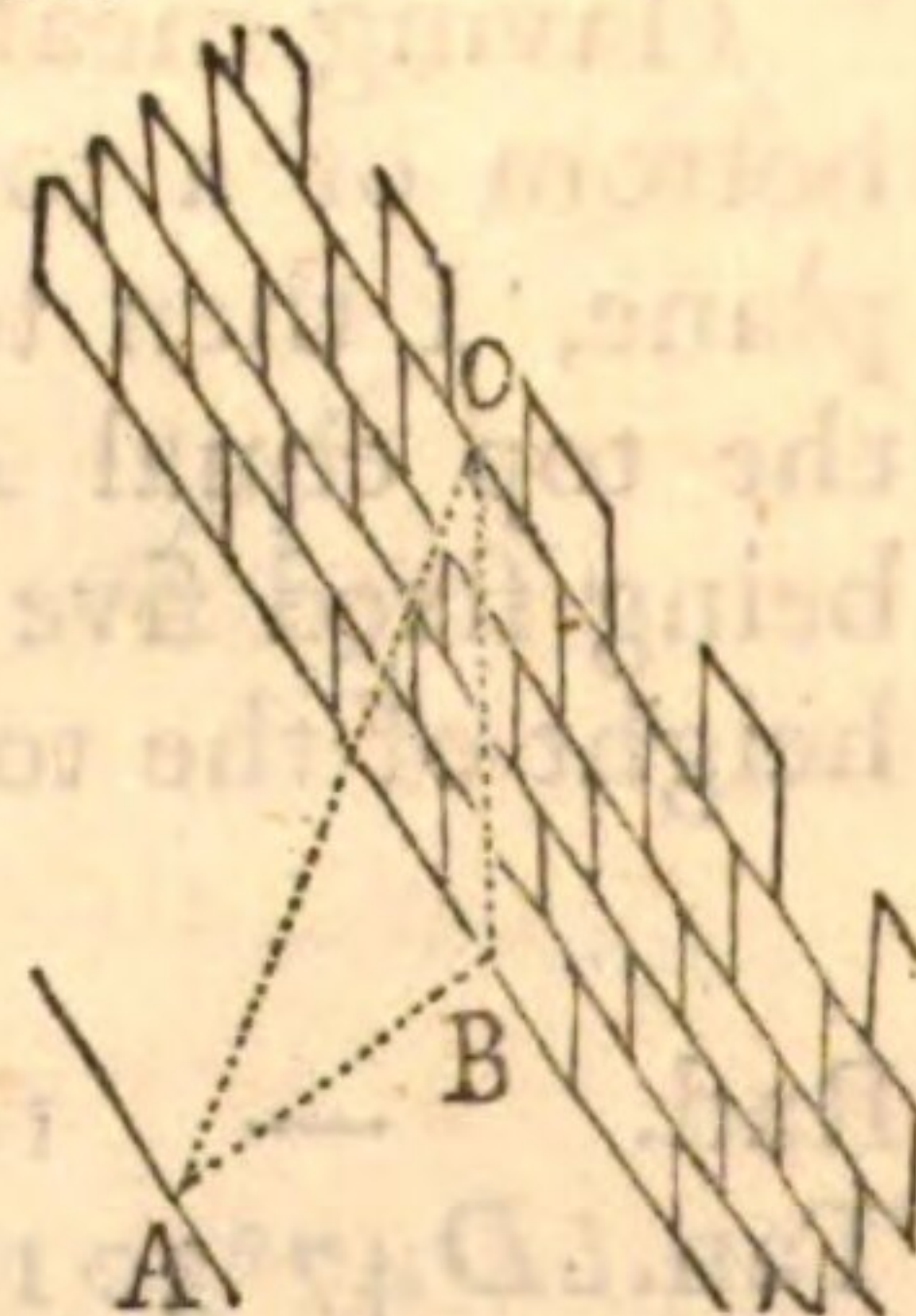
#### EXAMPLE II.

From the edge of a ditch of 18 feet wide, surrounding a fort, I took the angle of elevation of the top of the wall equal to  $62^{\circ} 40'$ : required the height of the wall and the length of a ladder necessary to reach from my station to the top of it.

First, As 1: 1.934702 (tan.  $62^{\circ} 40'$ )  
 :: 18:  $1.934702 \times 18 = 34.824636$ .

Then  $\sqrt{18^2 + 34.824636^2} =$   
 $\sqrt{1536.755272} = 39.2014 = AC$ . Or  
 as 1: 18 :: 2.1778594 (secant of  $62^{\circ}$   
 $40'$ ): 39.204692.

*Ans.* The height  $BC = 34.82$  and  
 the length of the ladder  $AC = 39.2$ .

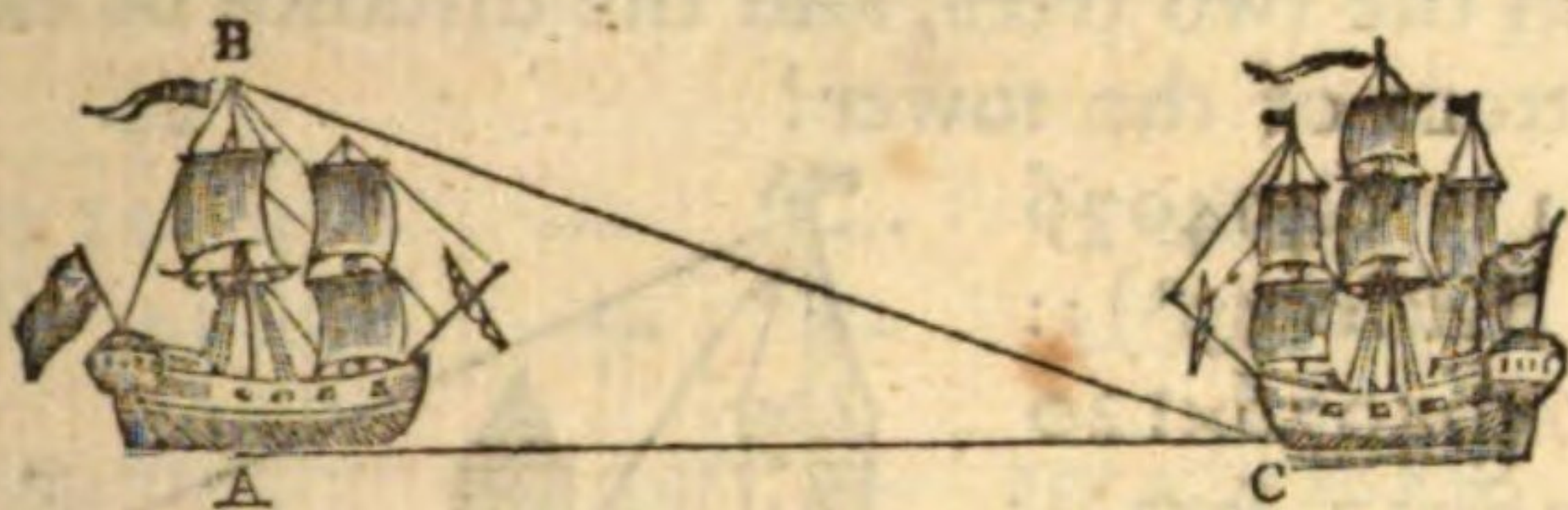


#### EXAMPLE III.

From the top of a ship's mast, which was 80 feet above the water, the angle of depression of another ship's hull upon the water at a distance is  $20^{\circ}$ ; what is their distance?

As





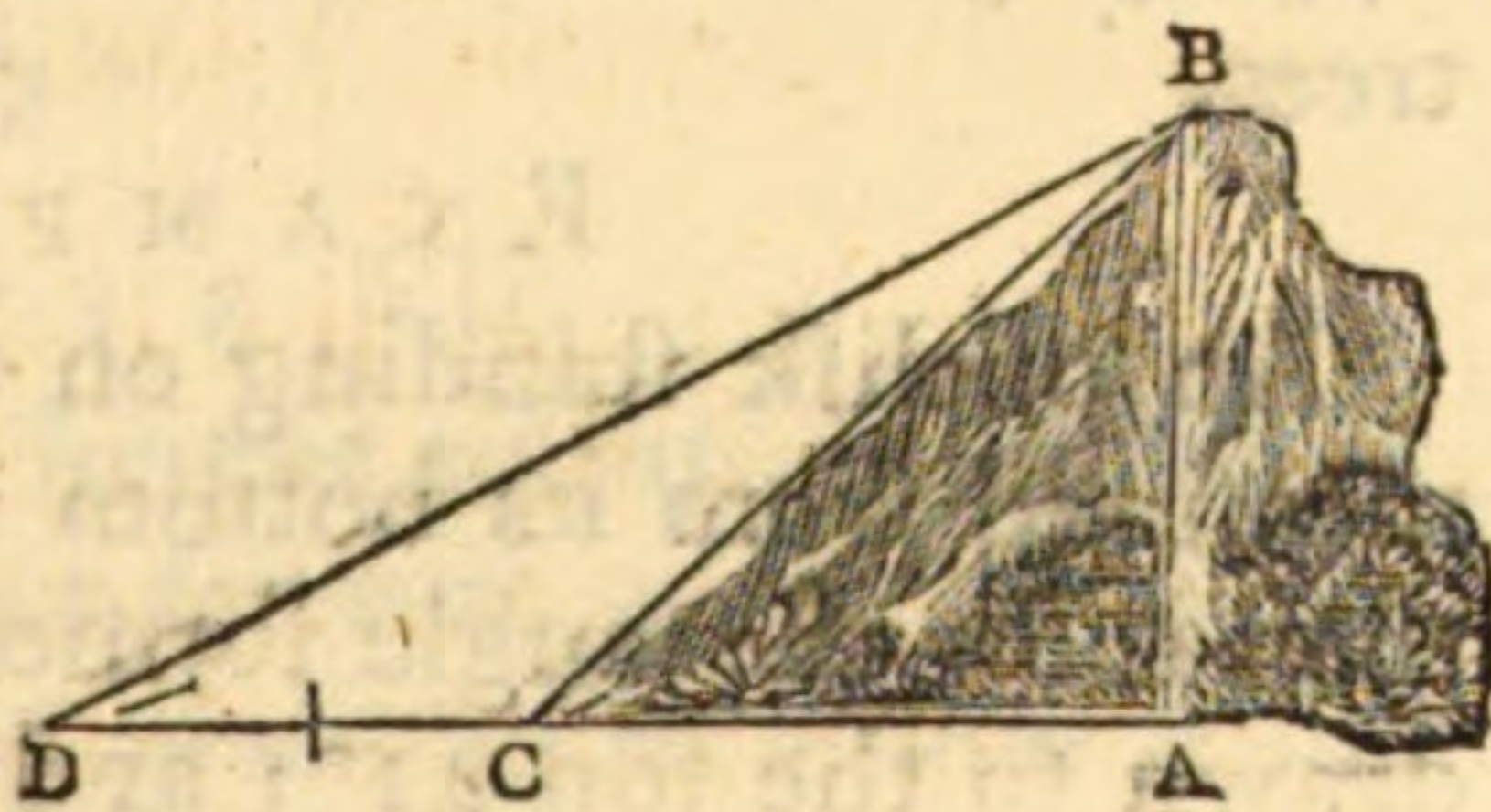
As  $1 : 2.7474774 (\tan. 70^\circ) :: 80 : 2.7473774 \times 80$   
 $= 219.790192 \text{ feet} = AC \text{ the distance required.}$

## EXAMPLE IV.

What is the perpendicular height of a hill whose angle of elevation taken at the bottom of it was  $46^\circ$ , and 100 yards farther off, on a level with the bottom of it, the angle was  $31^\circ$ ?

$46^\circ$  } subtract  
 $31^\circ$  }

$$\begin{array}{r} LDBC = 15^\circ - 9.4129962 \\ DC = 100 - 2.0000000 \\ LD = 31^\circ - 9.7118393 \\ \hline BC = \quad - 2.2988431 \end{array}$$



$$\begin{array}{r} LA = 90^\circ - \quad - 10.0000000 \\ BC - \quad - \quad - 2.2988431 \\ LC = 46^\circ - \quad - 9.8569341 \\ \hline AB = 143.14 - \quad - 2.1557772 \end{array}$$

## EXAMPLE V.

From the top of a tower, whose height was 120 feet, I took the angle of depression of two trees which lay in a direct line, upon the same horizontal plane, with the bottom of the tower, viz. that of the nearer  $57^\circ$ , and of the farther  $25\frac{1}{2}^\circ$ : What is the distance

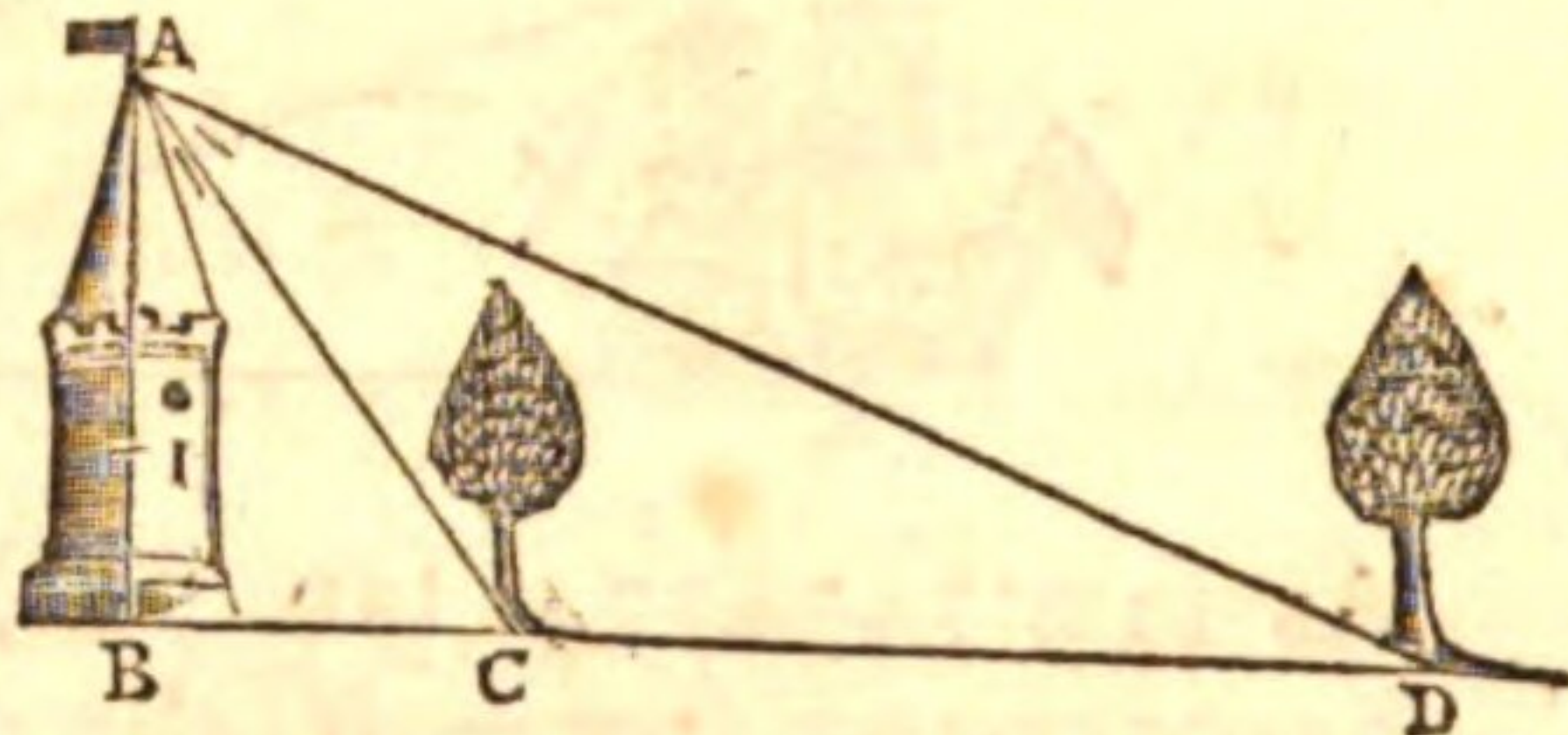
K

between



between the two trees, and the distance of each from the bottom of the tower?

As  $1 : .6494076$   
 $(\tan. \angle BAC = 33^\circ) ::$   
 $120 : .6494076 \times 120$   
 $= 77.928912 \text{ feet} =$   
 $BC$  the distance from  
 the bottom of the  
 tower to the nearest tree.



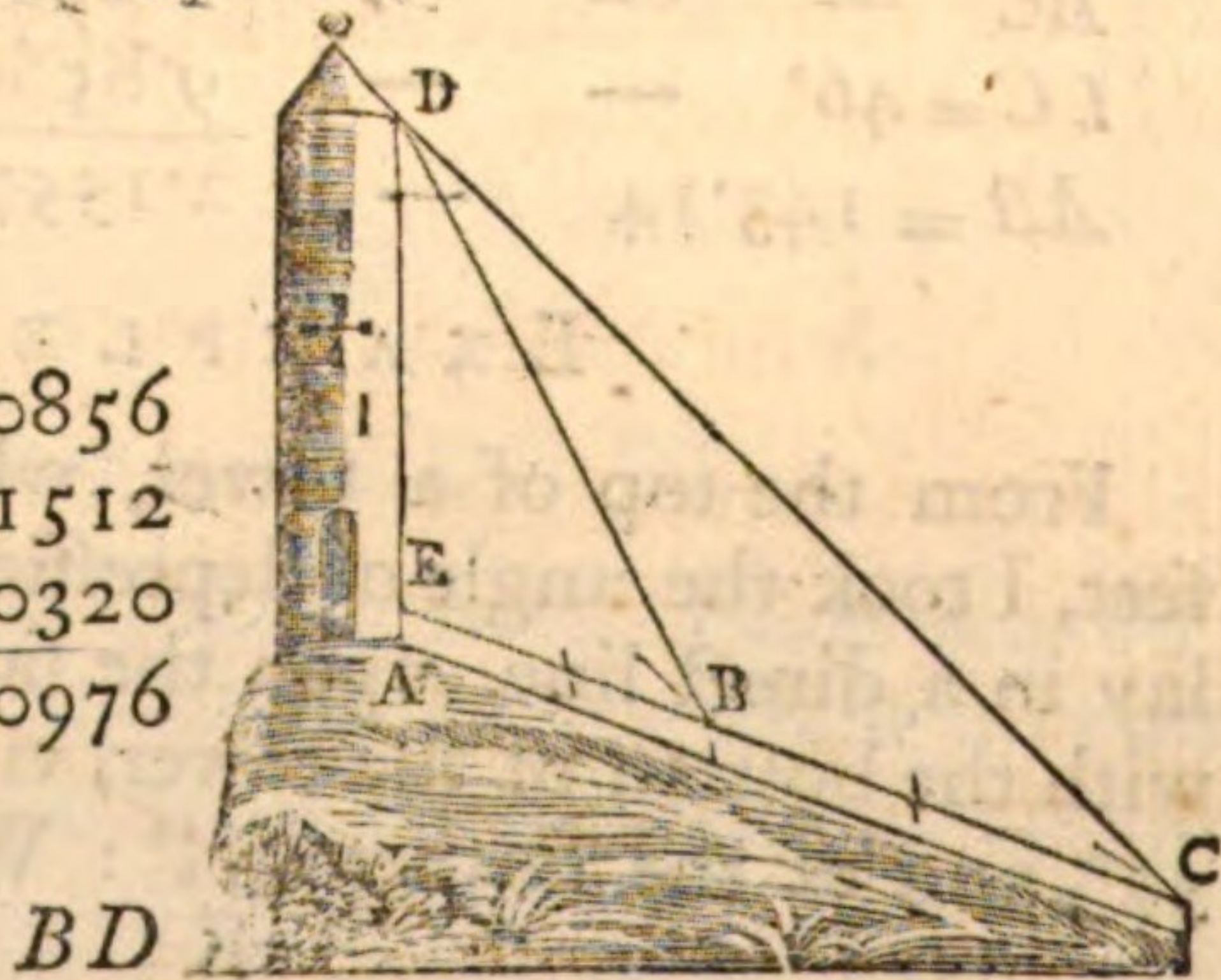
And as  $1 : 2.0965436$  ( $\tan. \angle BAD = 64\frac{1}{2}^\circ$ ) ::  $120 :$   
 $2.0965436 \times 120 = 251.585232 \text{ feet} = BD$  the distance  
 of the farther tree.

Therefore,  $DB - BC = 251.585232 - 77.928912 =$   
 $173.65632 \text{ feet} = CD$  the distance between the two  
 trees.

#### EXAMPLE VI.

An obelisk standing on the top of a declivity, I measured from its bottom a distance of 40 feet, and then took the angle formed by the plane and a line drawn to the top  $41^\circ$ ; and going on in the same direction 60 feet farther, the same angle was  $23^\circ 45'$ , the height of the instrument being five feet: What was the height of the obelisk?

fub.  $\begin{cases} 41^\circ \\ 23 & 45' \end{cases}$   
 $\angle CDB \ 17 \ 15 - 9.4720856$   
 $BC = 60 \quad \text{---} \quad 1.7781512$   
 $\angle C = 23^\circ 45' - 9.6050320$   
 $BD = 81.488 - 1.9110976$



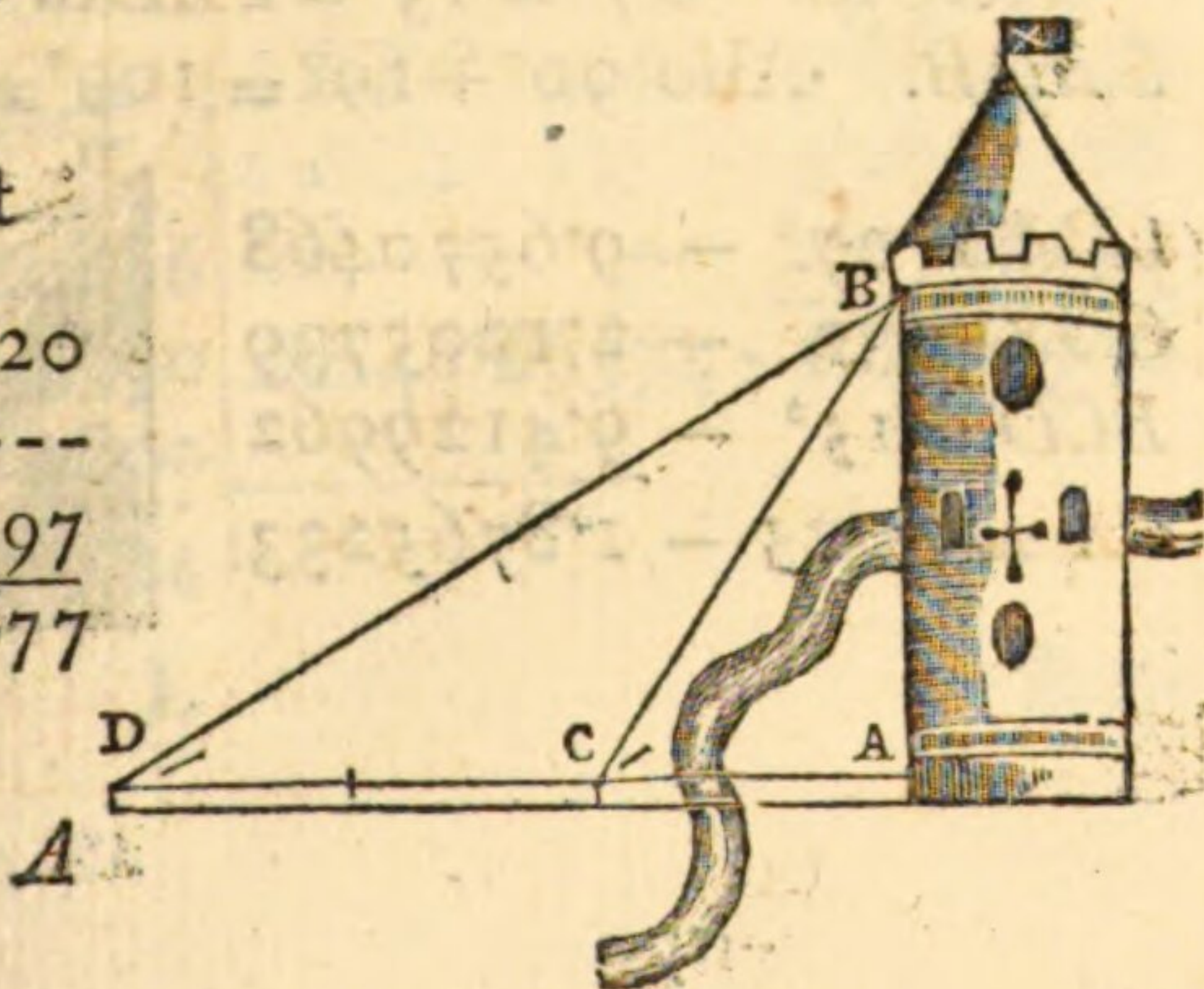


|              |                               |         |   |            |
|--------------|-------------------------------|---------|---|------------|
| $BD$         | 81.488                        |         |   |            |
| $BE$         | 40                            |         |   |            |
| Sum          | 121.488                       | —       | — | 2.0845048  |
| Diff.        | 41.488                        | —       | — | 1.6179225  |
| Tang.        | $\frac{E+EDB}{2}$             | 69° 30' | — | 10.4272623 |
| Tang.        | $\frac{E-EDB}{2}$             | 42 25   | — | 9.9606800  |
| Dif.         | $\angle EDB = 27^{\circ} 05'$ |         |   |            |
| $\angle EDB$ | $= 27^{\circ} 05'$            | —       | — | 9.6582842  |
| $BE$         | $= 40$                        | —       | — | 1.6020600  |
| $\angle B$   | $= 41^{\circ}$                | —       | — | 9.8169429  |
| $ED$         | $= 57.639$                    | —       | — | 1.7607187  |
|              | $\frac{5}{62.639} = AD$       |         |   |            |

## EXAMPLE VII.

Wanting to know the height of an inaccessible object; at the least distance from it, upon the same horizontal plane, I took its angle of elevation equal to  $58^{\circ}$ , and going 100 yards directly farther from it, found the angle there to be only  $32^{\circ}$ : required its height and my distance from it at the first station, the instrument being five feet above the ground at each observation.

|              |                          |
|--------------|--------------------------|
| $58^{\circ}$ | } subtract               |
| $32$         |                          |
| $\angle DBC$ | $26^{\circ} - 9.6418420$ |
| $DC$         | $100 - 2.0823677$        |
| $\angle D$   | $32^{\circ} - 9.7242097$ |
| $BC$         | $— - 2.0823677$          |





|                 |   |                       |                 |   |           |
|-----------------|---|-----------------------|-----------------|---|-----------|
| $LA 90^\circ$   | — | 10°-----              | $LA 90^\circ$   | — | 10°-----  |
| $BC$            | — | 2.0823677             | $BC$            | — | 2.0823677 |
| $LC 58^\circ$   | — | 9.9284205             | $LCBA 32^\circ$ | — | 9.7242097 |
| $AB 102.51$ yds | — | 2.0107882             | $AC 64.05$ yds  | — | 1.8065774 |
|                 |   | 1.66 &c. yds = 5 feet |                 |   |           |
|                 |   | 104.17 whole height   |                 |   |           |

## EXAMPLE VIII.

Wanting to know the height of, and my distance from, an object on the other side of a river, which seemed to be upon a level with the place where I stood close by the side of the river; and not having room to go backwards, on the same plane, on account of the immediate rise of the bank, I placed a mark where I stood, and measured, in a direct line from the object, up the hill, whose ascent was so regular that I might account it for a right line, to the distance of 132 yards, where I perceived that I was above the level of the top of the object; I there took the angle of depression of the mark by the river's side equal  $42^\circ$ , of the bottom of the object equal  $27^\circ$ , and of its top  $19^\circ$ : required the height of the object and the distance of the mark from its bottom.

Here  $42^\circ - 27^\circ = 15^\circ = LADC$ . And  $27^\circ - 19^\circ = 8^\circ = LAB$ . Also  $90^\circ + 19^\circ = 109^\circ = LB$ .

|                  |   |           |
|------------------|---|-----------|
| $LCAD 27^\circ$  | — | 9.6570468 |
| $CD = 132$       | — | 2.1205739 |
| $BCD = 15^\circ$ | — | 9.4129962 |
| $CA = 75.25$     | — | 1.8765233 |



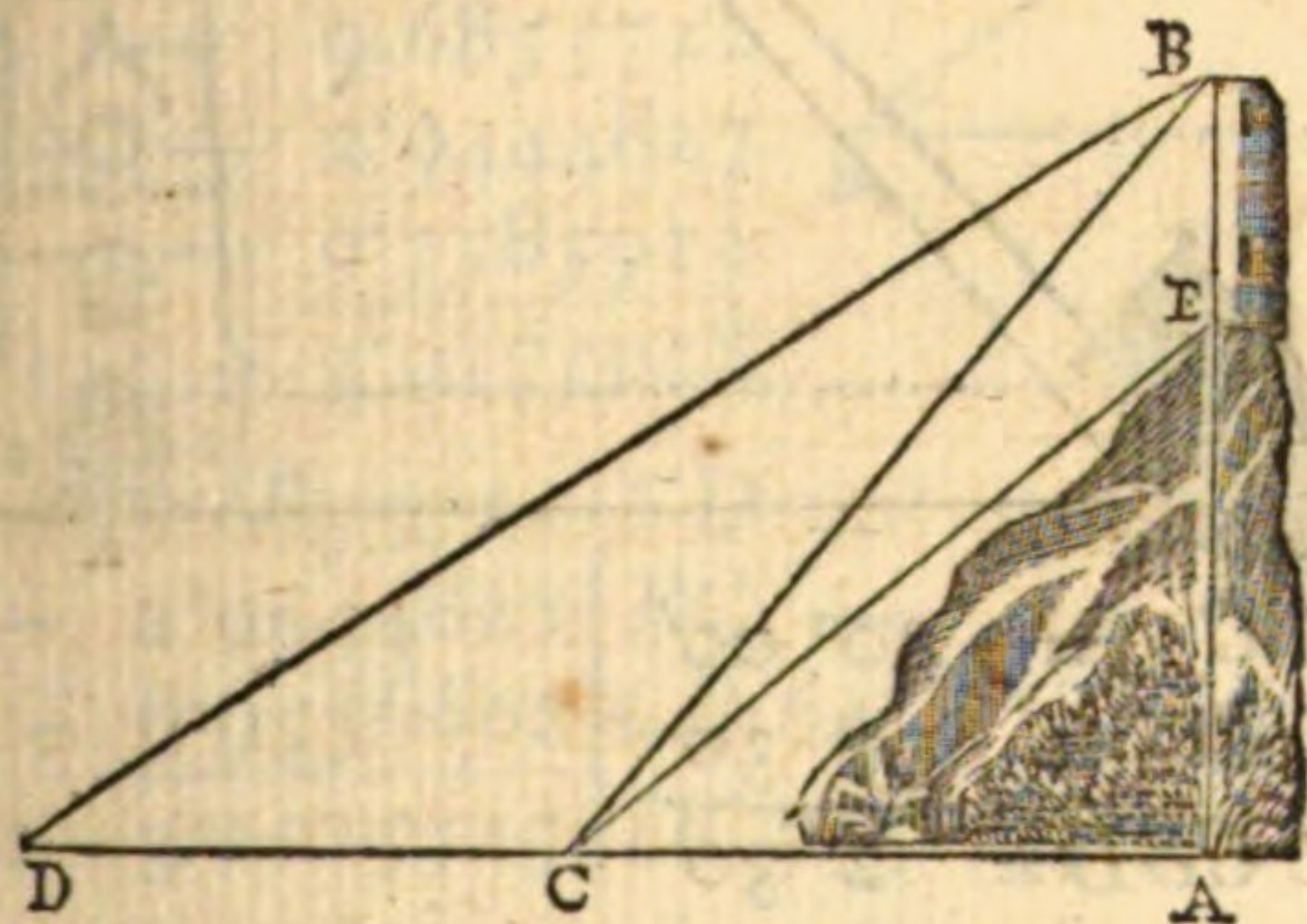
$LAD$



|                               |     |             |                  |     |             |
|-------------------------------|-----|-------------|------------------|-----|-------------|
| $LCAD\ 27^\circ$              | $—$ | $9.6570468$ | $LB = 109^\circ$ | $—$ | $9.9756701$ |
| $CD\ 132$                     | $—$ | $2.1205739$ | $DA$             | $—$ | $2.2890380$ |
| $LC\ 138^\circ$ or $42^\circ$ | $—$ | $9.8255109$ | $LADB = 8^\circ$ | $—$ | $9.1435553$ |
| $AD$                          | $—$ | $2.2890380$ | $AB = 28.63$     | $—$ | $1.4569232$ |

## EXAMPLE IX.

Being upon a horizontal plane, and wanting to know the height of an object on the top of an inaccessible hill; I took the angle of elevation of the top of the hill equal  $40^\circ$ , and of the top of the object equal  $51^\circ$ ; going, then, in a direct line from it to the distance of 100 yards farther, I found the angle of the top of the object to be  $33^\circ 45'$ : what is the object's height?



|                      |            |                     |     |             |
|----------------------|------------|---------------------|-----|-------------|
| $LBCA\ 51^\circ 00'$ | } subtract | $L\ BEC\ 130^\circ$ | $—$ | $9.8842540$ |
| $LD\ 33\ 45$         |            | $L\ BCE\ 11$        | $—$ | $9.2805988$ |
| $LDCB\ 17\ 15$       |            | $CB$                | $—$ | $2.2726534$ |
| $LD\ 33\ 45$         |            | $BE\ 46.66574$      | $—$ | $1.6689982$ |
| $DC\ 100$            |            |                     |     |             |
| $CB$                 |            |                     |     |             |

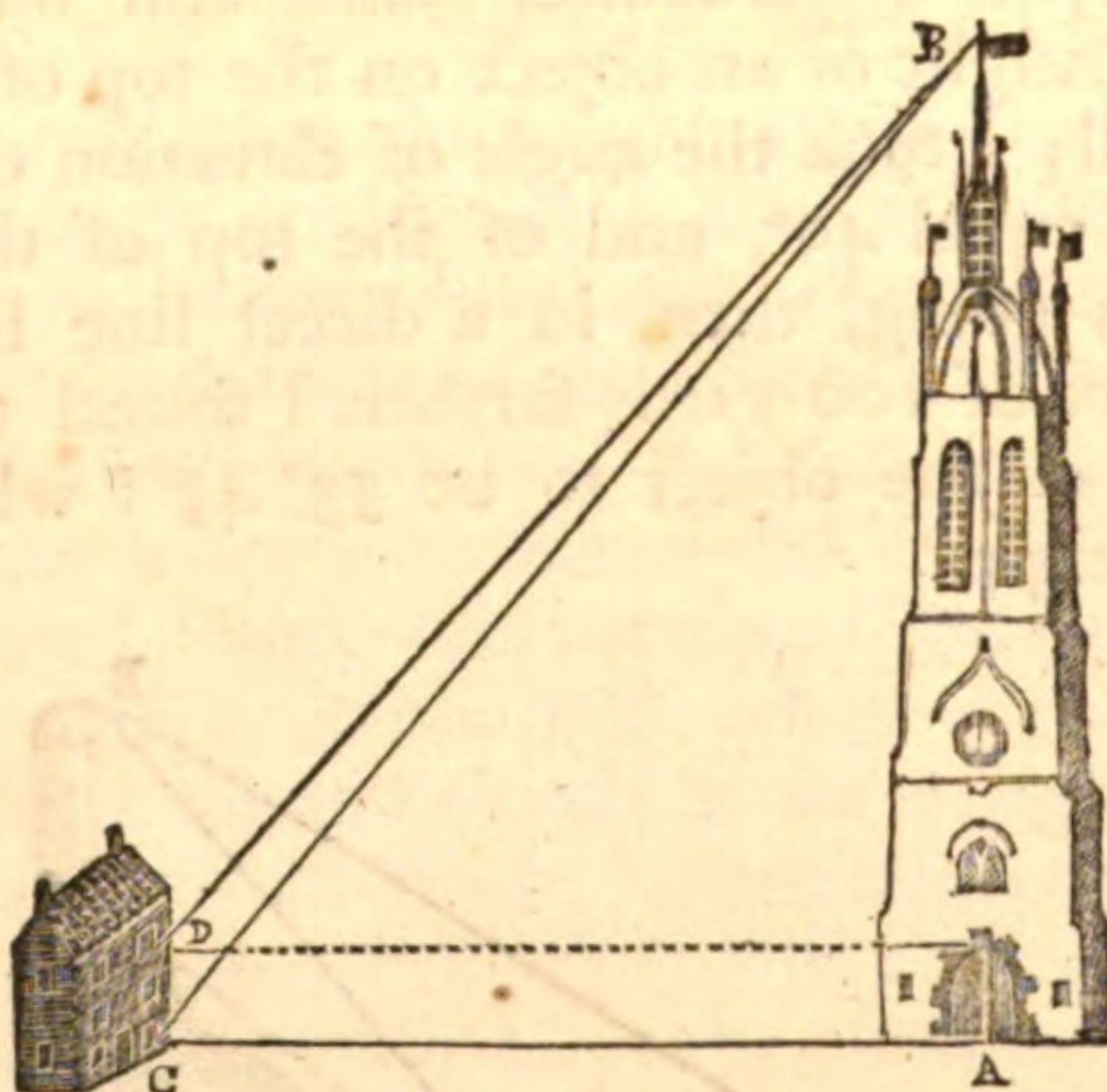
L

Ex-



## EXAMPLE X.

From a window near the bottom of a house, which seemed to be upon a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal  $40^\circ$ , and from another window 18 feet directly above the former, the same angle was  $37^\circ 30'$ : what then is the height and distance of the steeple?



$$\text{From } \angle C = 40^\circ 00'$$

$$\text{Subt. } \angle D = 37^\circ 30'$$

$$\text{Rem. } \angle CBD = 2^\circ 30'$$

$$\angle DBC = 2^\circ 30' \quad \text{---} \quad \text{---} \quad \text{---} \quad 8.6396796$$

$$DC = 18 \text{ feet} \quad \text{---} \quad \text{---} \quad \text{---} \quad 1.2552725$$

$$\angle CDB = 127\frac{1}{2}^\circ \text{ or } 52\frac{1}{2}^\circ \quad \text{---} \quad \text{---} \quad 9.8994667$$

$$CB \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad 2.5150596$$

$$\angle A 90^\circ \quad \text{---} \quad 10.0000000 \quad \left| \quad \angle A 90^\circ \quad \text{---} \quad 10.0000000$$

$$CB \quad \text{---} \quad \text{---} \quad 2.5150596 \quad \left| \quad CB \quad \text{---} \quad \text{---} \quad 2.5150596$$

$$\angle ACB 40^\circ \quad \text{---} \quad 9.8080675 \quad \left| \quad \angle CBA 50^\circ \quad \text{---} \quad 9.8842540$$

$$AB 210.44 \text{ f.} \quad \text{---} \quad 2.3231271 \quad \left| \quad CA 250.79 \quad \text{---} \quad 2.3993136$$

E x-



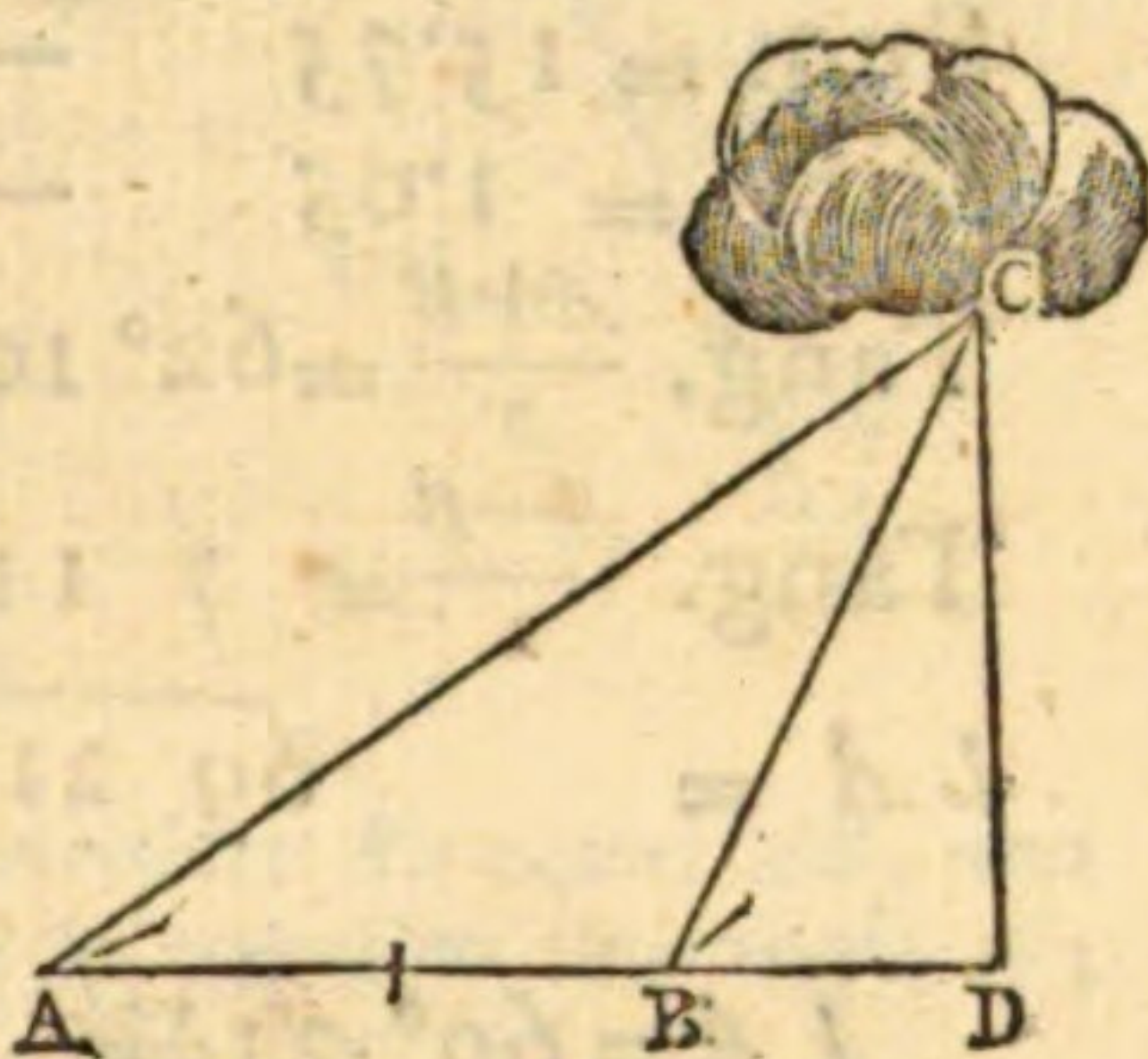
## EXAMPLE XI.

What is the perpendicular height of a cloud whose angles of elevation are  $35^\circ$  and  $64^\circ$ , taken by two observers, at the same time, both on the same side of the cloud, and at the distance of 880 yards the one from the other, so placed that a vertical plane would pass through both their stations and the cloud; and what is its distance from the two places of observation?

$$\begin{array}{r} 90^\circ \quad 90^\circ \\ 64 \quad 35 \\ \hline LBCD = 26 \quad 55 = LACD \\ 26 = LBCD \\ \hline 29 = LACB \end{array}$$

$$\begin{array}{r} LACB = 29^\circ \quad - \quad 9.6855712 \\ AB = 880 \quad - \quad - \quad 2.9444827 \\ LA = 35^\circ \quad - \quad - \quad 9.7585913 \\ BC = 1041.125 \quad - \quad 3.0175028 \end{array}$$

$$\begin{array}{r} LACB \ 29^\circ \quad - \quad 9.6855712 \quad | \quad LD \ 90^\circ \quad - \quad 10.0000000 \\ AB \ 880 \quad - \quad 2.9444827 \quad | \quad BC \quad - \quad - \quad 3.0175028 \\ LABC \ 116^\circ \quad - \quad 9.9536602 \quad | \quad LB \ 64^\circ \quad - \quad 9.9536602 \\ AC \ 1631.442 \quad - \quad 3.2125717 \quad | \quad DC \ 935.757 \quad - \quad 2.9711630 \end{array}$$



*Note.* In finding the distances of inaccessible objects, if they be of such a height as to admit of a pretty large angle of elevation, their distance will be found as in some of the foregoing examples.—If not, the theodolite or some such instrument generally is used to take the angles of the distances, or the horizontal angles, of objects as in the following examples.

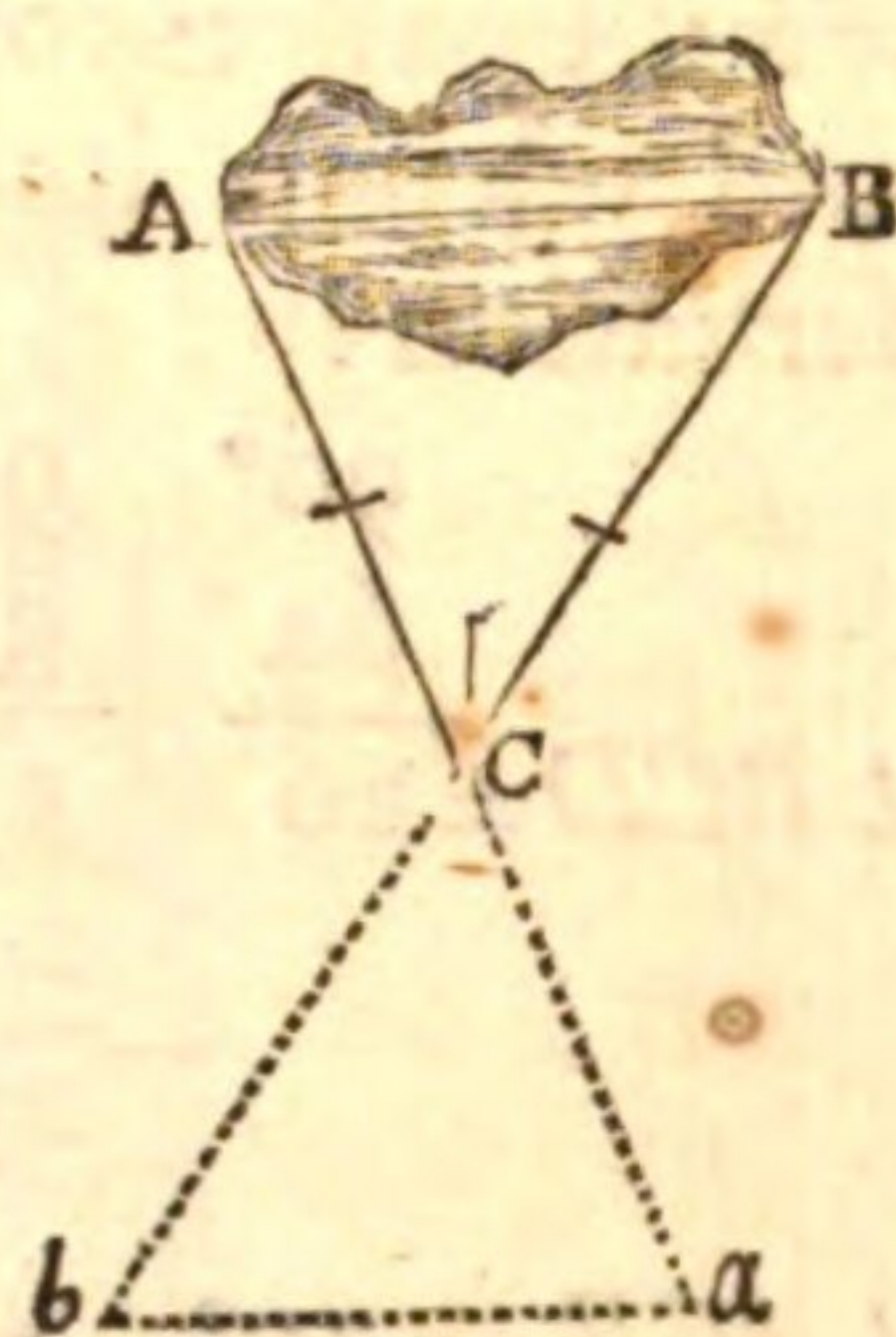
## EXAMPLE XII.

Suppose that I wanted to know the distance of the two places *A*, *B*, to whose ends there was free access,



cess, but not to the intermediate parts, because of a hill, precipice, or water between  $A$  and  $B$ ; and that therefore I measured from  $A, B$ , to any convenient place  $C$ , the distance  $AC$  equal 7.35 chains, and  $CB$  equal 8.4 chains, and found the angle  $ACB$  equal  $55^{\circ} 40'$ . What is the distance of the places  $A, B$ ?

$$\begin{array}{rcl}
 & 8.04 & \\
 & \underline{7.35} & \\
 \text{Sum} = & 15.75 & \text{---} \quad 1.1972806 \\
 \text{Diff.} = & 1.05 & \text{---} \quad 0.0211893 \\
 \text{Tang. } \frac{A+B}{2} = & 62^{\circ} 10' & \text{---} \quad 10.2773793 \\
 \text{Tang. } \frac{A-B}{2} = & 7 \ 11 \frac{1}{4} & \text{---} \quad 9.1012880 \\
 LA = & 69 \ 21 \frac{1}{4} &
 \end{array}$$



$$\begin{array}{rcl}
 LA = 69^{\circ} 21 \frac{1}{4}' & \text{---} & 9.9711982 \\
 CB = 8.4 \text{ chains} & \text{---} & 0.9242793 \\
 LC = 55^{\circ} 40' & \text{---} & 9.9168593 \\
 AB = 7.412 \text{ chains} & \text{---} & 0.8699404
 \end{array}$$

*Note.* If the lines  $AC, BC$ , be produced to  $a$  and  $b$  till  $Ca, Cb$ , be equal to  $CA, CB$ , or equal to  $CB, CA$ ; then the distance  $ba$ , will be equal to the distance  $AB$ , and therefore  $AB$  will be got, without any calculation, by only measuring  $ba$ .

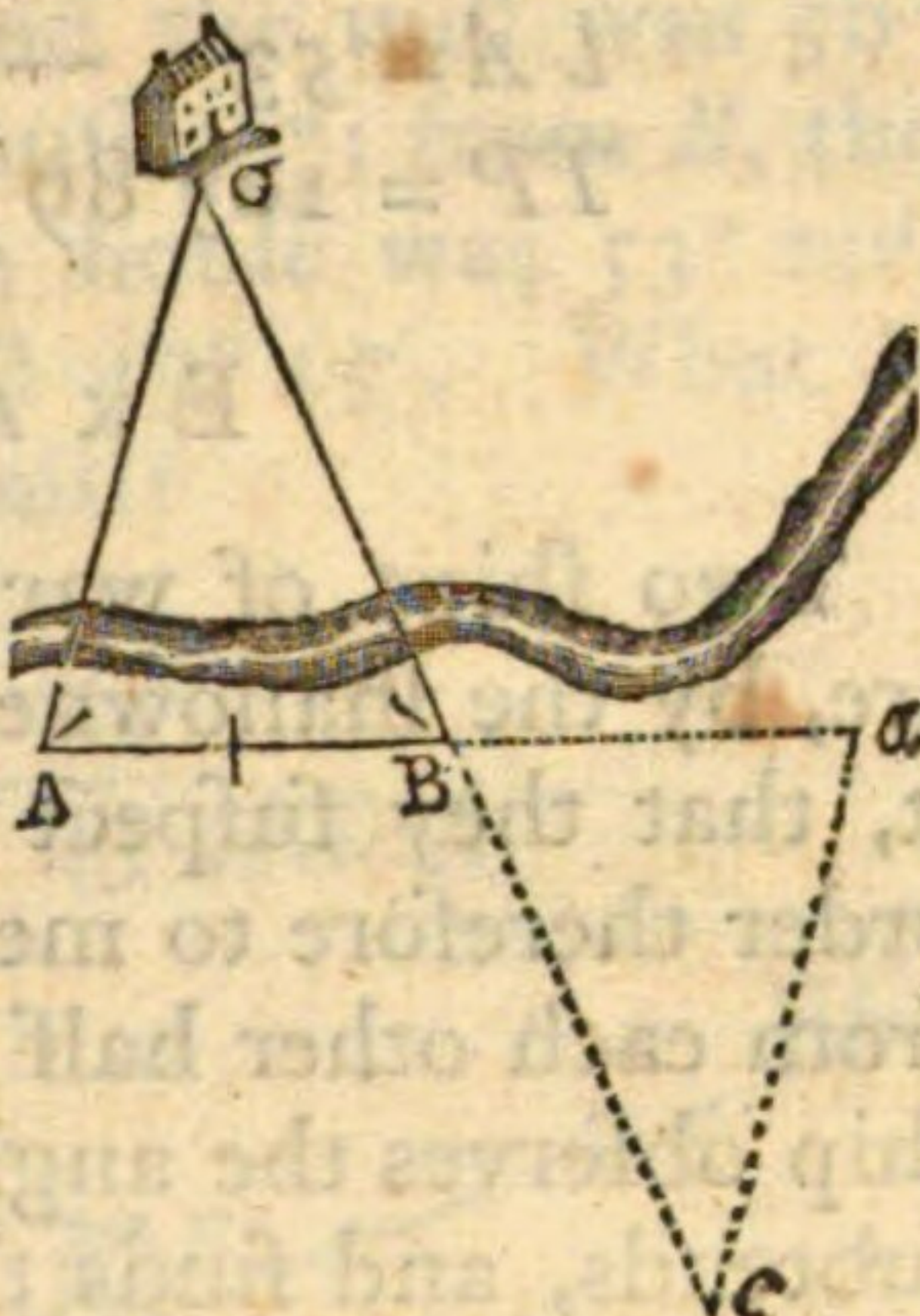
### EXAMPLE XIII.

Being on the side of a river, and wanting to know the distance to a house which stood on the other side, I measured 200 yards in a right line by the side of the river, and found that the two angles at each end of this line, formed by the other end and the



the house, were  $73^{\circ} 15'$  and  $68^{\circ} 02'$ : What was the distance between each station and the house?

$$\begin{array}{r}
 73^{\circ} 15' \\
 68 \ 02 \\
 \hline
 141 \ 17 \\
 180 \ 00 \\
 \hline
 LC = 38 \ 43 \text{ — } 9.7962062 \\
 AB = 200 \text{ — } 2.3010300 \\
 LA = 73^{\circ} 15' \text{ — } 9.9811711 \\
 BC = 306.19 \text{ — } 2.4859949 \\
 \\ 
 LC = 38^{\circ} 43' \text{ — } 9.7962062 \\
 AB = 200 \text{ — } 2.3010300 \\
 LB = 68^{\circ} 02' \text{ — } 9.9672679 \\
 AC = 296.54 \text{ — } 2.4720917
 \end{array}$$

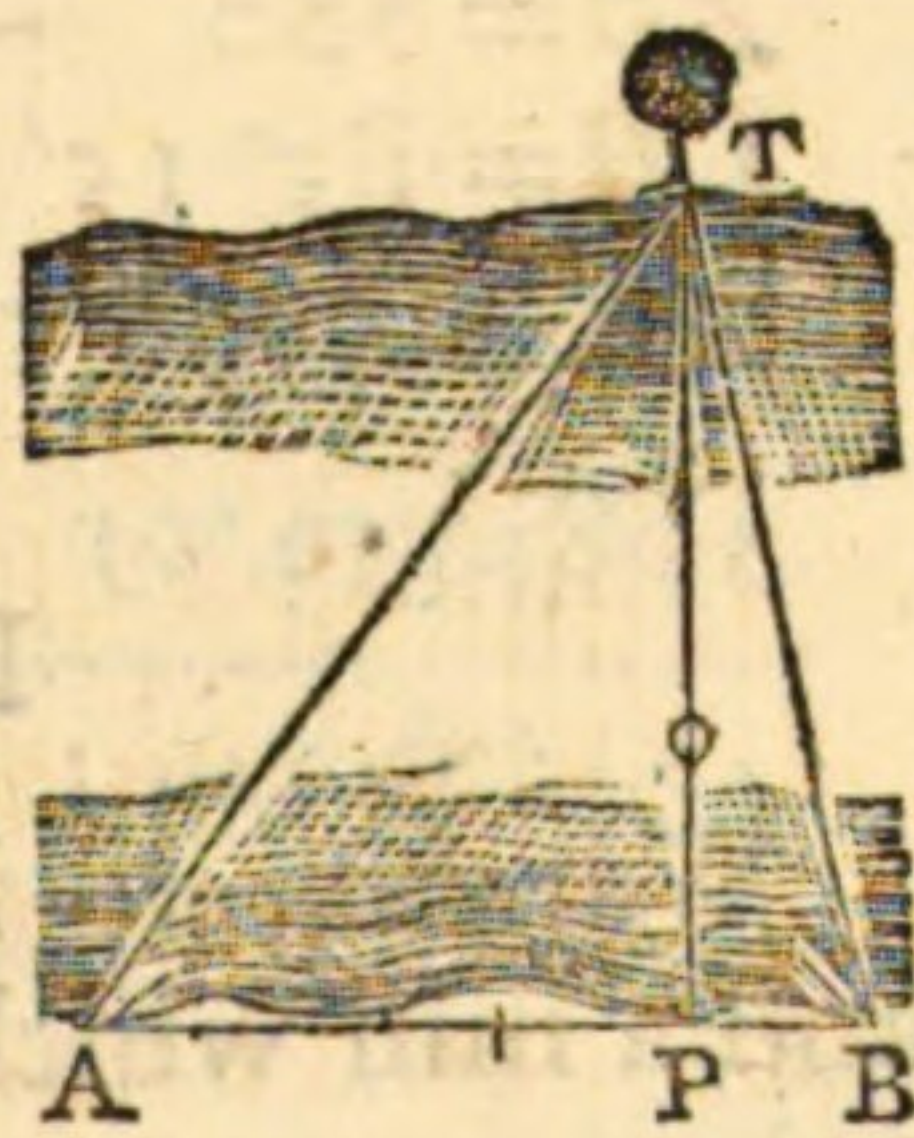


*Note.* If, in the right lines  $ABa$ , you measure  $Ba$  equal  $AB$ , and the line  $CBa$  be produced until the angle  $a$  be equal to the angle  $A$ ; the distances  $Bc$ ,  $ac$ , will be equal to  $BC$ ,  $AC$ .

## EXAMPLE XIV.

Wanting to know the breadth of a river, I measured 100 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree close by the other side of the river, to be  $53^{\circ}$  and  $79^{\circ} 12'$ . What is the perpendicular breadth?

$$\begin{array}{r}
 53^{\circ} 00' \\
 79 \ 12 \\
 \hline
 132 \ 12 \\
 180 \\
 \hline
 LATB = 47 \ 48 \text{ — } 9.8697037 \\
 AB = 100 \text{ — } 2.0000000 \\
 LB = 79^{\circ} 12' \text{ — } 9.9922385 \\
 AT \text{ — } \text{ — } 2.1225348
 \end{array}$$



M

LP



|                 |   |   |                  |
|-----------------|---|---|------------------|
| $LP = 90^\circ$ | — | — | 10.0000000       |
| $AT$            | — | — | 2.1225348        |
| $LA = 53$       | — | — | <u>9.9023486</u> |
| $TP = 105.89$   | — | — | 2.0248834        |

## EXAMPLE XV.

Two ships of war intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it; in order therefore to measure the distance they separate from each other half a mile or 880 yards; then each ship observes the angles which the other and the fort subtends, and finds them to be  $85^\circ 15'$  and  $83^\circ 45'$ : What is the distance between each ship and the fort?

|                     |   |   |   |                  |
|---------------------|---|---|---|------------------|
|                     |   |   |   |                  |
|                     |   |   |   | $83^\circ 45'$   |
|                     |   |   |   | $85 \quad 15$    |
|                     |   |   |   | <u>169</u>       |
|                     |   |   |   | 180              |
| $LF = 11$           | — | — | — | 9.2805988        |
| $AB = 880$          | — | — | — | 2.9444827        |
| $LA = 83^\circ 45'$ | — | — | — | <u>9.9974110</u> |
| $BF = 4584.5$       | — | — | — | 3.6612949        |
| $LF = 11^\circ$     | — | — | — | 9.2805988        |
| $AB = 880$          | — | — | — | 2.9444827        |
| $LB = 85^\circ 15'$ | — | — | — | <u>9.9985058</u> |
| $AF = 4596.1$       | — | — | — | 3.6623897        |



## EXAMPLE XVI.

Wanting to know the distance between a house and a mill which were separated from me by a river, I took

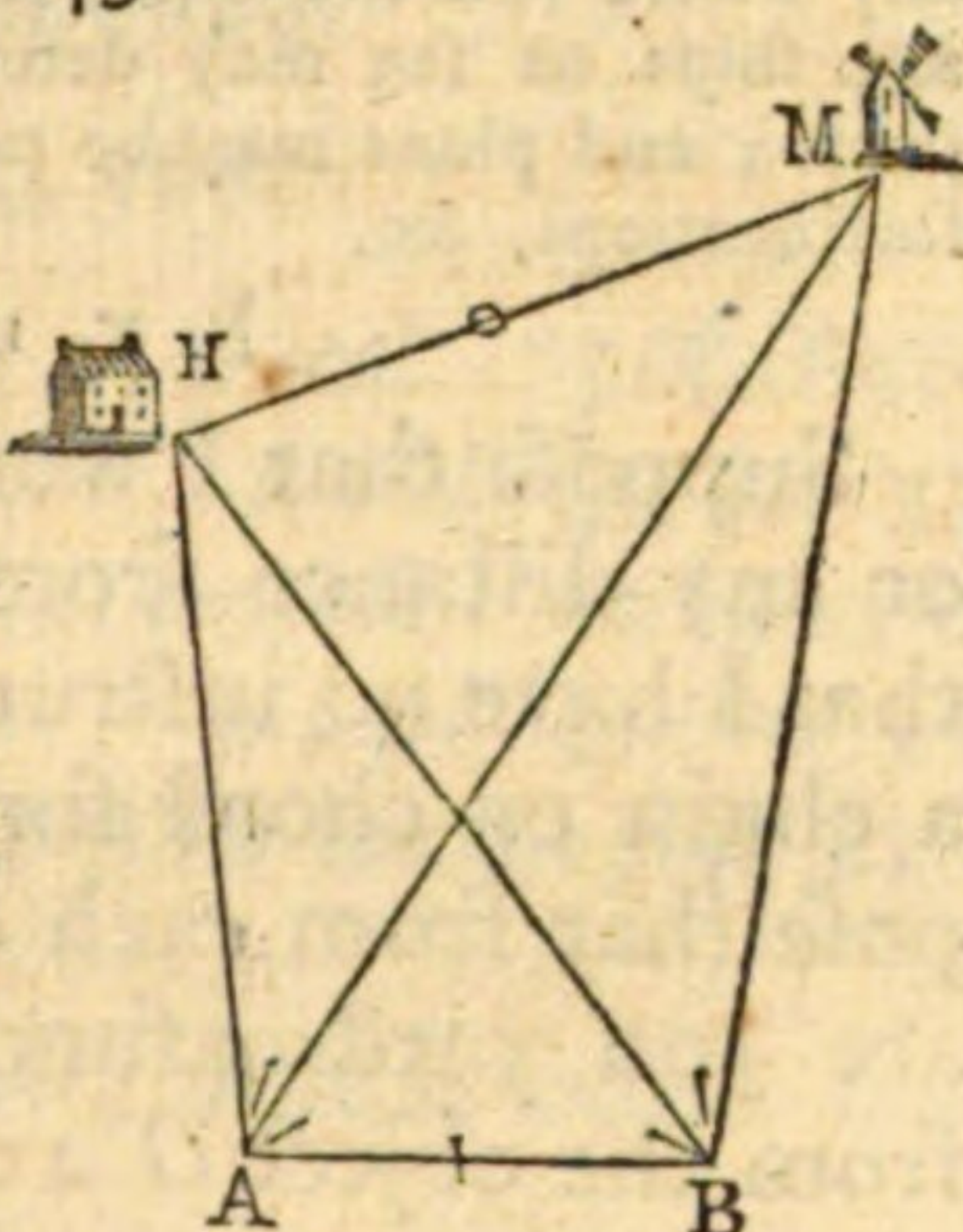


I took another station *B* at the distance of 300 yards from the first station *A*: now from the first station *A*, the angle subtended by *B* and the mill was  $58^{\circ} 20'$ , and by the mill and the house  $37^{\circ}$ ; from *B*, the angle subtended by *A* and the house was  $53^{\circ} 20'$ , and by the house and the mill  $45^{\circ} 15'$ . What is the distance of the house and mill?

|                  |                  |                      |
|------------------|------------------|----------------------|
| $37^{\circ} 00'$ | $58^{\circ} 20'$ | $53^{\circ} 30'$     |
| $58 \ 20$        | $53 \ 30$        | $45 \ 15$            |
| $53 \ 30$        | $45 \ 15$        | $98 \ 45 \angle ABM$ |
| $148 \ 50$       | $157 \ 05$       |                      |
| $180 \ 00$       | $180 \ 00$       |                      |

$22 \ 55 \angle AMB$

|                               |     |                     |
|-------------------------------|-----|---------------------|
| $\angle AHB = 31 \ 10$        | $-$ | $9^{\circ} 7139349$ |
| $AB = 300$                    | $-$ | $2^{\circ} 4771213$ |
| $\angle ABH = 53^{\circ} 20'$ | $-$ | $9^{\circ} 9042411$ |
| $AH = 464.79$                 | $-$ | $2^{\circ} 6674275$ |



|                                                            |     |     |     |                      |
|------------------------------------------------------------|-----|-----|-----|----------------------|
| $\angle AMB = 22^{\circ} 55'$                              | $-$ | $-$ | $-$ | $9^{\circ} 5903869$  |
| $AB = 300$                                                 | $-$ | $-$ | $-$ | $2^{\circ} 4771213$  |
| $\angle ABM = 98^{\circ} 45'$ or $81^{\circ} 15'$          | $-$ | $-$ | $-$ | $9^{\circ} 9949158$  |
| $AM = 761.46$                                              | $-$ | $-$ | $-$ | $2^{\circ} 8816502$  |
| $AH = 464.79$                                              |     |     |     |                      |
| Sum = $1226.43$                                            | $-$ | $-$ | $-$ | $3^{\circ} 0886427$  |
| Dif. = $296.49$                                            | $-$ | $-$ | $-$ | $2^{\circ} 4720100$  |
| Tan. $\angle \frac{AHM + AMH}{2} = 71^{\circ} \frac{1}{2}$ | $-$ | $-$ | $-$ | $10^{\circ} 4754801$ |
| Tan. $\angle \frac{AHM - AMH}{2} = 35^{\circ} 51'$         | $-$ | $-$ | $-$ | $9^{\circ} 8588474$  |
| $35 \ 39 = \angle AMH$                                     |     |     |     |                      |

$\angle AMH$



|                             |   |   |   |           |
|-----------------------------|---|---|---|-----------|
| $\angle AMH = 35^\circ 39'$ | — | — | — | 9.7655436 |
| AH                          | — | — | — | 2.6674275 |
| $\angle HAM = 37^\circ$     | — | — | — | 9.7794630 |
| HM = 480.11 yds             | — | — | — | 2.6813469 |

*Note.* Pretty much after the manner of these last examples many curious and useful problems may be solved: for if we can determine our distance from one remote object we can do the same for any number of objects; or if the distance between two remote objects can be determined, those between any number of objects may be determined likewise: so, we may determine the angles and sides of fields, or of very large tracts of land, and that whether we be within them, or any where without them, from whence the angles can be seen; hence also ships at sea may determine their distances from known visible ports; and plans may be taken of countries, towns, harbours, fleets, fortifications, &c.

## EXAMPLE XVII.

Suppose that I would know the breadth of a river, or my distance from an inaccessible object  $O$ , and that I have no instrument for taking angles, but only a chain or chord for measuring distances; and suppose that from each of the two stations  $A, B$ , which are 500 yards asunder, I measure in a direct line from the object  $O$  100 yards, viz.  $Aa$  and  $Bb$  each equal 100 yards, and that the diagonal  $Ab$  measures 550 yards, and the diagonal  $aB$  measures 560: What then is the distance of the object from each station  $A$  and  $B$ ?

Let fall the perpendiculars  $AP$ ,  $BQ$ . Then, in the triangle  $ABa$ ,  $aB = 560$ :  $BA + Aa = 600$ :  $BA - Aa = 400$ :  $\frac{240000}{560} = \frac{3000}{7} = 428\frac{4}{7} = BP - Pa$ .

And  $\begin{cases} BP = \frac{560 + 428\frac{4}{7}}{2} = \frac{988\frac{4}{7}}{2} = 494\frac{2}{7}, \\ Pa = \frac{560 - 428\frac{4}{7}}{2} = \frac{131\frac{3}{7}}{2} = 65\frac{5}{7}. \end{cases}$



Whence



$$\text{Whence } \begin{cases} Aa = 100 : aP = 65\frac{1}{2} :: 1 = \text{rad.} \cdot 6571428 = \\ \quad s. L aAP = 41^\circ 5', \\ AB = 500 : BP = 494\frac{2}{7} :: 1 = \text{rad.} \cdot 9885714 = \\ \quad s. L BAP = 81^\circ 20', \end{cases}$$

whose sum  $122^\circ 25'$  taken from  $180^\circ$  leaves  $57^\circ 35' = \angle BAO$ .

Again, in the triangle  $ABb$ ,  $Ab = 550 : AB + Bb = 600 :: AB - Bb = 400 : \frac{240000}{550} = \frac{4800}{11} = 436\frac{4}{11} = A\mathcal{Q} - \mathcal{Q}b$ .

$$\text{And } \begin{cases} A\mathcal{Q} = \frac{550 + 436\frac{4}{11}}{2} = \frac{986\frac{4}{11}}{2} = 493\frac{2}{11}, \\ \mathcal{Q}b = \frac{550 - 436\frac{4}{11}}{2} = \frac{113\frac{7}{11}}{2} = 56\frac{2}{11}. \end{cases}$$

$$\text{Whence } \begin{cases} Bb = 100 : b\mathcal{Q} = 56\frac{2}{11} :: 1 = \text{rad.} \cdot 5681818 = \\ \quad s. L bB\mathcal{Q} = 34^\circ 37', \\ BA = 500 : A\mathcal{Q} = 493\frac{2}{11} :: 1 = \text{rad.} \cdot 9863636 = \\ \quad s. L AB\mathcal{Q} = 80^\circ 32', \end{cases}$$

whose sum  $115^\circ 9'$  taken from  $180$  leaves  $64^\circ 51' = \angle ABO$ .

Then  $ABO + OAB = 122^\circ 26'$ , which taken from  $180^\circ$  leaves  $57^\circ 34' = \angle O$ .\*

N

Whence

\* OR the angles  $ABO$  and  $BAO$  may be otherwise found thus :

Draw the perpendiculars  $Ag$ ,  $Bb$  : Then since  $(Ag^2 =) AB^2 - Bg^2 = (Ab^2 - bg^2 =) Ab^2 - bB^2 - 2bB \times Bg - Bg^2$ , or  $AB^2 = Ab^2 - bB^2 - 2bB \times Bg$ , we have  $Bg = \frac{Ab^2 - AB^2 - bB^2}{2bB} = 425\frac{1}{2}$ .

And  $AB : 1 :: Bg : \frac{Bg}{AB} = .425 = \text{cofine of } 64^\circ 51' \text{ the } \angle ABO$ .

In like manner  $Ab = \frac{Ba^2 - AB^2 - aA^2}{2aA} = 268$ ,

And  $AB : 1 :: Ab : \frac{Ab}{AB} = .536 \text{ the cofine of } 57^\circ 35' \text{ the } \angle BAO$ , the same as above.



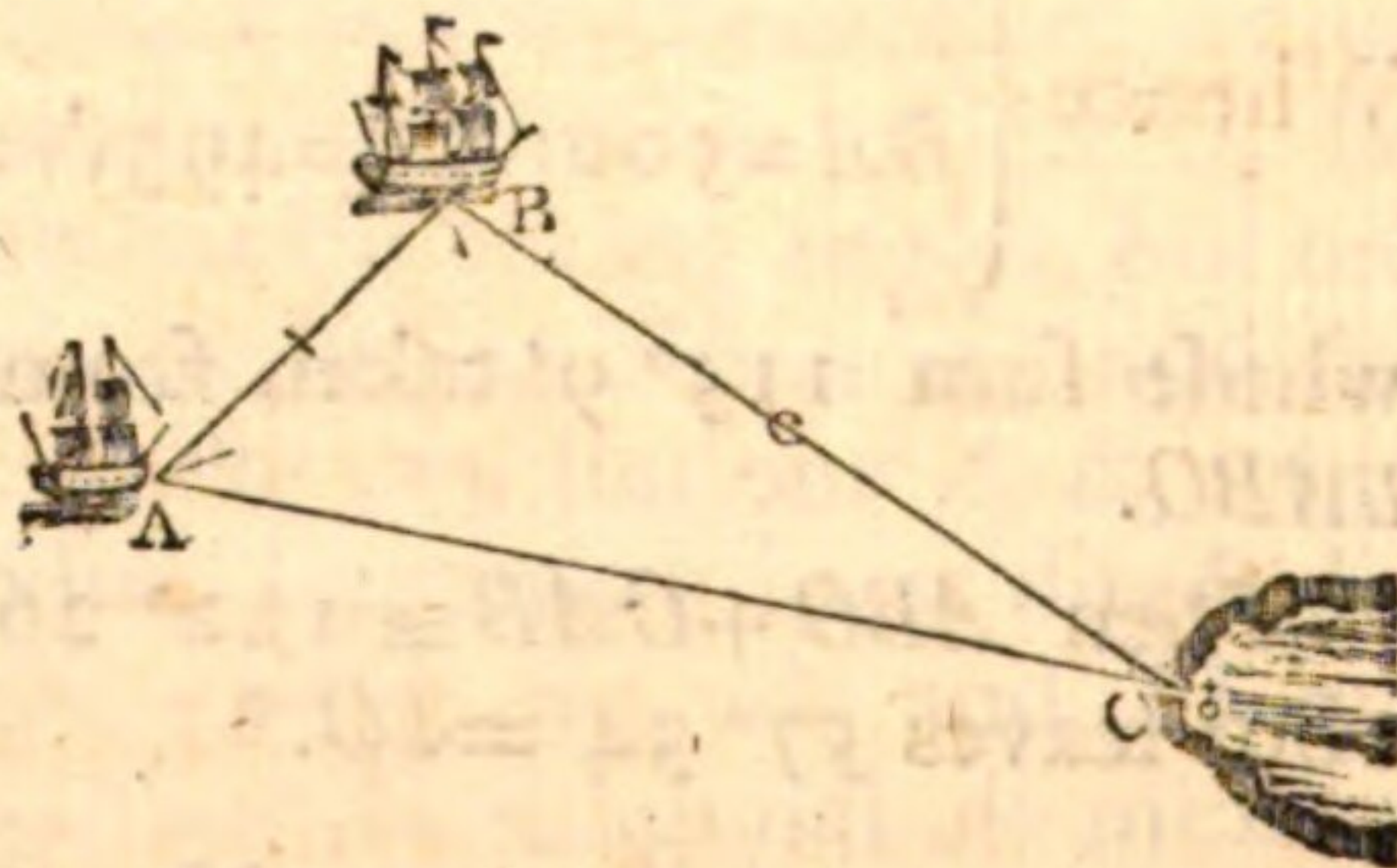
Whence

|                                   |                                   |
|-----------------------------------|-----------------------------------|
| $LO\ 57^{\circ}\ 34' - 9.9263507$ | $LO\ 57^{\circ}\ 34' - 9.9263507$ |
| $AB\ 500 - 2.6989700$             | $AB\ 500 - 2.6989700$             |
| $LA\ 57^{\circ}\ 35' - 9.9264310$ | $LB\ 64^{\circ}\ 51' - 9.9567437$ |
| $BO\ 500.1 - 2.6990503$           | $AO\ 536.25 - 2.7293630$          |

## EXAMPLE XVIII.

From a ship at sea I observed a point of land to bear *E.* by *S.* and after sailing *N. E.* 12 miles, I set it, again, and found its bearing to be *S. E.* by *E.* How far was the last observation made from the point of land?

Here are given the angle *A*, at the place of the first observation, 5 points or  $56^{\circ}\ 15'$ ; the angle *B*, 9 points or  $101^{\circ}\ 15'$ ; and the angle *C*, 2 points or  $22^{\circ}\ 30'$ ;



also the side *AB* 12 miles: to find the side *BC*.

|                              |   |   |           |
|------------------------------|---|---|-----------|
| As $s. LC = 22^{\circ}\ 30'$ | — | — | 9.5828397 |
| To $s. LA = 56\ 15$          | — | — | 9.9198464 |
| So is $AB = 12$ miles        | — | — | 1.0791812 |
| To $BC = 26.07281$ miles     | — | — | 1.4161879 |

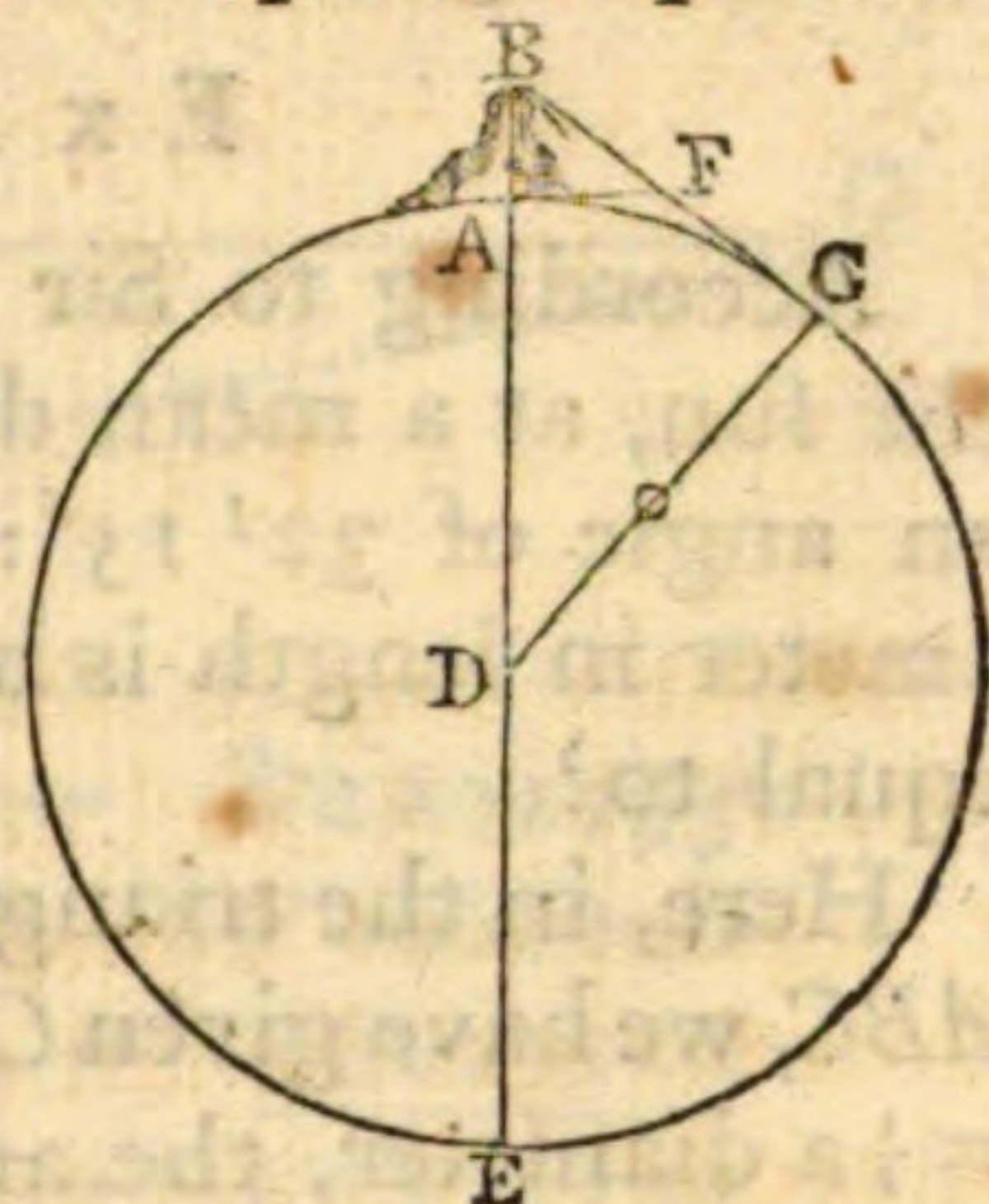
## EXAMPLE XIX.

If the height *AB* of the mountain called the pike of teneriff be 4 miles, and the angle *ABC* made by a plumb line and a line *BC* conceived to touch the earth in the farthest visible point *C*, be  $87^{\circ}\ 25'\ 55''$ ;  
re-



required the diameter  $AE$  of the earth, and the utmost distance  $BC$  that can be seen from the top of the mountain, supposing the earth to be a perfect sphere.

Draw  $AF$  perpendicular to  $AB$ , then from the principles of geometry it is known that  $CF$  is  $= FA$ ; draw, also, the radius  $CD$ , which will be perpendicular to  $CB$ .



Hence  $\angle D = \angle F = (90^\circ - 87^\circ 25' 55'' =) 2^\circ 34' 05''$ .

|                                           |   |                  |
|-------------------------------------------|---|------------------|
| Then, As $s. \angle F = 2^\circ 34' 05''$ | — | 8.6513363        |
| To $s. \angle A = 90^\circ$               | — | 10.0000000       |
| So is $AB = 4$ miles                      | — | 0.6020600        |
| To $BF = 89.27373$ miles                  | — | <u>1.9507237</u> |

|                                            |   |                  |
|--------------------------------------------|---|------------------|
| And, As radius                             | — | 10.0000000       |
| To $t. \angle B = 87^\circ 25' 55''$       | — | 11.3482278       |
| So is $BA = 4$ miles                       | — | 0.6020600        |
| To $AF = FC = 89.18419$                    | — | <u>1.9502878</u> |
| add $BF = 89.27373$                        |   |                  |
| gives $BC = 178.45792$ greatest vis. dist. |   |                  |

|                     |   |                  |
|---------------------|---|------------------|
| Lastly, As radius   | — | 10.0000000       |
| To $t. \angle B$    | — | 11.3482278       |
| So $BC = 178.45792$ | — | 2.2515359        |
| To $CD = 3978.9064$ | — | <u>3.5997637</u> |

$\frac{2}{7957.8128}$  diameter of the earth.

*Note.* This method of determining the magnitude of the earth is very easy and simple; but to do it with any tolerable degree of exactness,

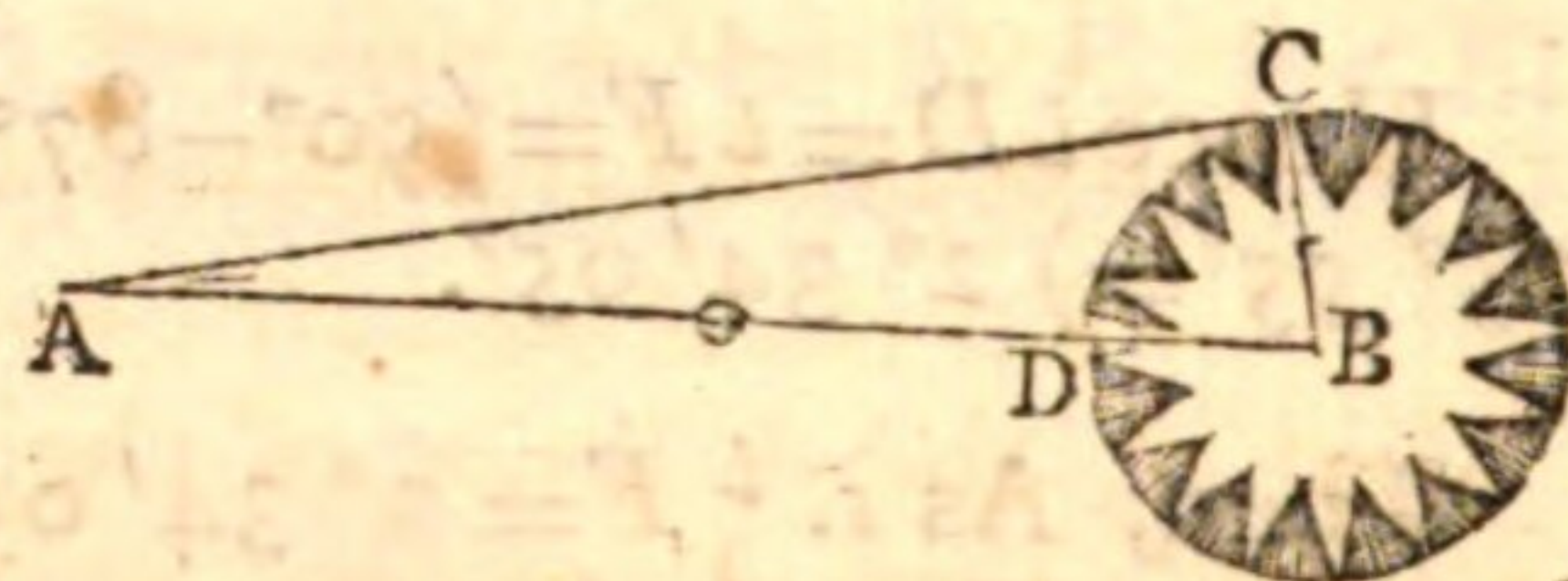


actness, the height of the mountain must be pretty accurately ascertained, and the angle at the top must be taken with a quadrant divided into minutes and seconds, and furnished with a telescope instead of the common sights.

## EXAMPLE XX.

According to Sir Isaac Newton, the diameter of the sun, at a mean distance from the earth, subtends an angle of  $32' 15''$ : Then how many times his diameter in length is his mean distance from the earth equal to?

Here, in the triangle  $ABC$ , we have given  $CB = \frac{1}{2}$  a diameter, the angle  $C = 90^\circ$ , and the angle  $A = 16' 7\frac{1}{4}''$ .



|                                              |     |            |
|----------------------------------------------|-----|------------|
| Hence, As $s. \angle A = 16' 7\frac{1}{4}''$ | — — | 7.6712242  |
| To $s. \angle C = 90^\circ$                  | — — | 10.0000000 |
| So is $CB = 1$ semi-diam.                    | — — | 0.0000000  |
| To $BA = 213.1944$                           | — — | 2.3287758  |

That is, the mean distance of the sun's center is 213.1944 semi-diameters, or 106.5972 of his whole diameters; and if from  $AB$  be taken  $BD (\frac{1}{2})$ , we shall have remaining nearly 106 diameters for the distance of the surface.

*Note.* After the same manner may be calculated the proportion of the diameter to the distance of any other celestial body whose apparent diameter is large enough to be measured. And, in particular, the mean distance of the moon, whose mean apparent diameter is  $31' 16''$ , will be found to be 109.95 times her diameter.

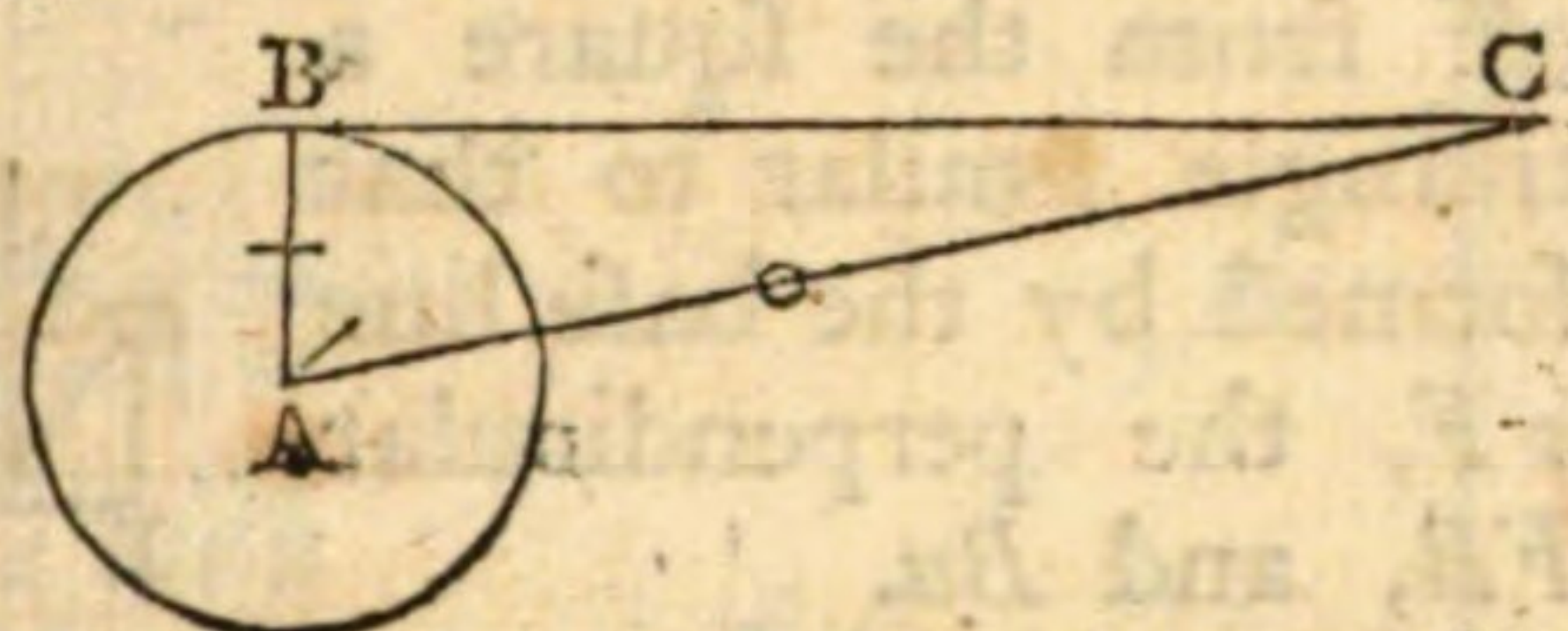
## EXAMPLE XXI.

If when the moon appears in the horizon to a spectator on the earth, at her mean distance from it, her zenith



zenith distance, as calculated from astronomical tables, be  $89^{\circ} 2' 55''$ ; it is required to find how many of the earth's semi-diameters the said mean distance of the moon is equal to.

Here  $AB$  is = one semi-diameter of the earth, and the  $\angle A = 89^{\circ} 2' 55''$ .



|                                   |     |            |
|-----------------------------------|-----|------------|
| Hence, As $s. \angle C = 57' 5''$ | — — | 8.2202155  |
| To $s. \angle B = 90^{\circ}$     | — — | 10.0000000 |
| So is $AB = 1$ semi-diam.         | —   | 0.0000000  |
| To $AC = 60.226$                  | — — | 1.7797845  |

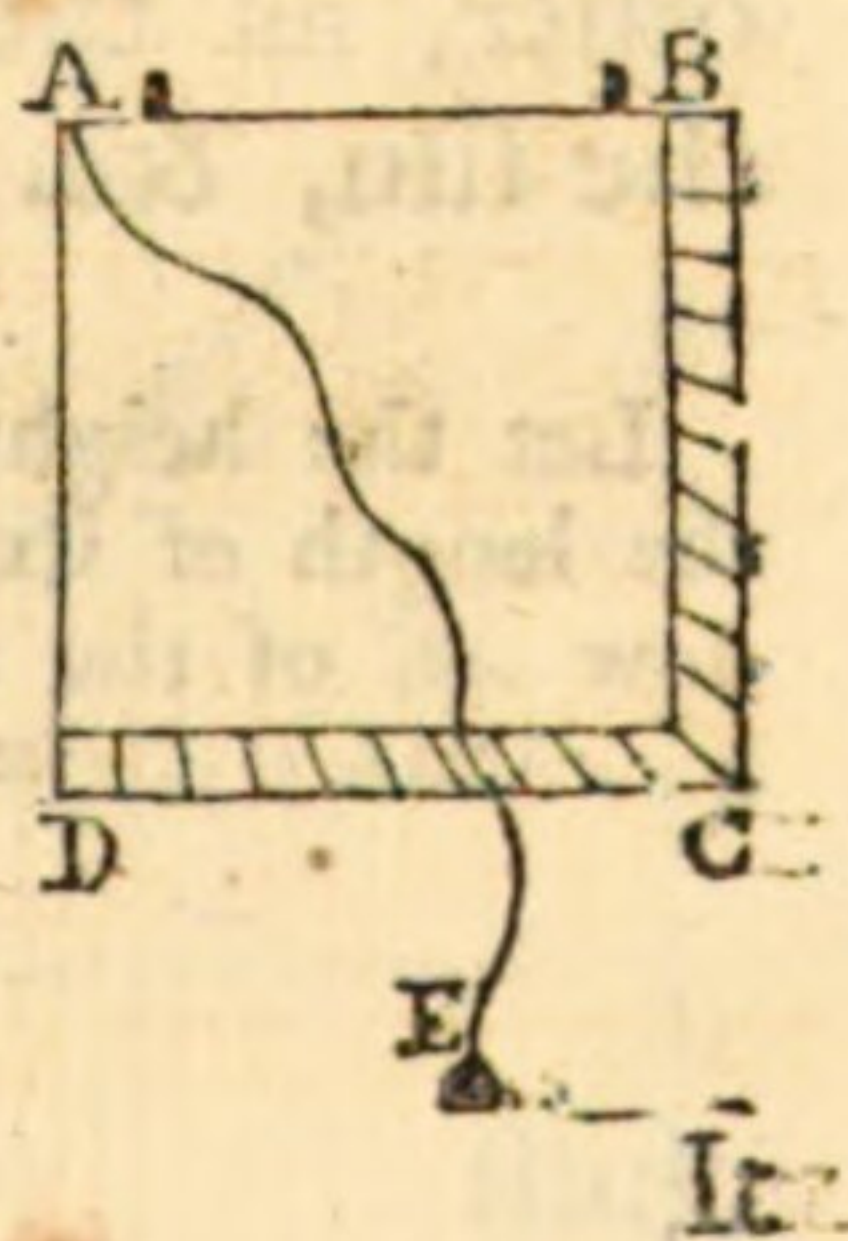
That is, the distance of their centers is  $60.226$  of the earth's semi-diameters, or  $30\frac{1}{2}$  whole diameters.

*Note.* Hence, and from the note to the last example, it appears that the diameter of the earth is to that of the moon as  $109.95$  to  $30\frac{1}{2}$ , or, as  $11$  to  $3$ , or as  $3\frac{2}{3}$  to  $1$  nearly.—Consequently, as  $3\frac{2}{3} : 1 :: 7958$  (the earth's diameter in miles) :  $2170$  miles = the moon's diameter.—Likewise, the surface of the earth is to that of the moon as  $(3\frac{2}{3} \times 3\frac{2}{3})$  or  $13\frac{4}{9}$  to  $1$  nearly; or, the earth reflects upon the moon about  $13\frac{4}{9}$  times as much light as the moon does upon the earth.—Moreover, the bulk of the earth is to that of the moon as  $(3\frac{2}{3} \times 3\frac{2}{3} \times 3\frac{2}{3})$  or  $48$  to  $1$  nearly.

#### GENERAL SCHOLIUM.

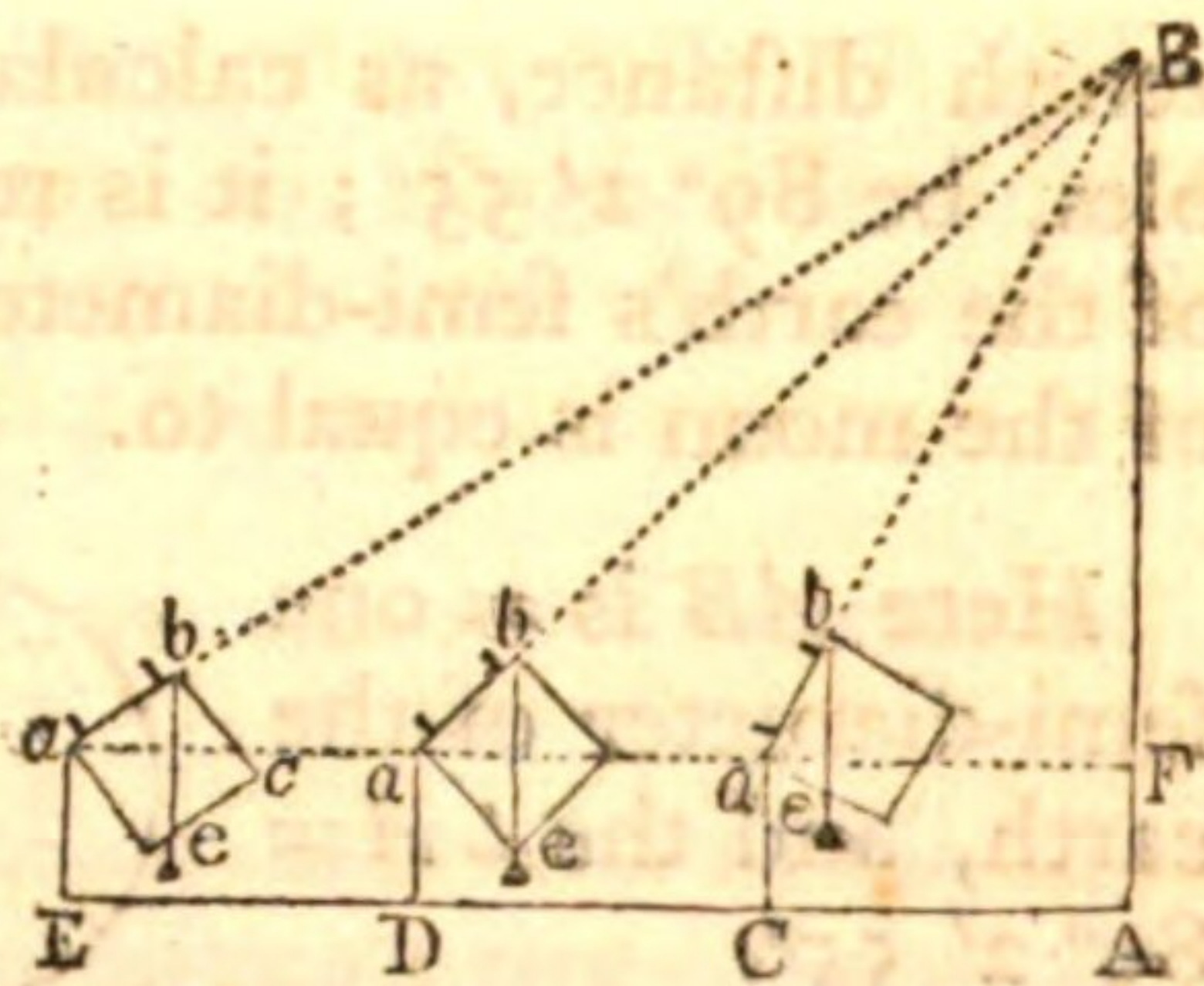
There are many other methods and instruments used to find the altitudes and distances of objects, some of which are here subjoined.

I. One very easy method is by a square with a plummet  $AE$  suspended from one corner  $A$ , and the two sides  $BC$ ,  $CD$ , meeting in the opposite angle  $C$ , divided in  $10$ ,  $100$ , or  $1000$  equal parts; and two sights on the side  $AB$ .





It is evident that, in taking any altitude  $AB$  with the square, the plummet will always cut off from the square a triangle similar to that formed by the base line  $aF$ , the perpendicular  $FB$ , and  $Ba$ .



If the angle  $BaF$  be equal to  $45^\circ$ , then the plumb line passeth through the opposite angle of the square, and the distance  $DA$  will be equal to the altitude  $BF$ : So if  $DA$  be 60, then  $BF$  will be 60 also.

If the angle be greater than  $45^\circ$ , as at the station  $C$ , then the part of the side cut off  $ae$ , will be to the whole side  $ab$ ; as  $aF$  to  $FB$ : So if  $ae$  be 6 divisions of which  $ab$  is 10, and the distance  $CA$  be 36 feet;

then  $6:10::36:\frac{360}{6}=60$  feet = the altitude  $BF$ .—Very often in

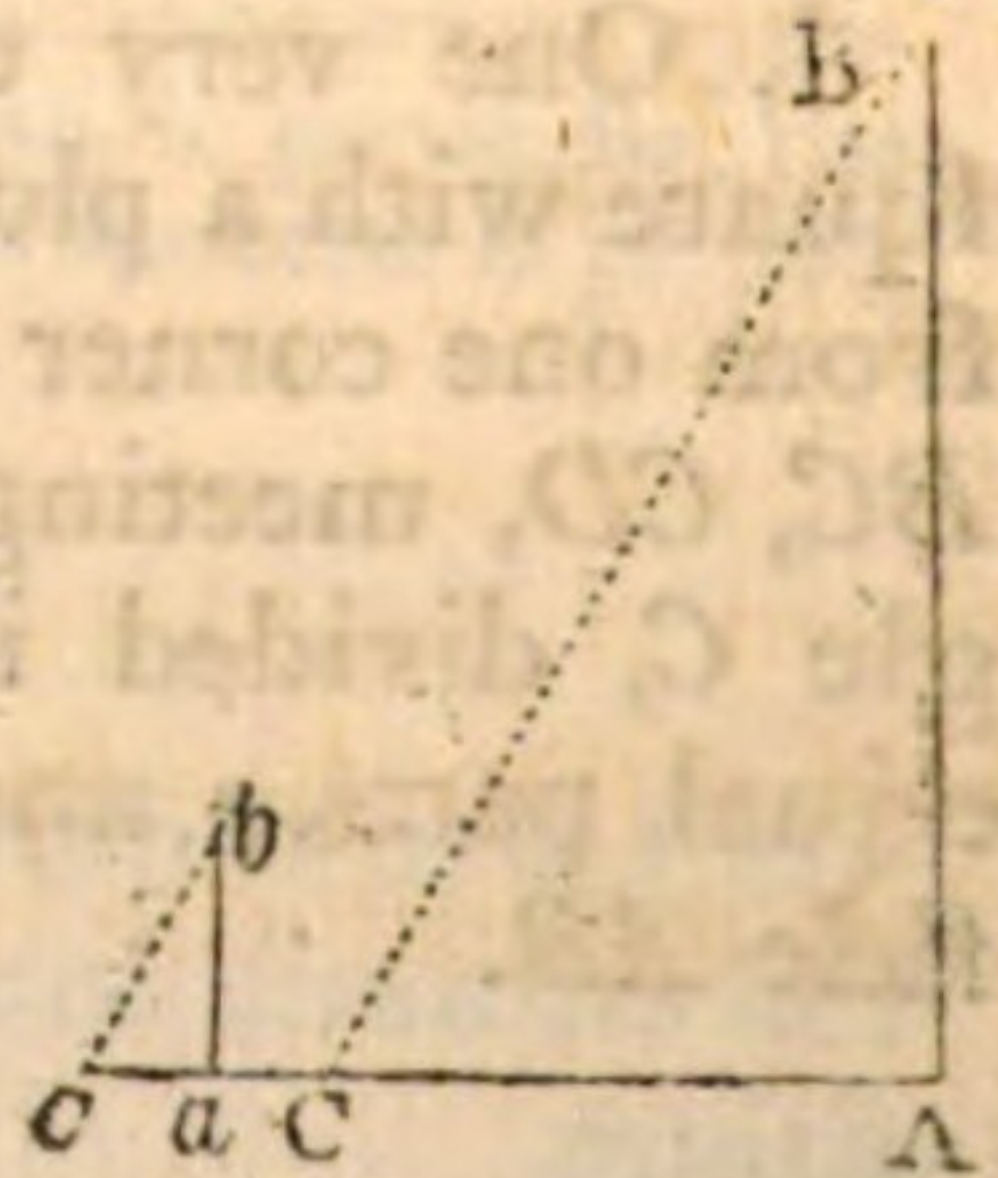
this case the table of multipliers, at the end of the book, will much facilitate the calculation; for the altitude will be found by only multiplying the distance  $AC$  into the decimal in the table put opposite the reciprocal of the parts cut off.

If the angle be less than  $45^\circ$ , as at the station  $E$ , then the part is cut off the other decimated side, and  $bc:ce::EA:BF$ ; so if the parts cut off be 6, and the distance 100 feet, then  $10:6::100:\frac{600}{10}=60$  feet =  $BF$ .

II. Another method is by shadows, from the property of similar triangles also; for any object and a stick set up parallel to it, are in proportion to each other, as the lengths of their shadows, formed by the sun, &c.

Let the height of the stick  $ab$  be 6 feet, the length of its shadow  $ac$  4, and the shadow  $AC$  of the altitude 40 feet; then  $ca=4:ab=6::CA=40:AB=60$ .

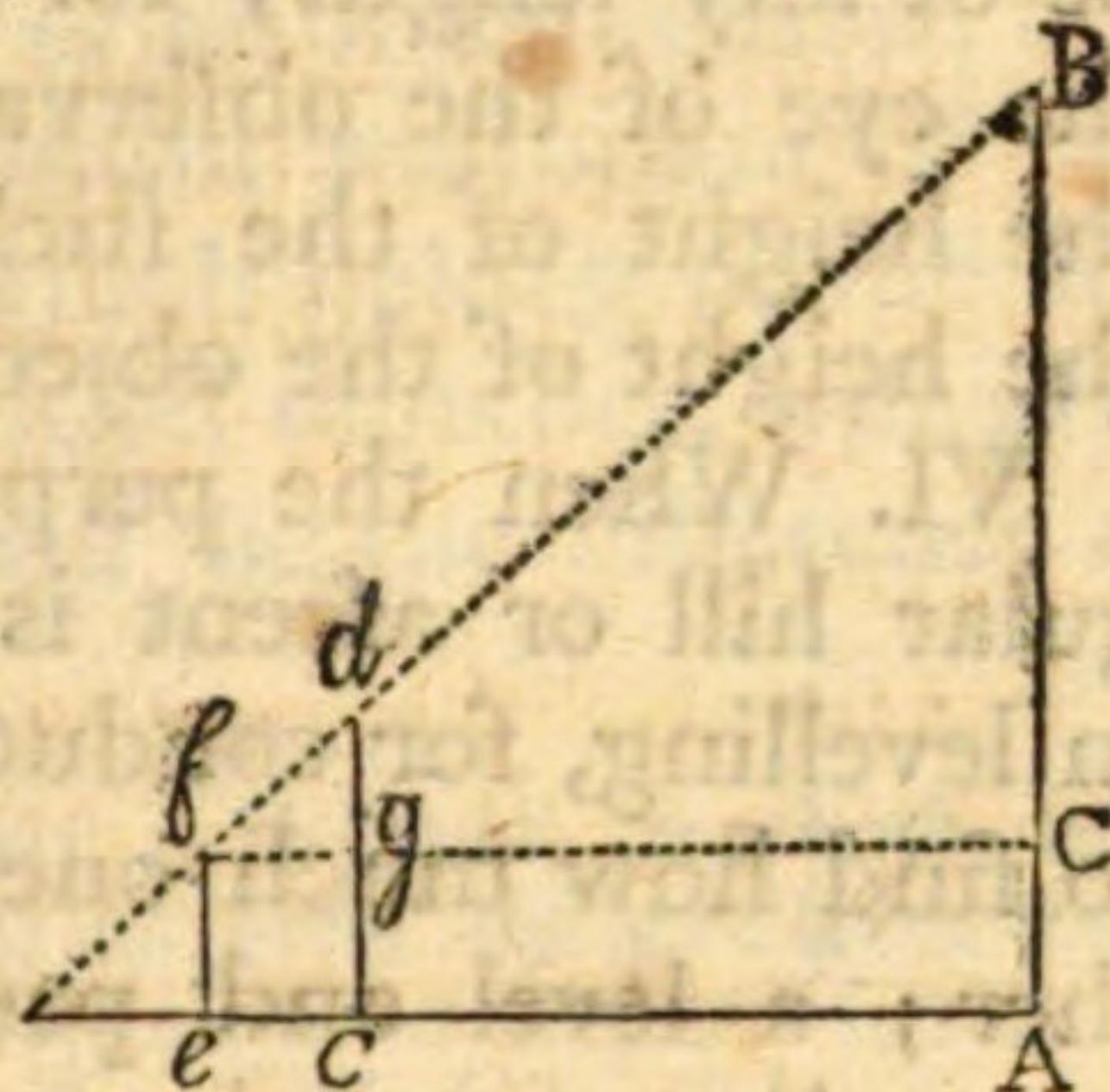
III. An-





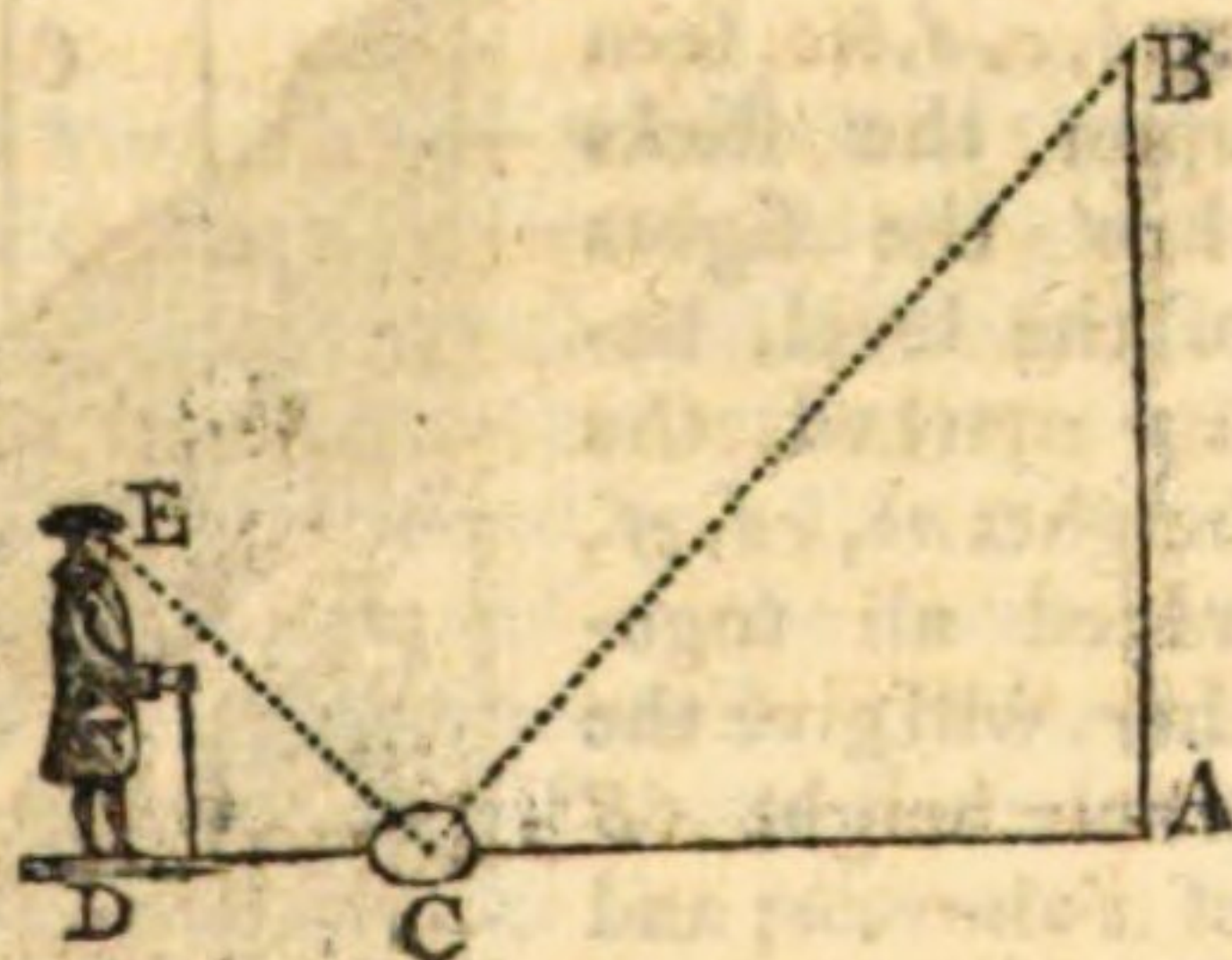
III. Another method is by two sticks set up parallel to the object, the one longer than the other, so that the observer may see the top of the object over the tops of both the sticks.

Let the stick  $ef$  be  $= 4$  feet,  $cd = 7$  feet, their distance asunder  $ec = fg = 8$  feet, and the distance  $eA$  of the shorter stick from the object  $= 160$  feet: then, the triangles  $fgd$ ,  $fCB$  being similar,  $fg:gd::fC:CB$ , that is  $8:3 (=7-4)::160:60$  feet  $= BC$ ; hence  $BC+CA = BC+fe = 60+4 = 64 = AB$ .

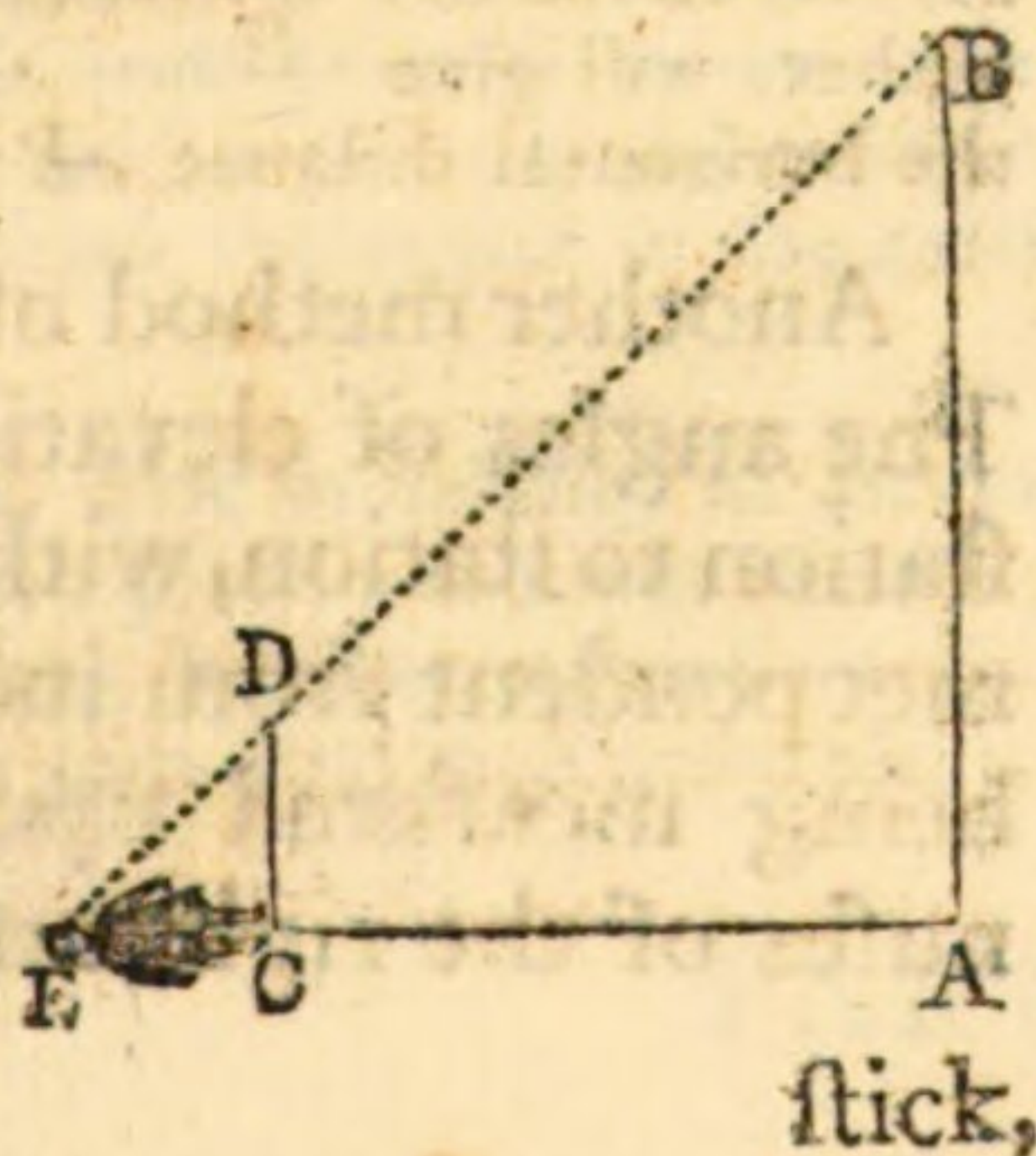


IV. A fourth method is by viewing the image of the top of the object reflected from a smooth surface, as a mirror placed horizontally, or a vessel of water.

Let  $C$  be the reflecting surface, at the distance of  $84$  feet from the bottom of the object  $AB$ ; and let a person at  $D$ ,  $7$  feet from  $C$ , with his eye  $5$  feet above the ground, view the image of the object at  $C$ ; then, because the triangles  $CDE$ ,  $CAB$  are similar, agreeably to an assumption of the opticians, we shall have  $CD:DE::CA:AB$ , that is  $7:5::84:60$  feet  $= AB$ .



V. A fifth method is for the observer to fix a stick  $CD$ , equal, in length, to the height of his eye, perpendicularly at  $C$ , by trials, at such a distance from  $A$ , that having laid himself upon his back, with his feet against the bottom of the

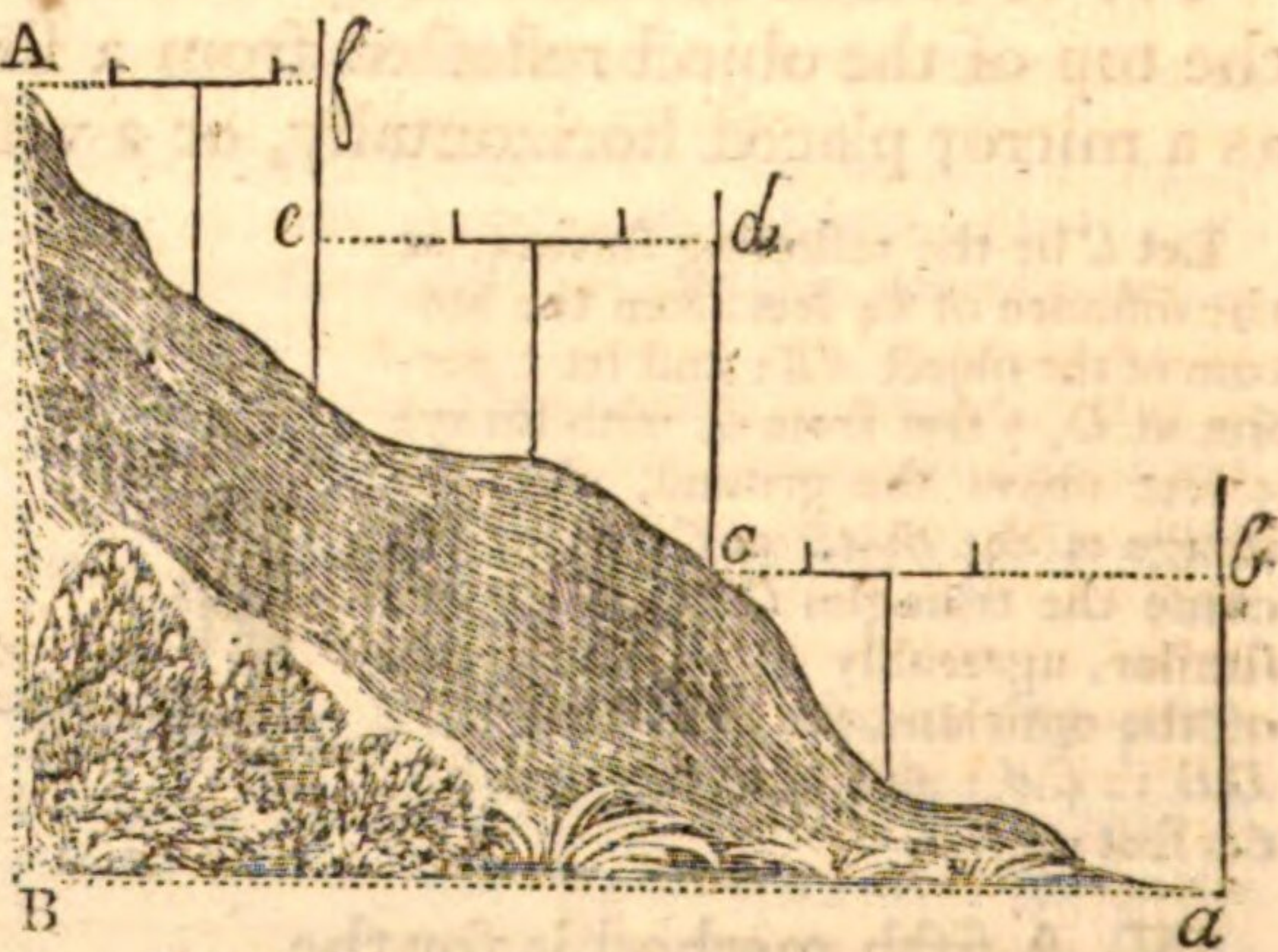




stick, he may see the tops  $D$  and  $B$  of the stick and object in the same line: For then  $EA$  will be equal to  $AB$ , because  $EC$  is equal to  $CD$ .—Or the stick may be of any length; for the distance from the foot to the eye of the observer will be in proportion to the height of the stick, as the whole distance is to the height of the object required.

VI. When the perpendicular altitude of an irregular hill or ascent is to be ascertained; or when, in levelling, for conducting water, &c. it is required to find how much one assigned place is above another; a level and perpendicular sticks, or objects, generally are used.

So the level being fixed horizontally, and the places  $b, c, d$ , &c. seen upon the sticks thro' the sights of the level, being marked, the heights  $ab, cd, ef$ , added all together, will give the whole height  $AB$  of  $A$  above  $a$ ; and the distances  $bc, de, fA$ , added together, will give the horizontal distance  $aB$  of  $A$  from  $a$ .



Another method of levelling is sometimes used, viz. The angles of elevation or depression are taken, from station to station, with an arc of a circle having a plummet pendent from its center, and the several distances, being measured upon the ground, are the hypothenuses of the right-angled triangles of which the angles



gles at the bases are expressed by the aforesaid observed angles; and consequently, the perpendiculars of these triangles being calculated, their sum will be the difference of level between the extreme places, if the angles were all of elevation or all of depression; but if the angles be some of elevation and some of depression, the perpendiculars which belong to the angles of the same kind being added together, will give two sums whose difference will be the difference of level required.

These two methods, when accurately performed, are both very true, at least where the stations are at no great distance from each other; yet the former is more to be depended on than the latter, in as much as the observations required by it are less susceptible of error; for besides the inaccuracy of the measured hypotenuses, on account of the unevenness of the ground, the angles themselves can hardly be taken, with the generality of instruments for this purpose, to less than quarters of degrees.

But when the places, of which we would know whether of them is the higher, are far distant from each other, instruments of greater accuracy are to be used, such as very exact levels with telescopes; but in such cases an allowance must be made for the rotundity of the earth; for the true water-level course is determined by a line of which every part is at the same distance from the center of the earth, and which is, therefore, an arc of a great circle of the earth, considering it not as an oblate spheroid, but as a sphere, its difference from which form having, in this case, no sensible effect; and the allowance necessary to be made in the level is at the rate of 8 inches nearly to a mile measured upon the earth; for the visual line, when the level is properly fixed, being a tangent to the earth, or at least parallel to one, that line at the distance of one mile from the station is 7.96 inches above the water-level line, or so much farther from the center of the earth than the instrument is.

It may be necessary farther to observe, that in taking, at several stations, the difference of level between two places, it is not necessary to go in a line from the one to the other, but every two successive stations may be taken in any direction that may seem most convenient.



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## PART II.

### *Of superficial Mensuration, or the Mensuration of plane Figures.*

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#### SECTION I.

##### *Of the Areas of right-lined and circular Figures.*

THE measure of a plane figure is called its area. By the mensuration of plane figures is determined the extension of bodies as to length and breadth; such as the quantities of lands, and the quantities of the works of many artificers.

Plane figures, and the surfaces of bodies, are measured by squares; as square inches, square feet, square yards, &c. that is, squares whose sides are inches, feet, yards, &c. Our least superficial measure is the square inch, other squares being taken from it according to the proportion in the following

TABLE of SQUARE MEASURE.

| Square inch. | Sq. feet         |                 |          |         |       |         |
|--------------|------------------|-----------------|----------|---------|-------|---------|
| 144          | 1                | S. yards        |          |         |       |         |
| 1296         | 9                | 1               | S. poles |         |       |         |
| 39204        | $272\frac{1}{4}$ | $30\frac{1}{4}$ | 1        | S. cha. |       |         |
| 627264       | 4356             | 484             | 16       | 1       | Acres |         |
| 6272640      | 43560            | 4840            | 160      | 10      | 1     | S. mile |
| 4014489600   | 27878400         | 3097600         | 102400   | 6400    | 640   | 1       |

P R O-

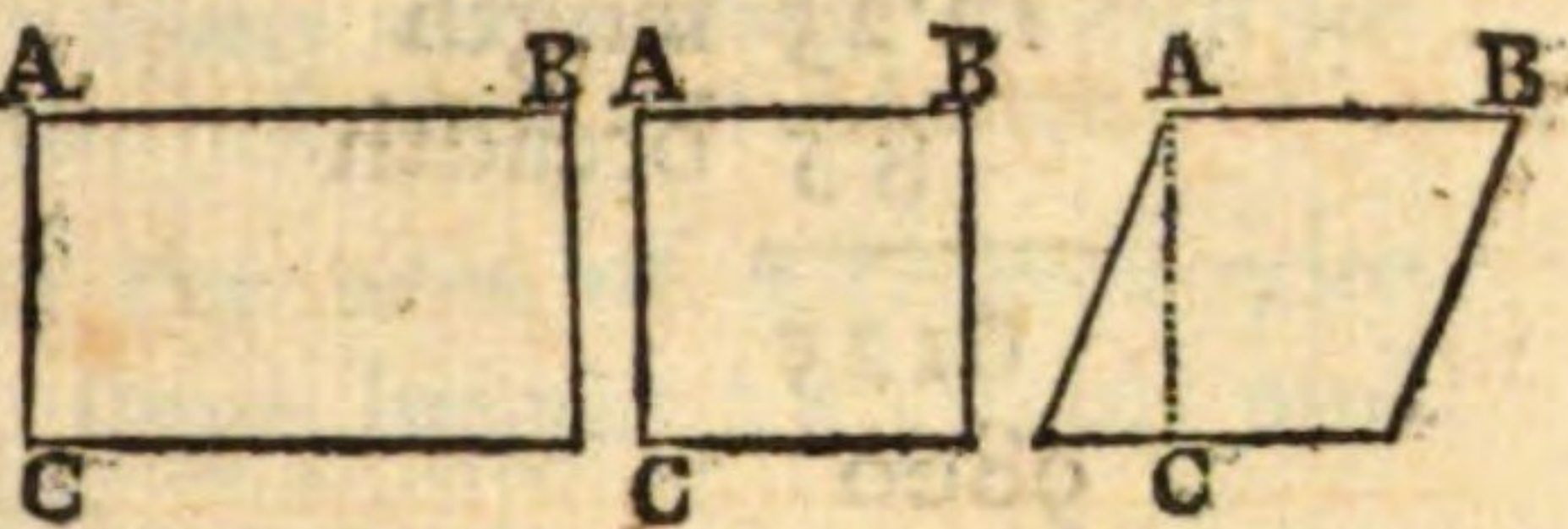


## PROBLEM I.

To find the area of a parallelogram, whether it be a square, a rectangle, a rhombus, or a rhomboides.

## RULE I.

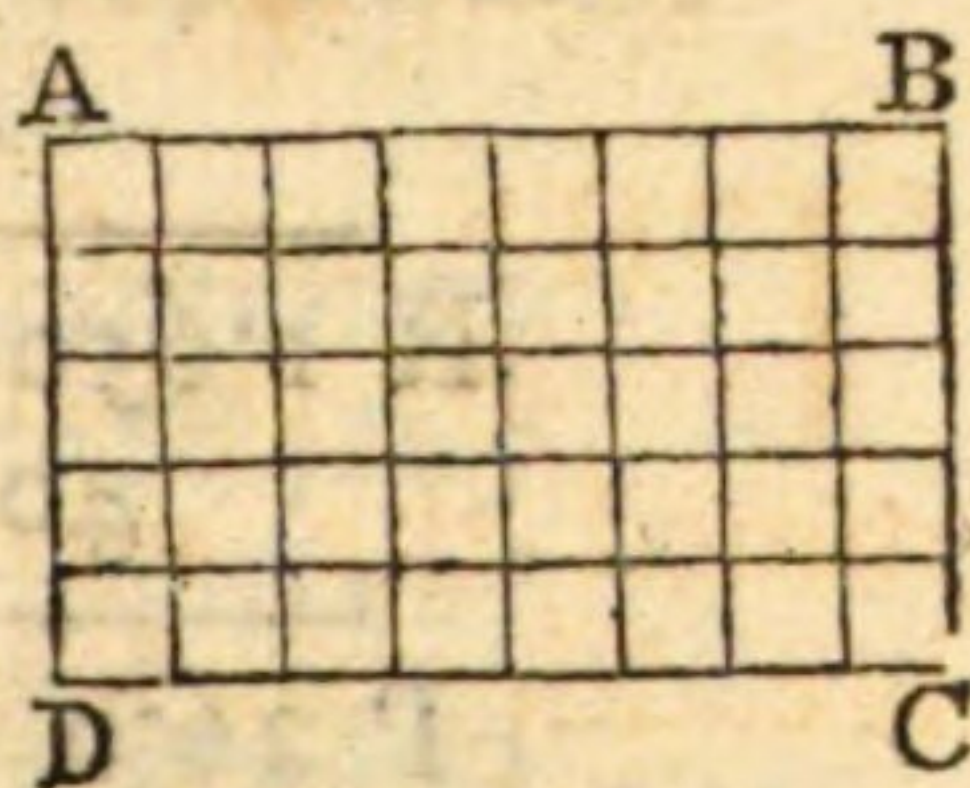
\* Multiply the length  $AB$  by the height or perpendicular breadth, and the product will be the area.



That is,  $AB \times AC = \text{the area.}$

Note.

\* For let  $ABCD$  be a rectangle, and let its length  $AB$  and  $CD$ , and its breadth  $AD$  and  $BC$ , be each divided into so many equal parts as is expressed by the number of times they contain the linear measuring unit; and let all the opposite points of division be connected by right lines.--Then, it is evident that, these lines divide the rectangle into a number of squares each equal to the superficial measuring unit, and that the number of these squares is equal to the number of linear measuring units in the length so often repeated as there are linear measuring units in the breadth or height, that is equal to the length drawn into the breadth. But the area is equal to the number of squares or superficial measuring units; and therefore the area of a rectangle is equal to the product of its length and breadth.



Again, a rectangle is equal to an oblique parallelogram of an equal length and perpendicular height; that is, parallelograms of equal bases and between the same parallels are equal.



For let  $EG$  be the right parallelogram, and  $GI$  the oblique one. Then because of the equal sides  $HI$  and  $GK$ ,  $HE$  and  $GF$ ,  $EI$  and  $FK$  (for  $EI = EF - FI = IK - FI = FK$ ), the triangles  $EHI$ ,  $FGK$  will be congruous and equal in every respect; and therefore, if to each be added the common space  $HIFG$ , the sums  $EFGH$ ,  $HIKG$ , also will be equal. But the former is equal to the product of its length and breadth, which are equal to the length and perpendicular height of the latter.

Wherefore the area of every parallelogram is equal to the product of its length and height. Q.E.D.



*Note.* Because that the length of a square is equal to its height, its area will be found by multiplying its side by itself.—That is  $AB \times AB$  or  $AB^2$  is the area of the square.

EXAMPLE I. What is the area of a parallelogram whose length is 12.25 chains, and its height 8.5 chains?

12.25 length  
8.5 breadth

6125  
9800

10)104.125 square chains

A 10.4125 Acres

4

A. Ro. Per.

Anf. 10 1 26

R 1.6500 Roods

40

P 26.0 Perches

*Note.* 4 Roods are equal to an acre, and therefore 40 Perches or square Poles make a rood.

EXAMPLE II. What is the area of a square whose side is 35.25 chains? Anf. 124 ac. 1r. 1 p.

EXAMPLE III. What is the area of a rectangular board, whose length is 12.5 feet and breadth 9 inches? Anf. 9.375 sq. feet.

EXAMPLE IV. How many square yards of painting are in a rhombus or a rhomboides whose length is 37 feet and perpendicular breadth 5.25 feet?

Anf. 21.58 $\frac{1}{2}$ .

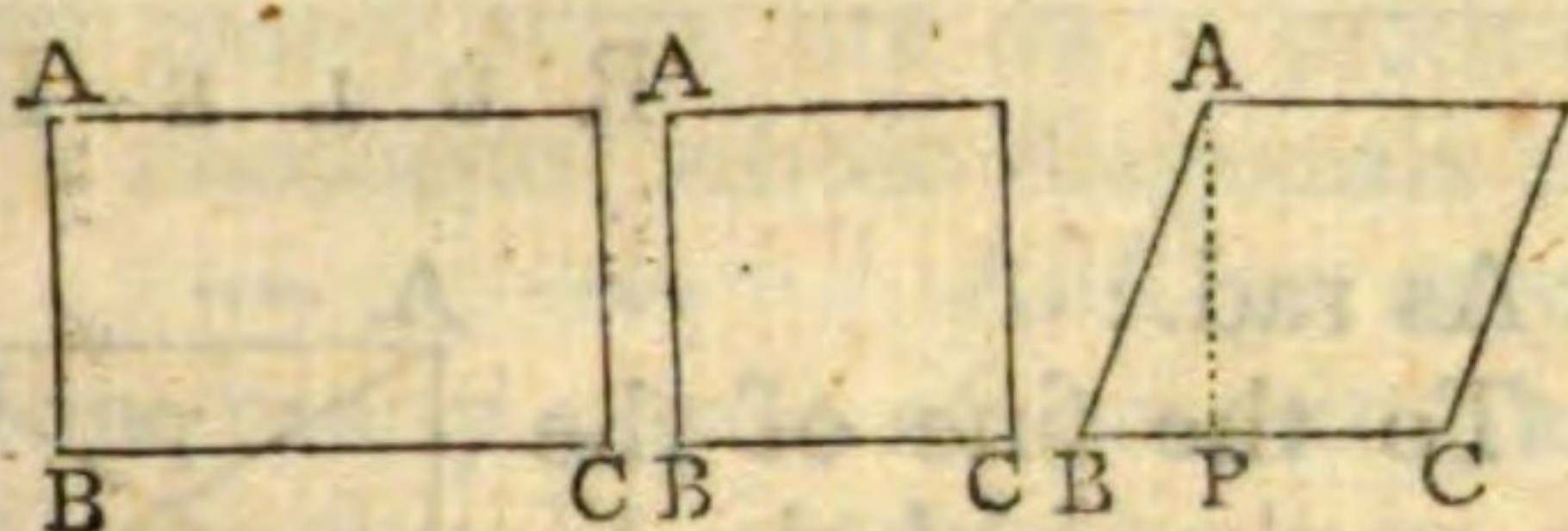
#### R U L E II.

As radius (viz. sine of 90° or tang. of 45°):  
Is to the sine of any angle of a parallelogram ::  
So is the product of the sides including the angle:  
To the area of the parallelogram.

That



\* That is,  $AB \times BC \times \text{nat. fine of the angle } B = \text{the area.}$



*Note.* Because the angles of a square and rectangle are each  $90^\circ$  whose fine is 1, this rule, for them, is the same as the former.

EXAMPLE I. What is the area of a rhomboides whose length is 36 feet, flope height 25.5 feet, and one of the less angles  $58^\circ$ ?

(Rad.)  $1 : .8480481$  (nat. fine of  $58^\circ$ ) ::  $918 (= 25.5 \times 36) : 778.5081558$  the area.

*Or, to use the logarithms.*

|                    |   |   |   |             |
|--------------------|---|---|---|-------------|
| Rad.               | — | — | — | 10.00000000 |
| Sine of $58^\circ$ | — | — | — | 9.9284205   |
| 918                | — | — | — | 2.9628427   |
| 778.5081           | — | — | — | 2.8912632   |

EXAMPLE II. What is the area of a parallelogram whose angle is  $90^\circ$ , and the including sides 20 and 12.25 chains? — — — Ans. 245 acres.

EXAMPLE III. What is the area of a rhombus each of whose sides is 21 feet 3 inches, and each of the less angles  $53^\circ 20'$ ? — — — Ans. 362.208757 feet.

EXAMPLE IV. How many acres are in a rhomboides whose less angle is  $30^\circ$ , and the including sides 25.35 and 10.4 chains? — — — Ans. 13 ac. 29.12 per.

Q

R U L E

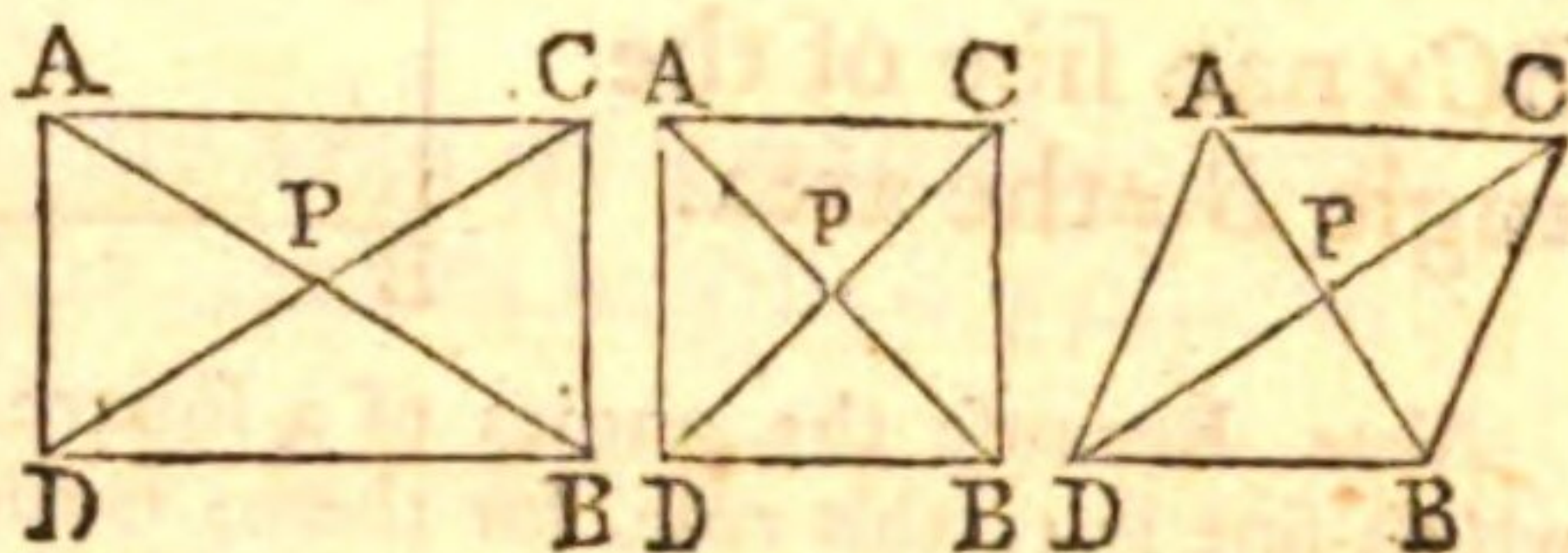
\* For, having drawn the perpendicular  $AP$ , the area, by the first rule, is  $AP \times BC$ ; but as rad. 1 ( $s. L P$ ) :  $s. L B :: BA : AP = s. L B \times BA$ ; therefore  $AP \times BC = CB \times s. L B \times BA$ , is the area; or  $1 : s. L B :: AB \times BC : s. L B \times AB \times BC = \text{the area of the parallelogram. Q.E.D.}$



## R U L E III.

\* As rad.:

To the sine of the  
angle which the di-  
agonals of a paral-  
lelogram make with  
each other::



So is  $\frac{1}{2}$  the product of the diagonals:  
To the area.

That is,  $\frac{AB \times CD \times \text{nat. s. } \angle P}{2} = \text{the area.}$

*Note.* Because the diagonals of a square and rhombus intersect at a right angle, whose sine is 1, therefore half the product of their diagonals is the area.

That is,  $\frac{1}{2} AB^2$  in the square and  $\frac{1}{2} AB \times CD$  in the rhombus, is the area.

EXAMPLE I. How many square yards of pavement are in a square whose diagonal is 27 feet 6 inches?

$$\begin{array}{r}
 27.5 \\
 27.5 \\
 \hline
 1375 \\
 1925 \\
 550 \\
 \hline
 2)756.25 \\
 9)378.125 \text{ feet} \\
 \hline
 42.013\frac{8}{9} \text{ yards}
 \end{array}$$

## EXAMPLE

\* This rule is common to all quadrilaterals, and is proved at case 2 of prob. 3.



EXAMPLE II. How many acres are in a piece of land, in the form of a rhombus, whose diagonals are 30 and 20 chains? — — — Anf. 30 acres.

EXAMPLE III. How many yards of painting are in a rectangle whose diagonals, intersecting in an angle of  $30^\circ$ , are each 32 feet? — — — Anf.  $28\frac{4}{5}$ .

EXAMPLE IV. What is the area of a rhomboides whose diagonals, making an angle of  $60^\circ$ , are 30 and 25 feet? — — — Anf. 324.7595 feet.

### PROBLEM II.

*To find the area of a triangle.*

#### \* RULE I.

Multiply one of its sides by the perpendicular let fall upon it from its opposite angle, and half the product will be the area.

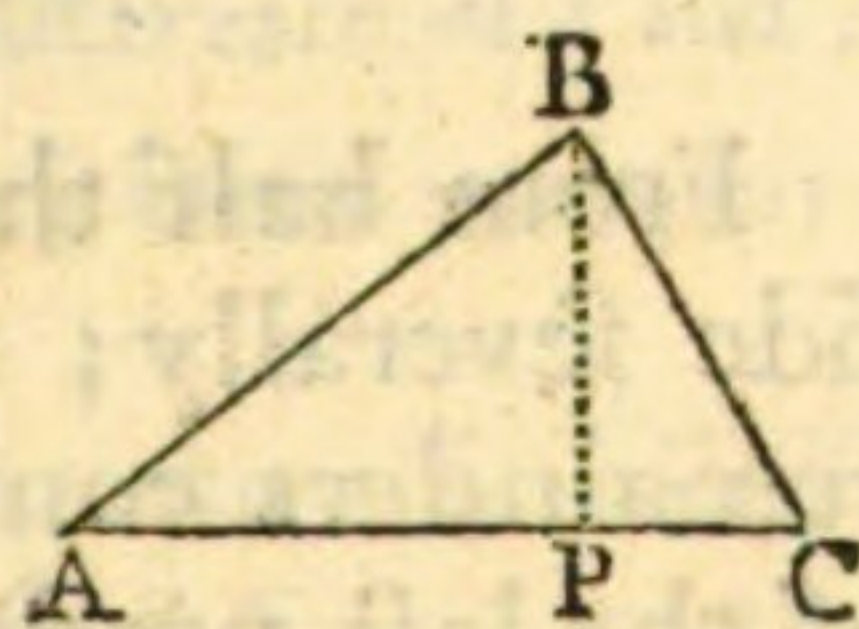
That is,  $\frac{AC \times BP}{2} = \text{the area.}$

EXAMPLE I. What is the area of a triangle whose three sides are 20, 30, 40 chains?

First, by the third prob. in trigonometry,

$$AC = 40 : AB + BC = 50 :: AB - BC = 10 : AP - PC = \frac{10 \times 50}{40} = \frac{50}{4} = 12\frac{1}{2}.$$

$$\text{And } \frac{40 + 12\frac{1}{2}}{2} = \frac{52\frac{1}{2}}{2} = 26\frac{1}{4} = AP.$$



Then, because that the sum of the squares of the base and perpendicular of a right-angled triangle ( $APB$ )

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\* This follows from rule 1 prob. 1, because that a triangle is, evidently, half a parallelogram of the same base and height.



( $APB$ ) is equal to the square of the hypotenuse, we have  $BP = \sqrt{AB^2 - AP^2} = \sqrt{900 - 689.0625} = \sqrt{210.9375} = 14.52368$  the perpendicular.

Wherefore, by the rule,  $\frac{40 \times 14.52368}{2} = 290.4736$  square chains, or 29.04736 acres

$$\begin{array}{r} 4 \\ \cdot 18944 \\ 40 \\ \hline \end{array}$$

Anf. 29 ac. 7 per.

$$7.57760$$

EXAMPLE II. How many square feet are in a right-angled triangle whose base is 40 and perpendicular 30 feet? — — — — — Anf. 600.

EXAMPLE III. How many square yards are in a triangle whose base is 49 feet, and perpendicular  $25\frac{1}{4}$  feet? — — — — — Anf.  $68\frac{736}{9}$ .

EXAMPLE IV. How many acres are in a triangular field of which two sides are 25 and 21.25 chains, and their included angle  $45^\circ$ ? — Anf. 18 a. 3 r. 5 p.

### \* RULE II.

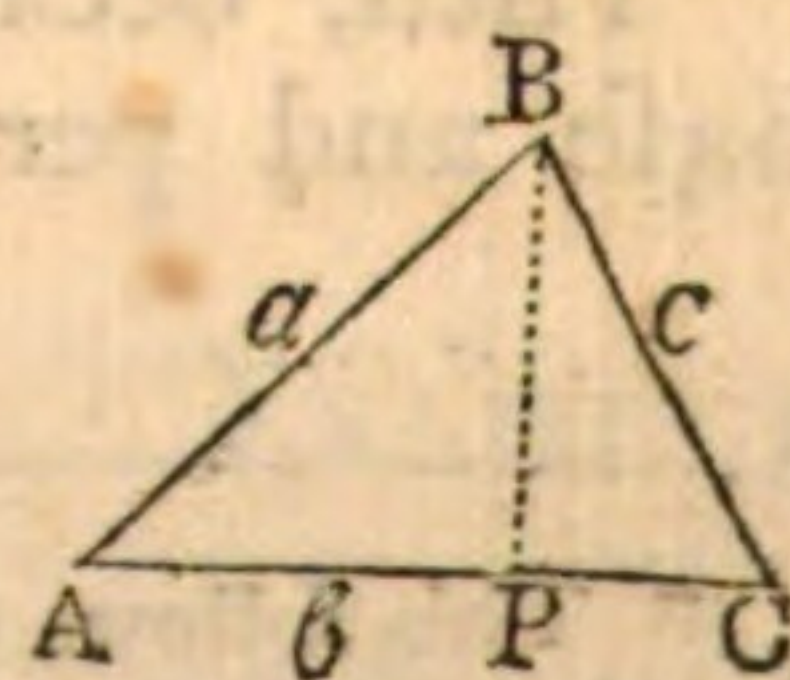
From half the sum of the three sides subtract each side severally; multiply the half sum and the three remainders continually together, and the square root of the last product will be the area of the triangle.

That

\* For,  $b : a+c :: a-c : \frac{aa-cc}{b} = AP-PC$ ; therefore  $\frac{1}{2}b + \frac{aa-cc}{2b} = \frac{bb+aa-cc}{2b} = AP$ : hence

$$\sqrt{aa - \frac{bb+aa-cc}{2b}} =$$

$$= \sqrt$$





That is,  $\sqrt{\frac{a+b+c}{2} \times \frac{a+b+c}{2} - a \times \frac{a+b+c}{2} - b \times \frac{a+b+c}{2} - c}$  or  
 $\sqrt{\frac{a+b+c}{2} \times \frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2}} = \text{the area.}$

Or, if  $s$  be half the sum of the sides, then

$\sqrt{s \times s - a \times s - b \times s - c}$  is the area.

R

EXAMPLE

$$\begin{aligned} \sqrt{\frac{2a^2b^2 - b^4 + 2b^2c^2 - a^4 + 2a^2c^2 - c^4}{4bb}} &= BP : \text{then } BP \times \frac{1}{2} AC = BP \times \\ \frac{1}{2} b &= \sqrt{\frac{2a^2b^2 - b^4 + 2b^2c^2 - a^4 + 2a^2c^2 - c^4}{16}} = \\ \dagger \sqrt{\frac{-aa + bb + cc + 2bc}{4} \times \frac{aa - bb - cc + 2bc}{4}} &= \\ \sqrt{\frac{a+b+c}{2} \times \frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2}} &= \\ \sqrt{\frac{a+b+c}{2} \times \frac{a+b+c}{2} - a \times \frac{a+b+c}{2} - b \times \frac{a+b+c}{2} - c} &= \text{the area.} \end{aligned}$$

Q. E. D.

Cor. 1. The expression  $\dagger$  is also  $= \frac{1}{4} \sqrt{(b+c)^2 - a^2 \times a^2 - b \times c)^2} =$   
 (putting  $s = b+c$  the sum, and  $d = b \times c$  the difference of  $b$  and  $c$ )  $\frac{1}{4}$   
 $\sqrt{ss - aa \times aa - dd}$ ; which is another rule, very useful on many oc-  
 casions.

Cor. 2. If all the sides be equal, the rule will become  $\sqrt{\frac{1}{2}a \times \frac{1}{2}a \times \frac{1}{2}a \times \frac{1}{2}a} =$   
 $\frac{1}{4} aa \sqrt{3}$  for the equilateral triangle whose side is  $a$ .

Cor. 3. If the triangle be right-angled,  $a$  being the hypotenuse,  
 the rule will become  $\frac{a+b+c}{2} \times \frac{-a+b+c}{2}$  or  $\frac{1}{2} p \times \frac{1}{2} p - a$ , putting  $p$  for

the perimeter. For the two quantities  $\frac{a+b+c}{2} \times \frac{-a+b+c}{2}$  and  
 $\frac{a-b+c}{2} \times \frac{a+b-c}{2}$  or  $\frac{-aa + bb + cc + 2bc}{4}$  and  $\frac{aa - bb - cc + 2bc}{4}$  are then  
 $=$  one another, being each  $= \frac{1}{2} bc$ , because  $aa$  is  $= bb + cc$ .



EXAMPLE I. What is the area of a triangle whose three sides are 20, 30, 40 chains?

|              |              |                |                      |
|--------------|--------------|----------------|----------------------|
| 40           | 45           | 45             |                      |
| 30           | 30           | 20             |                      |
| 20           | 15 sec. dif. | 25 third dif.  |                      |
| 2)90         |              | 15 second dif. |                      |
| 45 half sum  |              | 375            |                      |
| 40           |              | 5 first dif.   |                      |
| 5 first dif. |              | 1875           |                      |
|              |              | 45 half sum    |                      |
|              |              | 9375           |                      |
|              |              | 7500           |                      |
|              |              | 84375          | (290.4737 sq. chains |
|              |              | 4              | or 29.04737 acres    |
| 49   443     |              | 4              |                      |
| 9   441      |              | .18948         |                      |
| 5804   27500 |              | 40             |                      |
| 4   23216    |              | 7.5792         |                      |
| 5808   4284  |              |                |                      |
|              | 4066         |                |                      |
|              | 218          |                |                      |
|              | 174          |                |                      |
|              | 44           |                | Ans. 29 ac. 7 per.   |

EXAMPLE II. How many square yards of plaistering are in a triangle whose sides are 30, 40, 50 feet?

Ans.  $66\frac{2}{3}$ .

Ex-



EXAMPLE III. How many acres are in a triangle whose sides are 49, 50.25, 25.69 chains?

Ans. 61 ac. 3 r. 36.8064 p.

EXAMPLE IV. How many square feet are in a triangle of which one angle is  $45^\circ$ , and its including sides 25 and  $21\frac{1}{4}$  feet? — — Ans. 187.82524.

\* RULE III.

As radius:

To the sine of any angle of a triangle::

So is half the product of the sides including the angle:

To the area of the triangle.

That is,  $\frac{AB \times AC \times \text{nat. s. of } \angle A}{2} = \text{the area}$  (See the last figure.)

EXAMPLE I.

What is the area of a triangle whose three sides are 20, 30, 40 chains?

First, by the third prob. of trigonometry,  $AC=40$ :  
 $AB + BC = 50 :: AB - BC = 10 :: \frac{10 \times 50}{40} = \frac{50}{4} = 12\frac{1}{2} = AP - PC$ , and  $AP = \frac{40 + 12\frac{1}{2}}{2} = \frac{52\frac{1}{2}}{2} = 26\frac{1}{4}$ . Whence  $BP = \sqrt{AB^2 - AP^2} = \sqrt{900 - 689.0625} = 14.52368$ .

Ex.

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\* This follows from rule 2 prob. 1, because that a triangle is half a parallelogram of the same base and height.

Corol. 1. Similar  $\Delta$ s are as the squares of their like sides: for let  $A, B$ , be two sides of one  $\Delta$ , and  $a, b$ , two like sides of another similar to it; then because the included  $\angle$ s are equal, the  $\Delta$ s will be as  $A \times B$  to  $a \times b$ ; but  $\left\{ \begin{array}{l} A:a :: B:b \\ \& A:a :: A:a \end{array} \right\}$ , by mult.  $A \times B : a \times b :: A^2 : a^2$ .

Corol. 2. All similar figures are as the squares of their like sides. For they are composed of an equal number of similar  $\Delta$ s, alike placed.



Again, by the first prob. of trigonometry,  $AB = 30:1 = \text{fine } \angle P = 90^\circ :: BP = 14.52368 : \frac{14.52368}{30} = \text{fine of the angle } A$ .

Therefore, by the rule,  $\frac{30 \times 40}{2} \times \frac{14.52368}{30} = 290.4736$  chains = 29.04736 acres.

EXAMPLE II. How many square yards are in a triangle of which one angle is  $45^\circ$  and its including sides 25 and  $21\frac{1}{4}$  feet? — — Anf. 20.86947.

EXAMPLE III. How many acres are in a right-angled triangle whose base is 40 and perpendicular 30 chains? — — — — Anf. 60.

EXAMPLE IV. How many square feet of glazing are in a triangle whose three sides are 25.69, 49, 50.25 feet? — — — — Anf. 619.8004.

### PROBLEM III.

*To find the area of a trapezium.*

### GENERAL RULE.

Divide it into triangles according to the manner which you judge most convenient; then the sum of the areas of the triangles, calculated by the last problem, will be the area of the trapezium.

Observe, also, the following particular rules.

### \* RULE I.

If the trapezium be divided into two triangles by a diagonal joining two opposite angles, and perpendiculars

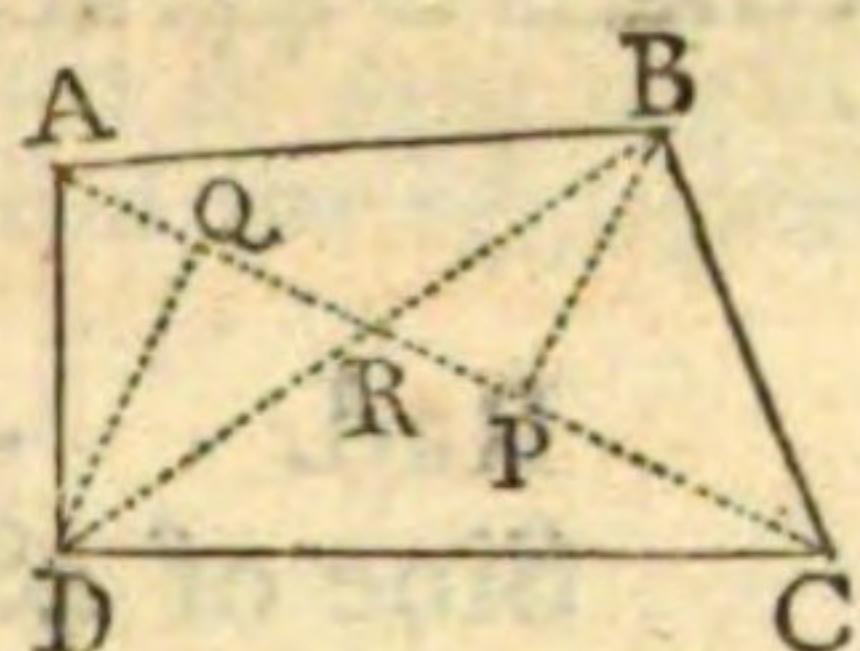
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$$* \text{ For the trapezium } = \triangle ABC + \triangle CDA = \frac{AC \times BP}{2} + \frac{AC \times DQ}{2} = \frac{BP + DQ}{2} \times AC. \quad Q.E.D.$$



diculars be drawn to it from the other angles: Then the sum of the perpendiculars being multiplied by the diagonal, half the product will be the area.

That is,  $\frac{BP+DQ}{2} \times AC = \text{the area.}$



## E X A M P L E.

If the diagonal  $AC$  be 20 feet, and the perpendiculars  $BP$  equal 4.2 feet, and  $DQ$  equal 3.8 feet.

Then  $\frac{4.2+3.8}{2} \times 20 = 8 \times 10 = 80$  square feet, the area.

## \* R U L E II.

If there be drawn two diagonals cutting each other: The product of the diagonals multiplied by half the natural sine of the angle of intersection of the diagonals will be the area. And this rule, as well as the former, is common to all quadrilaterals.

That is,  $\frac{AC \times BD \times \text{nat. s. } \angle R}{2} = \text{the area.}$

Or, as radius: s.  $\angle R$  ::  $\frac{1}{2} AC \times BD$ : the area.

## E X A M P L E.

Suppose that the two diagonals are 40 and 30 chains, and that at their intersection one of the less angles is  $48^\circ$ .

S

Then,

## \* DEMONSTRATION.

For the trapez. = the four  $\Delta$ s  $ARB, BRC, CRD, DRA$ , =  
 $\frac{AR \times RB + BR \times RC + CR \times RD + DR \times RA}{2} \times \text{s. } \angle R =$   
 $\frac{AR + RC}{2} \times \frac{BR + RD}{2} \times \text{s. } \angle R = \frac{AC \times BD}{4} \times \text{s. } \angle R = \frac{AC \times BD \times \text{s. } \angle R}{2} \quad \text{Q.E.D.}$



Then, since the nat. sine of  $48^\circ$  is  $\cdot 7431448$ , the  
 $\text{area} = \frac{40 \times 30 \times \cdot 7431448}{2} = 600 \times \cdot 7431448 = 445\cdot 88688 \text{ sq.}$   
 $\text{chains} = 44. \text{ ac. } 2 \text{ r. } 14\cdot 19008 \text{ p.}$

*By the logarithms.*

|                                         |   |   |   |             |
|-----------------------------------------|---|---|---|-------------|
| Rad.                                    | — | — | — | 10.00000000 |
| Sine of $48^\circ$                      | — | — | — | 9.8710735   |
| $\frac{40 \times 30}{2} = 600$          | — | — | — | 2.7781512   |
| Area $445\cdot 8869 \text{ sq. chains}$ | — | — | — | 2.6492247   |

\* R U L E III.

Square each side of the trapezium, add the squares of each pair of opposite sides together, subtract the greater sum from the less, and multiply the difference into one-fourth of the tangent of the angle formed by the diagonals, and the product will be the area.

That

\* D E M O N S T R A T I O N.

For call  $AR, m$ ;  $RB, n$ ;  $CR, p$ ;  $RD, q$ ;  $s, c, t$ , the sine, cosine, and tan.  $\angle R$ . (See the last fig.)

Then, by 12. and 13. II. *Eucl.*

$$BA^2 = mm + nn + 2mnc$$

$$AD^2 = mm + qq - 2mqc$$

$$BC^2 = nn + pp - 2npc$$

$$CD^2 = qq + pp + 2pqc$$

And by add. and subtr.  $AB^2 + CD^2 - DA^2 - BC^2 = mn + pq + mq + np$   
 $\times 2c = \overline{m+p} \times \overline{n+q} \times 2c = AC \times BD \times 2c.$

But  $AC \times BD \times \frac{1}{2} s =$  the area by the last rule.

$$\text{Theref. } AB^2 + CD^2 - DA^2 - BC^2 = \frac{4c}{s} \times \text{area} = \frac{4}{t} \times \text{area}.$$

$$\text{And } \overline{AB^2 + CD^2 - DA^2 - BC^2} \times \frac{1}{4} t = \text{area. } \text{Q.E.D.}$$

COROLLARY. It appears from the demonstration, that the sum of the squares is greater in that pair of opposite sides which subtend the greater or obtuse angles at the intersection  $R$ ; and, of consequence, when all the angles at  $R$  are equal to each other, that is, when the diagonals intersect at right angles, the sums of the squares of both pairs of opposite sides will be equal to each other.



That is,  $\overline{AB^2 + CD^2} \propto \overline{DA^2 + BC^2} \times \frac{1}{4} \tan. LR.$

Or As radius :  $\tan. LR :: \frac{\overline{AB^2 + CD^2} \propto \overline{DA^2 + BC^2}}{4} : \text{area.}$

*Note.* This rule fails when the diagonals intersect at right angles, for then the tangent is infinite and the difference of the aggregates of the squares is nothing.

### EXAMPLE.

If the sides be  $AB=10$ ,  $BC=9$ ,  $CD=8$ ,  $DA=7$  feet; and the angle made by the diagonals equal to  $100^\circ$  degrees; what is the area?

The  $\tan.$  of  $100^\circ$  or of  $80^\circ$  is  $5.6712818$ , therefore  $\overline{AB^2 + CD^2} - \overline{DA^2 + BC^2} \times \frac{1}{4} \tan. 100^\circ$  is  $= 164 - 130 \times \frac{1}{4} \times 5.6712818 = 34 \times 1.41782045 = 48.2058953$  square feet, the area.

Or, by the logarithms thus.

|                      |   |   |            |
|----------------------|---|---|------------|
| Radius               | — | — | 10.0000000 |
| Tan. of $100^\circ$  | — | — | 10.7536812 |
| $\frac{34}{4} = 8.5$ | — | — | 0.9294189  |
| Area = $48.20589$    | — | — | 1.6831001  |

### \* RULE IV.

If the trapezium can be inscribed in a circle, that is if the sum of any two opposite angles be equal to two right angles or  $180^\circ$ ; then, multiply any two adjacent

### \* DEMONSTRATION.

For this product will be equal to the two triangles  $ADC$ ,  $ABC$ , found by rule 3 prob. 2, since the sines of the opposite angles  $D$ ,  $B$ , of a trapezium inscribed in a circle are equal to each other.



adjacent sides together, and the other two sides together, and multiply the sum of these products by half the sine of the angle included by either of the pairs of sides which are multiplied together; so shall this last product be the area.

That is,  $\frac{AD \times DC + AB \times BC \times s. \angle D \text{ or } s. \angle B}{2} = \text{the area.}$  (See the last figure.)

Or As radius:  $s. \angle D$  or  $s. \angle B :: \frac{AD \times DC + AB \times BC}{2} : \text{area.}$

#### EXAMPLE.

If the sides be  $AB$  equal to 7.5,  $BC$  equal to 5.5,  $CD$  equal to 6,  $DA$  equal to 4; and the angle  $B$  equal to  $74^\circ 40\frac{1}{4}'$ ; and, therefore, the angle  $D$ , equal to  $105^\circ 19\frac{3}{4}'$ ; what is the area.

Here  $\frac{4 \times 6 + 7.5 \times 5.5}{2} \times .9644229$  ( $s. \text{ of } 74^\circ 40\frac{1}{4}'$ ) = 31.46429 &c. square feet, the area.

|                                                  |   |   |             |
|--------------------------------------------------|---|---|-------------|
| Or, Radius                                       | — | — | 10.00000000 |
| Sine $\angle B 74^\circ 40\frac{1}{4}'$          | — | — | 9.9842675   |
| $\frac{4 \times 6 + 7.5 \times 5.5}{2} = 32.625$ | — | — | 1.5135505   |
| Area = 31.4643                                   | — | — | 1.4978180   |

#### \* RULE V.

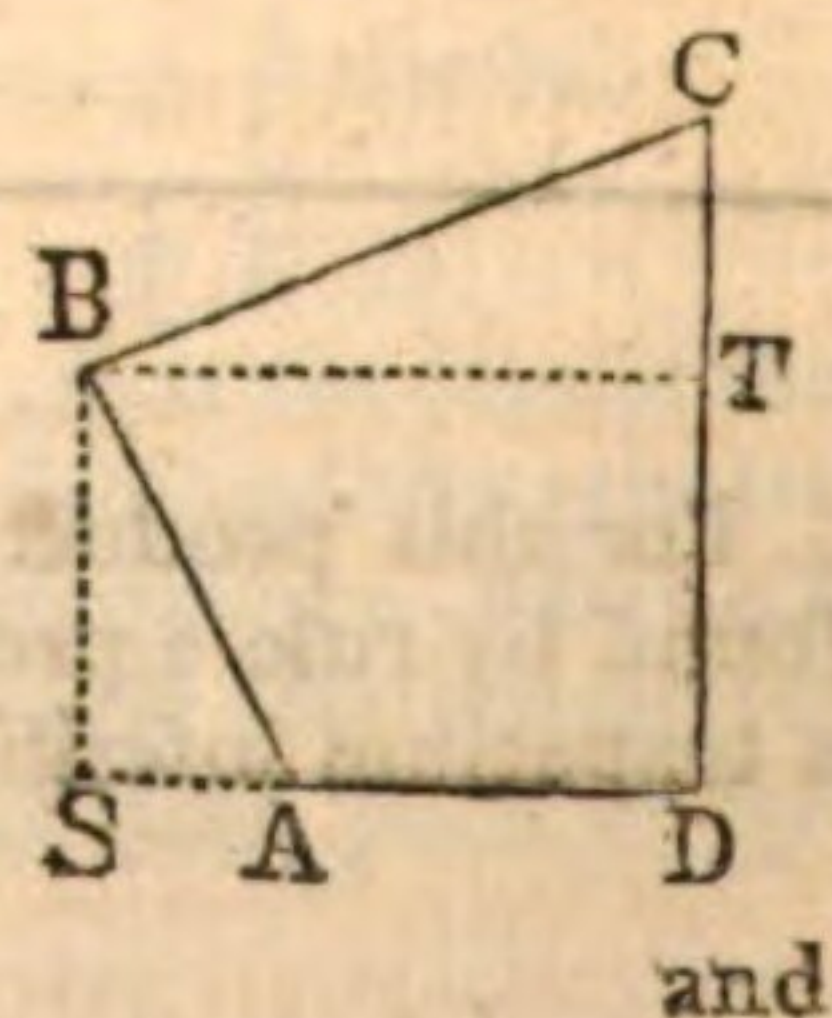
Or, when the trapezium can be inscribed in a circle, the area may be, otherwise, found thus:

From

#### \* DEMONSTRATION.

For, upon the sides  $AD$ ,  $DC$ , let fall the perpendiculars  $BS$ ,  $BT$ ; put  $s =$  the sine of the angle  $A$  or of the angle  $C$ , and  $a, b, c, d =$  the four sides  $AB, BC, CD, DA$  respectively.

Then  $1 : s :: \begin{cases} a : as = BS, \\ b : bs = BT, \end{cases}$





From half the sum of the four sides subtract each side severally; multiply the four remainders continually

$$\text{and } 1 : \sqrt{1-ss} :: \begin{cases} a : a\sqrt{1-ss} = AS, \\ b : b\sqrt{1-ss} = CT, \end{cases}$$

hence  $SD = d + a\sqrt{1-ss}$ , and  $DT = c - b\sqrt{1-ss}$ ; and, by right-angled triangles,  $BS^2 + SD^2 = DT^2 + TB^2$ , that is,  $dd + 2ad\sqrt{1-ss} + aa = cc - 2bc\sqrt{1-ss} + bb$ , and hence  $\sqrt{1-ss} = \frac{bb + cc - aa - dd}{2ad + 2bc}$ , and  $s =$

$\frac{\sqrt{(2ad + 2bc)^2 - bb + cc - aa - dd}}{2ad + 2bc}$ ; then, by the last rule, the area will be  $\frac{1}{4}\sqrt{(2ad + 2bc)^2 - bb + cc - aa - dd} = \frac{1}{4}\sqrt{(b+c)^2 - a-d)^2 \times (a+d)^2 - b-c)^2} = \frac{1}{4}\sqrt{(a+b+c-d) \times (a+b-c+d) \times (a-b+c+d) \times (-a+b+c+d)} = \sqrt{s-a} \times s-b \times s-c \times s-d$ , if  $s$  be half the sum of the four sides. Q. E. D.

## COROLLARY I.

The expression marked † is a very pretty theorem, and may be useful on many occasions.

## COROLLARY II.

Hence may be deduced rule 2 for the triangle; for, if we here suppose one of the sides, as  $d$ , to be nothing, or to decrease till it vanish, the rule will become  $\sqrt{s-a} \times s-b \times s-c \times s$ , the same as in the triangle.

## COROLLARY III.

If, in the trapezium,  $a=d$ , and  $b=c$ , the rule will be barely  $ab$ .

## COROLLARY IV.

When all the four sides are equal, the rule becomes  $\sqrt{axaxaxa} = aa$ .

## COROLLARY V.

If the sides be in arithmetical proportion, either continued or discontinued; the square root of the continual product of all the four sides will be the area. Or, take a mean proportional between any two sides, and also a mean proportional between the other two sides; and the figure will be equal to the rectangle of these mean proportionals. For, half the sum of the sides is equal to the sum of the extremes, or equal to the sum of the means; and each extreme and mean being taken from it, must leave the other extreme and mean; and consequently the four remainders will be equal to the four given sides. Wherefore, &c. That is,  $\sqrt{abcd} = \text{the area}$ .



nually together, and the square root of the last product will be the area.

That is,  $\sqrt{\frac{a+b+c-d}{2} \times \frac{a+b-c+d}{2} \times \frac{a-b+c+d}{2} \times \frac{-a+b+c+d}{2}}$   
 = the area. Or if  $s$  be half the sum of the sides, then  
 $\sqrt{s-a \times s-b \times s-c \times s-d}$  is the area.

The letters  $a, b, c, d$ , denoting the four sides of the trapezium.

#### EXAMPLE.

The four sides of a trapezium inscribed in a circle, are 6, 5.5, 7.5, 4 feet; required the area.

$$\sqrt{\frac{6+5.5+7.5+4}{2} \times \frac{6+5.5-7.5+4}{2} \times \frac{6-5.5+7.5+4}{2} \times \frac{-6+5.5+7.5+4}{2}}$$

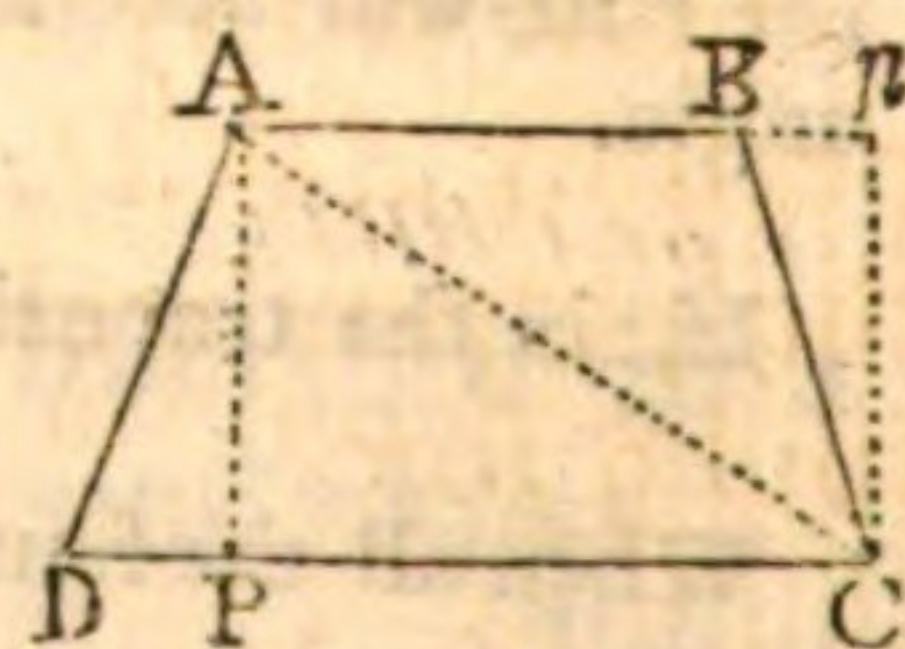
$$= \frac{1}{4} \sqrt{15 \times 8 \times 12 \times 11} = \frac{1}{4} \sqrt{15840} = \frac{125.85708}{4} = 31.46427$$

square feet the area.

#### RULE VI.

In a trapezoid, multiply the sum of the parallel sides by the distance between them, and half the product will be the area.\*

That is, If  $AB, DC$  be the parallel sides, and  $AP$  perpendicular to  $DC$ ; then  $\frac{AB+CD}{2}$  will be the area.



*Note.* This case is very useful in surveying; for many irregular pieces of land may be, very conveniently, divided into a number of trapezoids, of that kind in which the parallel sides are perpendicular to one of the other sides, which, in that case, is the distance of the parallel sides.

Ex-

\* For, the diagonal  $AC$  divides the trapezoid into two triangles whose bases are the two parallel sides, and heights each equal to the distance  $AP$ ; therefore, &c.



## EXAMPLE.

If  $AB$  be 7.5,  $CD$  12.25, and  $AP$  15.4 chains; what is the area?

$$\begin{array}{r}
 12.25 \\
 7.5 \\
 \hline
 19.75 \\
 15.4 \\
 \hline
 7900 \\
 9875 \\
 \hline
 1975 \\
 2) 304.150 \quad \text{Ans. 15 ac. 33.2 per.} \\
 15,2075 \\
 \hline
 4 \\
 \cdot 8300 \\
 40 \\
 \hline
 33.2000
 \end{array}$$

Some PROMISCUOUS EXAMPLES for PRACTICE.

## EXAMPLE I.

How many square yards of paving are in the trapezium  $ABCD$ , whose diagonal  $AC$  is 65 feet, and the perpendiculars  $BP$  equal to 33.5, and  $DQ$  equal to 28 feet?

By rule 1.

$$\begin{array}{r}
 33.5 \\
 28 \\
 \hline
 61.5 \\
 65 \\
 \hline
 3075 \\
 3690 \\
 \hline
 2) 3997.5 \\
 9) 1998.75 \text{ feet} \\
 \text{Ans. } 222.08\frac{1}{3} \text{ yards.}
 \end{array}$$

Ex-



## EXAMPLE II.

What is the area of a trapezium whose south side is 27.4 chains, east side 35.75 chains, north side 37.55 chains, and west side 41.05 chains; also the diagonal from south-west to north-east 48.35 chains?

By rule 2 of prob. 2,

$$\frac{27.4 + 35.75 + 48.35}{2} = \frac{111.5}{2} = 55.75 = \text{half the sum of the}$$

sides of the south-east triangle. And  $\frac{37.55 + 41.05 + 48.35}{2}$

$$= \frac{126.95}{2} = 63.475 = \text{half the sum of the sides of the north-west triangle. Whence}$$

$$\begin{aligned} &\sqrt{55.75 \times 55.75 - 27.4 \times 55.75 - 35.75 \times 55.75 - 48.35} = \\ &\sqrt{55.75 \times 28.35 \times 20 \times 7.4} = \sqrt{233915.85} = 483.6485, \text{ the} \\ &\text{area of the former triangle. And} \end{aligned}$$

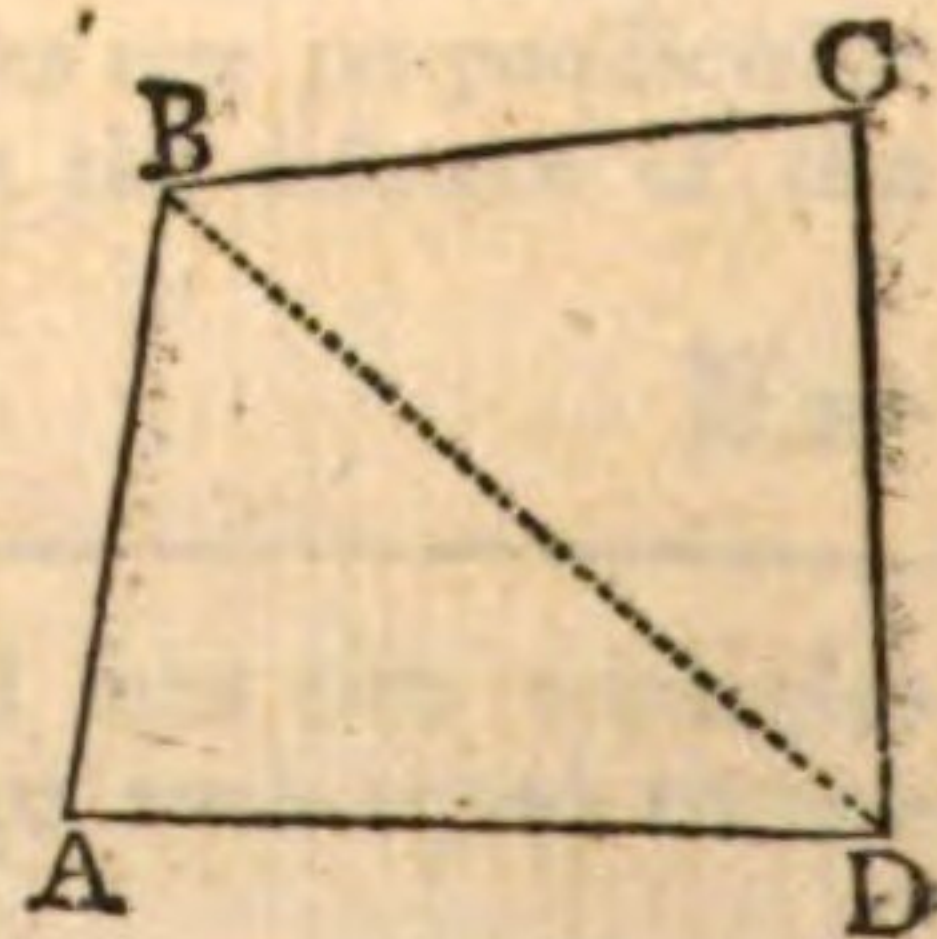
$$\begin{aligned} &\sqrt{63.475 \times 63.475 - 37.55 \times 63.475 - 41.05 \times 63.475 - 48.35} = \\ &= \sqrt{63.475 \times 25.925 \times 22.425 \times 15.125} = \sqrt{558148.1 \&c.} \\ &= 747.0932 \text{ the area of the latter. And their sum is} \\ &1230.7417 \text{ square chains} = 123 \text{ ac. } 11.8672 \text{ per. the} \\ &\text{area required.} \end{aligned}$$

## EXAMPLE III.

How many square feet are in a trapezium whose sides are  $AB$  equal to 12,  $BC$  equal to 13,  $CD$  equal to 14,  $DA$  equal to 15 feet, and the angle  $A$  equal to  $83^\circ 30'$ ?

First, by rule 3 prob. 2,  $\frac{.9935718 \times 12 \times 15}{2}$   
 $= .9935718 \times 90 = 89.421462 = \text{the}$   
 area of the triangle  $ABD$ .

Again, by trigonometry, As  $27 =$   
 $DA + AB : 3 = DA - AB :: 1.1204053$   
 (tang.





(tang.  $\frac{DBA+ADB}{2} = 48^\circ 15'$ ) :  $\frac{1.1204053}{9} = .1244895 =$   
 tang.  $\frac{DBA-ADB}{2} = 7^\circ 06'$ . Therefore  $48^\circ 15' - 7^\circ 06' = 41^\circ 09' = ADB$ .

And,  $\angle ADB = 41^\circ 09' = 9.8182474$   
 $\angle A = 83^\circ 30' = 9.9971993$   
 $AB = 12 = 1.0791812$   
 $BD = 18.119 = 1.2581331$

Then, by rule 2 prob. 2,

$$\sqrt{\frac{18.119+14+13}{2} \times \frac{18.119+14-13}{2} \times \frac{18.119-14+13}{2} \times \frac{18.119-14-13}{2}} = \sqrt{22.5595 \times 9.5595 \times 8.5595 \times 4.4405} = \sqrt{8196.812}$$

$$= 90.53625 = \text{the area of the triangle } DCB.$$

The sum of these two is  $179.957712 = \text{the area of the trapezium required.}$

#### EXAMPLE IV.

In a quadrangular field the south side is 23.4, the east side 19.75, and the north side 20.5 chains; also the south-east and north-east angles are  $73^\circ$  and  $87^\circ 30'$ : What is the area?

First, by rule 3 prob. 2,

$$\frac{.9990482 \times 19.75 \times 20.5}{2} = .4995241 \times 19.75 \times 20.5 = 202.24482 \text{ the area of the north-east triangle } BCD.$$

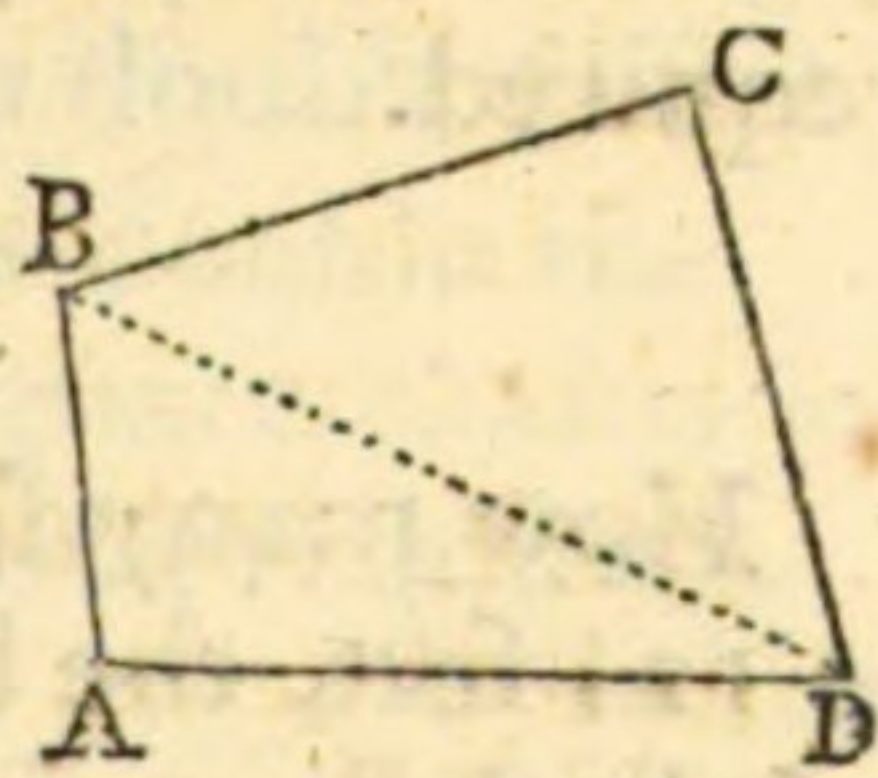
Again,  $40.25 = BC + CD : .75 =$   
 $BC - CD :: 1.0446136 \left( \text{tang. } \frac{BDC + CBD}{2} = 46^\circ 15' \right) :$   
 $\frac{1.0446136 \times 3}{161} = .01946485 = \text{the tang. of } \frac{BDC - CBD}{2} = 1^\circ 07'.$

Wherefore  $\angle BDC = 46^\circ 15' + 1^\circ 07' = 47^\circ 22'.$

And  $\angle ADB = (ADC - CDB = 73^\circ - 47^\circ 22' =) 25^\circ 38'.$

U

But,





|                   |   |                |   |             |
|-------------------|---|----------------|---|-------------|
| But, $\angle BDC$ | — | $47^\circ 22'$ | — | $9.8667026$ |
| $\angle C$        | — | $87^\circ 30'$ | — | $9.9995809$ |
| $CB$              | — | $20.5$         | — | $1.3117539$ |
| $BD$              | — | $27.83763$     | — | $1.4446322$ |

Whence, by rule 3 prob. 2, also,

|                                  |   |                |   |              |
|----------------------------------|---|----------------|---|--------------|
| Radius                           | — | —              | — | $10.0000000$ |
| s. $ADB$                         | — | $25^\circ 38'$ | — | $9.6360969$  |
| $\frac{27.83763 \times 23.4}{2}$ | — | —              | — | $2.5128181$  |

$140.9013$  the area of the  $\triangle ABD$  —  $2.1489150$

Their sum is  $343.1461$  square chains = 34 acres, 1 rood, and  $10.3376$  perches, the area required.

#### EXAMPLE V.

What is the area of a trapezium, inscribed in a circle, whose four sides are 24, 26, 28, 30 yards?

Here  $\frac{24+26+28+30}{2} = \frac{108}{2} = 54$  the half sum of the sides.

Then, by rule 5,  $\sqrt{54-24 \times 54-26 \times 54-28 \times 54-30} = \sqrt{30 \times 28 \times 26 \times 24} = 24 \sqrt{5 \times 7 \times 13 \times 2} = 24 \sqrt{910} = 24 \times 30.1662062 = 723.9889488$  square yards, the area required.

#### EXAMPLE VI.

How many square feet are in a board whose length is  $12\frac{1}{2}$  feet, the breadth of the greater end  $1\frac{1}{4}$  feet, and of the less 11 inches?

By rule 6,

$\frac{1\frac{1}{4} + 1\frac{1}{2}}{2} \times 12\frac{1}{2} = \frac{13}{12} \times 12\frac{1}{2} = 13 \frac{6\frac{1}{2}}{12} = 13.541\frac{2}{3}$  square feet, the area required.

Ex-



## EXAMPLE VII.

In a trapezoid, the length of the two parallel sides are 30, and 46 chains, and the length of one of the other sides, which is perpendicular to the parallel sides, is 60.37 chains: What is the area?

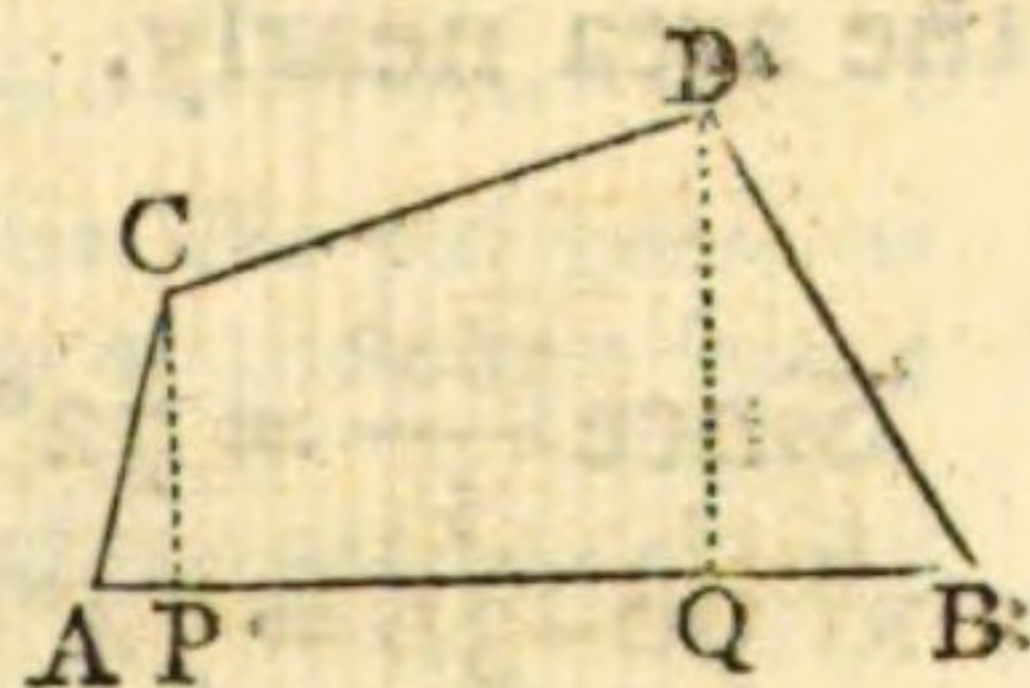
By rule 6,

$$\frac{30+46}{2} \times 60.37 = 38 \times 60.37 = 2294.06 \text{ square chains} \\ = 229 \text{ acres, 1 rood, 24.96 perches, the area required.}$$

## EXAMPLE VIII.

In measuring along one side  $AB$  of a quadrangular field, that side and the two perpendiculars upon it from the opposite corners, measured as in the field book below; required the area.

| FIELD BOOK.  |             |
|--------------|-------------|
| chains       | chains      |
| $AP = 1.10$  | $PC = 3.52$ |
| $AQ = 7.45$  | $QD = 5.95$ |
| $AB = 11.10$ |             |



$$\begin{aligned} AP \times PC &= 1.10 \times 3.52 = 3.872 = \text{double } APC \\ PQ \times PC + QD &= 6.35 \times 9.47 = 60.1345 = \text{double } CPQD \\ DQ \times QB &= 5.95 \times 3.65 = 21.7175 = \text{double } DQB \\ \text{the sum} &= 85.724 = \text{double the} \\ \text{whole figure, whose half} &= 42.862 \text{ square chains} = \\ 4 \text{ acres, 1 rood, 5.792 perches is the area required.} \end{aligned}$$

## PROBLEM IV.

To find the areas of regular figures or polygons.

## RULE I.

Multiply half the perimeter of the figure by the radius of its inscribed circle, or by the perpendicular demitted



demitted from its center to one of the sides, and the product will be the area.\*

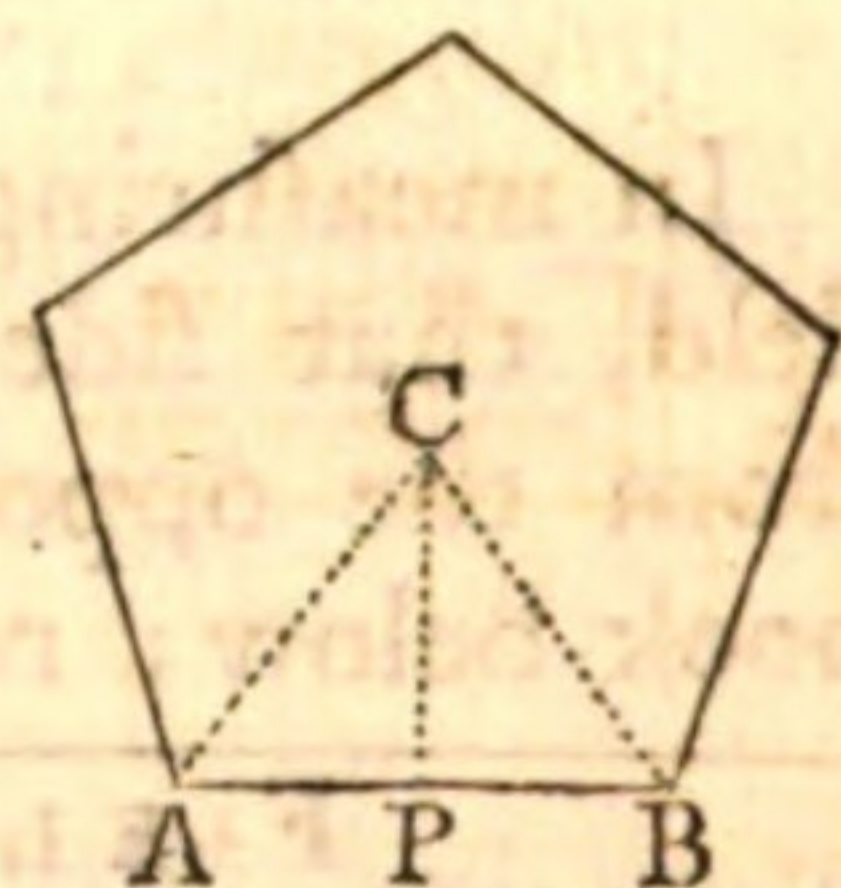
*Note 1.* The perimeter of a figure is the sum of its sides, and in a regular figure is equal to the length of one side multiplied by the number of sides.

2. The center of the figure is the same as the center of the inscribed or circumscribed circle.

#### EXAMPLE.

Required the area of a regular pentagon whose side is 25 yards.

By constructing the figure and measuring the perpendicular  $CP$ , I find it about 17.2 yards, wherefore  $\frac{5 \times 25 \times 17.2}{2} = 1075$  square yards equal the area nearly.



*Otherwise.*

Since  $\frac{360^\circ}{5} = 72^\circ = \angle ACB$ ; hence  $\frac{72^\circ}{2} = 36^\circ = \angle ACP$ , and  $90 - 36 = 54^\circ = \angle CAP$ .

Then, as  $1 = \text{rad.} : 1.3763819 = \text{tang. } 54^\circ :: 12\frac{1}{2} = AP : 1.3763819 \times 12\frac{1}{2} = 17.2047737 = PC$  the perpendicular.

Whence  $\frac{17.2047737 \times 25 \times 5}{2} = 1075.298356$  the area required.

#### RULE II.

Multiply the square of the side, of any regular figure, by the multiplier standing opposite to its

---

\* For, the polygon consisting of so many triangles as it hath sides, whose heights and bases are each equal to the perpendicular and the side of the polygon respectively; if  $s$  be a side of the polygon,  $p$  the perpendicular, and  $n$  the number of sides; then  $\frac{1}{2}ps$  will be one  $\Delta$ , and  $\frac{1}{2}nps$  all the  $\Delta$ s, or the polygon. Q.E.D.



its name, in the following table; and the product will be the area.\*

| N <sup>o</sup> of sides. | Names.                    | Multipliers, or area when the side is 1.                                                |
|--------------------------|---------------------------|-----------------------------------------------------------------------------------------|
| 3                        | Trigon or equil. $\Delta$ | $0.433013 - = \frac{1}{4} \tan. 30^\circ = \frac{1}{4} \sqrt{3}$                        |
| 4                        | Tetragon or square        | $1.000000 = \frac{1}{4} \tan. 45^\circ = 1 \times 1$                                    |
| 5                        | Pentagon                  | $1.720477 + = \frac{5}{4} \tan. 54^\circ = \frac{5}{4} \sqrt{1 + \frac{2}{3} \sqrt{5}}$ |
| 6                        | Hexagon                   | $2.598076 + = \frac{6}{4} \tan. 60^\circ = \frac{3}{2} \sqrt{3}$                        |
| 7                        | Heptagon                  | $3.633912 + = \frac{7}{4} \tan. 64\frac{1}{2}^\circ$                                    |
| 8                        | Octagon                   | $4.828427 + = \frac{8}{4} \tan. 67\frac{1}{2}^\circ = 2 \times 1 + \sqrt{2}$            |
| 9                        | Nonagon                   | $6.181824 + = \frac{9}{4} \tan. 70^\circ$                                               |
| 10                       | Decagon                   | $7.694209 - = \frac{10}{4} \tan. 72^\circ = \frac{5}{2} \sqrt{5 + 2\sqrt{5}}$           |
| 11                       | Undecagon                 | $9.365641 + = \frac{11}{4} \tan. 73\frac{1}{2}^\circ$                                   |
| 12                       | Duodecagon.               | $11.196152 + = \frac{12}{4} \tan. 75^\circ = 3 \times 2 + \sqrt{3}$                     |

X

EX-

\* This rule is founded upon the property that like polygons, as similar figures, are to one another as the squares of their like sides; which is proved in corol. 2 to rule 3 of prob. 2.

Now, the multipliers in the table are the areas of the polygons to the side 1; whence the rule is manifest.

## SCHOLIUM.

The table is formed thus: As radius = 1: tang.  $LCAP = t :: AP: PC = t \times AP = \frac{1}{2}t$ , supposing  $AB = 1$ ; then  $\frac{1}{4}t (= CP \times PA) =$  the  $\Delta ACB$ , and  $\frac{1}{4}nt =$  the polygon; where  $n$  is the number of sides: So that, by finding the tangent of the  $LCAP$ , by the tables of tangents, and multiplying it by the number of sides,  $\frac{1}{4}$  of the product will be the multiplier in the table.

All these multipliers may be derived from other methods, and, indeed, many of them from such as are very simple.

Thus, in the trigon or equilateral triangle,  $\sqrt{1^2 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$  = the perpendicular, and  $\frac{1}{4}\sqrt{3}$  is the area.—In the trigon the angle  $CAP$  is  $30^\circ$ , and therefore the tangent of  $30^\circ$  is  $= \sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$ .

In the tetragon or square, the side drawn into itself,  $1 \times 1 = 1$  is the area.

In the pentagon, the  $LCAP$  is  $54^\circ$ , whose sine is  $\frac{\sqrt{5+1}}{4}$ , and its cosine  $\frac{\sqrt{10-2\sqrt{5}}}{4}$ ; and, because the tangent is equal to the sine divided



## EXAMPLE I.

What is the area of a pentagon whose side is 25 feet?

The

vided by the cosine, we shall have the tangent of  $54^\circ$  or  $t = \frac{\sqrt{5+1}}{\sqrt{10-2\sqrt{5}}} = \sqrt{\frac{6+2\sqrt{5}}{10-2\sqrt{5}}} = \sqrt{1+\frac{2}{3}\sqrt{5}}$ ; wherefore the area of the pentagon is  $\frac{5}{4} \sqrt{1+\frac{2}{3}\sqrt{5}}$ .

In the hexagon, all the triangles are equilateral, and therefore it will be 6 times the trigon, or 6 times  $\frac{3}{4} \sqrt{3} = \frac{6}{4} \sqrt{3} = \frac{3}{2} \sqrt{3}$ .—And since, in it, the  $\angle CAP$  is  $= 60^\circ$ , the tangent of  $60^\circ$  will be  $= \sqrt{3}$ .

In the octagon, the  $\angle CAP$  is  $67\frac{1}{2}^\circ$ , whose sine is  $\frac{1}{2} \sqrt{2+\sqrt{2}}$ , and its cosine  $\frac{1}{2} \sqrt{2-\sqrt{2}}$ ; whence the tang. of  $67\frac{1}{2}^\circ$  or  $t = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = 1+\sqrt{2}$ ; and therefore the area of the octagon  $= \frac{8}{4} \times 1+\sqrt{2} = 2 \times 1+\sqrt{2}$ .

In the decagon, the angle  $CAP$  is  $72^\circ$ , whose sine is  $\frac{1}{4} \sqrt{10+2\sqrt{5}}$ , and its cosine  $\frac{1}{4} \times \sqrt{5-1}$ ; whence  $t$  the tangent of  $72^\circ$  is  $\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5-1}} = \sqrt{\frac{10+2\sqrt{5}}{6-2\sqrt{5}}} = \sqrt{5+2\sqrt{5}}$ ; and therefore the area of the decagon is  $\frac{10}{4} \sqrt{5+2\sqrt{5}} = \frac{5}{2} \sqrt{5+2\sqrt{5}}$ .

In the duodecagon, the angle  $CAP$  is  $75^\circ$ , whose sine is  $\frac{1}{2} \sqrt{2+\sqrt{3}}$ , and its cosine  $\frac{1}{2} \sqrt{2-\sqrt{3}}$ ; whence the tangent of  $75^\circ$  is  $\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = 2+\sqrt{3}$ ; and therefore the area of the duodecagon is  $\frac{12}{4} \times 2+\sqrt{3} = 3 \times 2+\sqrt{3}$ .

The tangents of the angle  $CAP$  in the heptagon, nonagon, and undecagon, are not so easily found, independent of a table of tangents; but we may find equations whose roots shall be the required tangent, in the following manner.

Put  $y$  and  $x$  for the sine and cosine of the angle  $ACB$  at the center, subtended by the side of any polygon, and  $n =$  the number of sides: Then it is known that the sine of  $n$  times that angle, or of  $360^\circ$ , the whole circumference of a circle, is  $nx^{n-1}y - \frac{n \times n - 1 \times n - 2}{1 \times 2 \times 3} x^{n-3}y^3 + \frac{n \times n - 1 \times n - 2 \times n - 3 \times n - 4}{1 \times 2 \times 3 \times 4 \times 5} x^{n-5}y^5$  &c. But the sine of the whole circum-

ference



The multiplier for the pentagon being 1.720477, we shall have  $1.720477 \times 25 \times 25 = 1075.298125$  the area required.

Ex-

ference is nothing, and therefore this series, or the same divided by  $ny$ , will be equal to nothing, also, viz.  $x^{n-1} - \frac{n-1 \times n-2}{2 \times 3} x^{n-3} y^2 +$

$\frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5} x^{n-5} y^4 \&c. = 0$ . And since  $n$  is an integer num-

ber, it is evident that the series will always terminate. Now exter-

minating  $y$  by its value  $\sqrt{1-xx}$ , the series will become  $x^{n-1} - \frac{n-1 \times n-2}{2 \times 3}$

$x^{n-3} \cdot \frac{1-xx}{xx} + \frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5} x^{n-5} \cdot \frac{1-xx}{xx}^2 \&c. = 0$ , or

$1 - \frac{n-1 \times n-2}{2 \times 3} \cdot \frac{1-xx}{xx} + \frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5} \cdot \frac{1-xx}{xx}^2 \&c. = 0$ . Or ex-

terminating  $x$  by its value  $\sqrt{1-y^2}$ , will give  $1 - y^2 \cdot \frac{n-1}{2} - \frac{n-1 \times n-2}{2 \times 3}$

$\frac{1-y^2}{1-y^2}^{\frac{n-3}{2}} y^2 + \frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5} \frac{1-y^2}{1-y^2}^{\frac{n-5}{2}} y^4 \&c. = 0$ , or  $1 - \frac{n-1 \times n-2}{2 \times 3}$

$\frac{1-y^2}{1-y^2} x y^2 + \frac{n-1 \times n-2 \times n-3 \times n-4}{2 \times 3 \times 4 \times 5} \frac{1-y^2}{1-y^2}^2 y^4 \&c. = 0$ . And by writing

any particular value for  $n$ , the root of the equation resulting will be the  $\left\{ \begin{smallmatrix} \text{cofine} \\ \text{fine} \end{smallmatrix} \right\}$  of the angle  $ACB$ .

Thus, if  $n$  be  $= 7$ , the equation will be  $\left\{ \begin{smallmatrix} x^6 - \frac{5}{4} x^4 + \frac{1}{8} x^2 - \frac{1}{64} = 0 \\ y^6 - \frac{7}{4} y^4 + \frac{7}{8} y^2 - \frac{7}{64} = 0 \end{smallmatrix} \right\}$

whose root  $\left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\}$  is the  $\left\{ \begin{smallmatrix} \text{cofine} \\ \text{fine} \end{smallmatrix} \right\}$  of the double of the complement of the  $L CAP$  ( $64\frac{3}{4}^\circ$ ) in the heptagon.

If  $n = 9$ , the series will become  $\left\{ \begin{smallmatrix} x^8 - \frac{7}{4} x^6 + \frac{15}{16} x^4 - \frac{5}{32} x^2 + \frac{1}{256} = 0 \\ y^8 - \frac{9}{4} y^6 + \frac{27}{16} y^4 - \frac{15}{32} y^2 + \frac{9}{256} = 0 \end{smallmatrix} \right\}$

whose root  $\left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\}$  is the  $\left\{ \begin{smallmatrix} \text{cofine} \\ \text{fine} \end{smallmatrix} \right\}$  of the double of the complement of the angle  $CAP$  ( $70^\circ$ ) in the nonagon.

If  $n = 11$ , there will come out the equation

$\left\{ \begin{smallmatrix} x^{10} - \frac{9}{4} x^8 + \frac{7}{4} x^6 - \frac{15}{64} x^4 + \frac{15}{256} x^2 - \frac{1}{1024} = 0 \\ y^{10} - \frac{11}{4} y^8 + \frac{11}{4} y^6 - \frac{77}{64} y^4 + \frac{55}{256} y^2 - \frac{11}{1024} = 0 \end{smallmatrix} \right\}$ , whose root is the

$\left\{ \begin{smallmatrix} \text{cofine} \\ \text{fine} \end{smallmatrix} \right\}$  of the double of the complement of the  $L CAP$  ( $73\frac{7}{11}^\circ$ ) in the undecagon.

When



## EXAMPLE II.

What is the area of a hexagon whose side is 20?  
 Here  $2.598076 \times 20 \times 20 = 1039.2304 =$  the area.

Ex-

When the root  $x$  of the equation is extracted, the tangent of the  $\angle CAP$  may be easily found thus. Call that tangent  $t$ ; now  $t$  being the cotangent of an angle of the double of which  $\left\{ \frac{x}{y} \right\}$  is the  $\left\{ \begin{smallmatrix} \text{cofine} \\ \text{fine} \end{smallmatrix} \right\}$ ,

$$\text{we shall have } t = \sqrt{\frac{1+x}{1-x}} = \frac{1+\sqrt{1-y^2}}{y} = \frac{y}{1-\sqrt{1-y^2}}.$$

Or, by writing  $\frac{t-1}{t+1}$  instead of  $x$ , or  $\frac{2t}{1+t}$  instead of  $y$ , in the three equations above, we shall have,

First,  $t^{12} - 26t^{10} + 143t^8 - 245t^6 + 143t^4 - 26t^2 + 1 = 0$ , an equation whose root is the tangent of the  $\angle CAP$  ( $64\frac{2}{7}^\circ$ ) in the heptagon.

Secondly,  $t^{16} - 45t^{14} + 476t^{12} - 1768t^{10} + 2701t^8 - 1768t^6 + 476t^4 - 45t^2 + 1 = 0$ , an equation whose root is the tangent of the  $\angle CAP$  ( $70^\circ$ ) in the nonagon.

And lastly,  $t^{20} - 70t^{18} + 1197t^{16} - 7752t^{14} + 22610t^{12} - 32065t^{10} + 22610t^8 - 7752t^6 + 1197t^4 - 70t^2 + 1 = 0$ , the root of which equation is the tangent of the angle  $CAP$  ( $73\frac{7}{11}^\circ$ ) in the undecagon.

MOREOVER, If in a square be inscribed a circle, and in the circle the several regular polygons; then

The square will be to 4 as the circle to  $3.14159265$  &c. and as the inscribed

|            |        |                                                                            |
|------------|--------|----------------------------------------------------------------------------|
| Trigon     | } to { | $\frac{3}{2} \times \text{fine of } 120^\circ = \frac{3}{2} \sqrt{3}$      |
| Tetragon   |        | $\frac{4}{2} \times \text{fine of } 90 = 2$                                |
| Pentagon   |        | $\frac{5}{2} \times \text{fine of } 72 = \frac{5}{8} \sqrt{10+2\sqrt{5}}$  |
| Hexagon    |        | $\frac{6}{2} \times \text{fine of } 60 = \frac{3}{2} \sqrt{3}$             |
| Heptagon   |        | $\frac{7}{2} \times \text{fine of } 51\frac{1}{7} = \frac{7}{2} x$         |
| Octagon    |        | $\frac{8}{2} \times \text{fine of } 45 = 2\sqrt{2}$                        |
| Nonagon    |        | $\frac{9}{2} \times \text{fine of } 40 = \frac{9}{2} y$                    |
| Decagon    |        | $\frac{10}{2} \times \text{fine of } 36 = \frac{5}{4} \sqrt{10-2\sqrt{5}}$ |
| Undecagon  |        | $\frac{11}{2} \times \text{fine of } 32\frac{8}{11} = \frac{11}{2} z$      |
| Duodecagon |        | $\frac{12}{2} \times \text{fine of } 30 = 3$                               |
| &c.        |        | &c.                                                                        |

Where  $x$  } is the root of equa.  $\begin{cases} x^6 - \frac{7}{2}x^4 + \frac{7}{8}x^2 - \frac{7}{64} = 0 \\ y^8 - \frac{9}{2}y^6 + \frac{17}{16}y^4 - \frac{15}{128}y^2 + \frac{9}{256} = 0 \\ z^{10} - \frac{11}{2}z^8 + \frac{11}{4}z^6 - \frac{77}{64}z^4 + \frac{55}{128}z^2 - \frac{11}{1024} = 0 \end{cases}$   
 as appears in page 83.

For,



## EXAMPLE III.

What is the area of a trigon whose side is 20?

Here  $433013 \times 20 \times 20 = 1732052 = \text{the area required.}$

## PROBLEM V.

*To find the diameter and circumference of a circle, the one from the other.\**

Y

RULE.

For, if the side of the square or diameter of the circle be 2, their areas will be 4 and  $3.14159$  &c. And, if radiuses be drawn to each angular point of the polygons, they will divide the polygons into so many equal triangles as they have sides; and if, in one of these triangles, a perpendicular be demitted from the extremity of one radius to the other, half this perpendicular will express the area of the triangle, because the base upon which it falls is equal to 1, and consequently half the number of sides drawn into the perpendicular will denote the area of the polygon; but the perpendiculars are the fines of the angles above expressed; and consequently the truth of the proportions is manifest.—As to the values of the fines, they are sufficiently well known, excepting those denoted by  $x, y, z$ , whose values are as above specified.

\* The proportion of the diameter of a circle to the circumference is best found from the tangent of some arc as follows.

Let  $AT$  be the tangent of the arc  $AB$ , and draw  $Ct$  indefinitely near  $CT$ , and  $Ta$  perpendicular to  $Ct$ . Put  $r = CA$ ,  $t = AT$ , and  $a =$  the arc  $AB$ . Then  $Tt = i$ ,  $Bb = a$ , and  $CT = \sqrt{rr + tt}$ .

In the similar  $\Delta$ s  $CAT$ ,  $Tat$ , we have  $TC :$

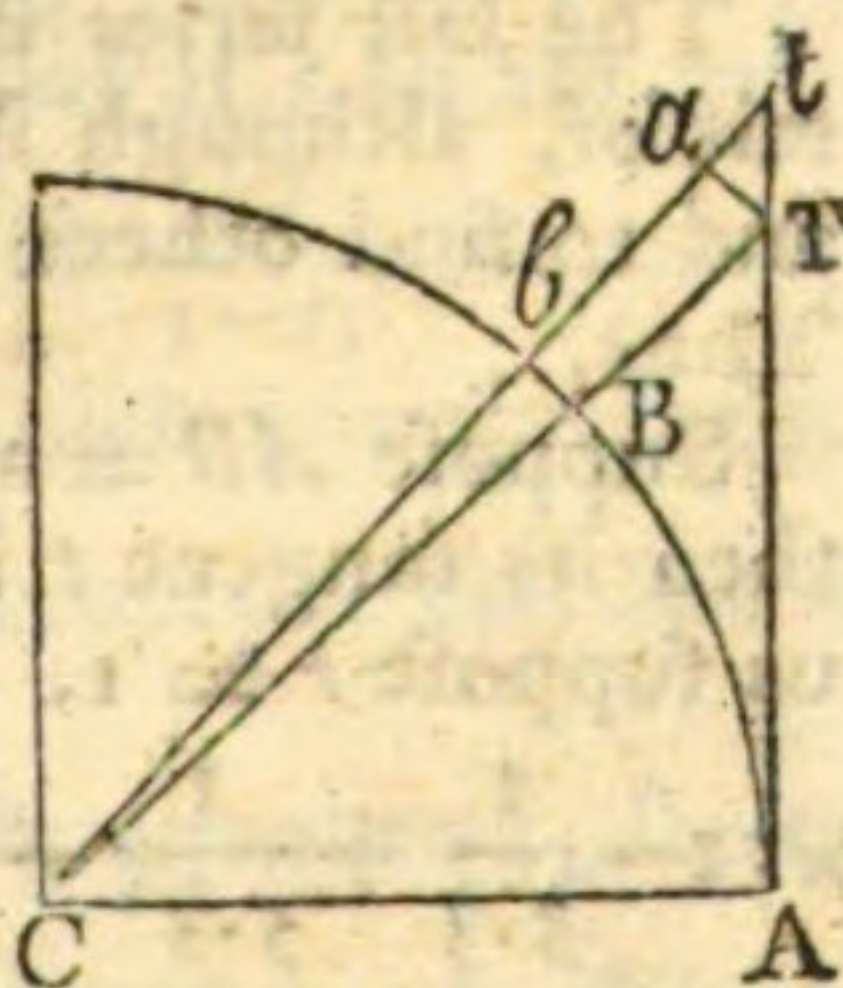
$CA :: tT : Ta = \frac{rt}{CT}$ ; and in the similar  $\Delta$ s  $bBC$ ,

$CTa$ , we have  $CT : Ta :: CB : Bb$ , or  $a =$

$\frac{CB \times Ta}{CT} = \frac{rrt}{CT^2} = \frac{rrt}{rr + tt} = i \times 1 - \frac{t^2}{r^2} + \frac{t^4}{r^4} - \frac{t^6}{r^6} + \frac{t^8}{r^8}$  &c. and the fluent

is  $a = t \times 1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8}$  &c. = the length of the arc  $AB$ .

Then, by taking  $AB =$  any given arc whose tangent can be found in terms





## R U L E.

1. Multiply the diameter by 3.1416, and the product will be the circumference. And, therefore,
2. Divide the circumference by 3.1416, and the quotient will be the diameter.

E x-

terms of the radius, the series will then become known; and being repeated as often as  $AB$  is contained in the whole circumference, we shall have the length of the circumference in terms of the diameter.

This series was first given, for this purpose, by Mr JAMES GREGORY, in a letter, of the 15th of Feb. 1671, sent to Mr COLLINS, and inserted in the *Commerc. Epistol.*—If the arc be assumed greater than half a quadrant, the series will diverge and become of no use here; because that the tangent will, in that case, be greater than the radius: But in every other case the series will converge, and become a proper expression of the arc.

Thus, if  $AB$  be the  $\frac{1}{8}$  part of the circumference, or  $45^\circ$ , its tangent will be equal to the radius, and the series will become  $r \times : 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \&c. =$  the arc of  $45^\circ$ , and therefore  $8 r \times : 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \&c.$  will be the whole circumference.—Hence the circumferences of circles are to one another as their radii or as their diameters; and, consequently, the circumference of a circle whose diameter is 1, being multiplied into the diameter of any other circle, will produce the circumference of this circle; which is our rule.

The last series is the simplest form that the general series will admit of, although it does not converge so quickly as may be required; and to find others converging quicker we must take the arc  $AB$  less. Thus,

Suppose  $AB =$  an arc of  $30^\circ$ , or the  $\frac{1}{6}$  part of the circumference; then its tangent  $t$  is  $= r \sqrt{\frac{1}{3}}$ , or to make the forms, still, simpler, let us suppose  $r = 1$ , then  $t = \sqrt{\frac{1}{3}}$ , and the series will become  $a = \sqrt{\frac{1}{3}} \times : 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} \&c. =$  the arc of  $30^\circ$ ; which converges so fast as that it may be very quickly summed, and being multiplied into 12 will give the whole circumference. And this is Doctor HALLEY's method.

And by taking, still, smaller arcs, series may be found, in the same manner, converging much quicker; but then they will be more compounded with surd quantities.

As



## EXAMPLE I.

If the diameter of a circle be 7, what is the circumference?

$3.1416 \times 7 = 21.9912$  the circumference  $= 22$  nearly.

E x-

As the famous quadrature of the late Mr JOHN MACHIN, *Professor of Astronomy in Gresham College*, is extremely expeditious, and but little known, I shall take this opportunity of explaining it as follows. — Since the chief advantage consists in taking small arcs whose tangents shall be numbers easy to manage, Mr Machin very properly considered, that since the tangent of  $45^\circ$  is 1; and that the tangent of any arc being given, the tangent of double that arc can easily be had; if there be assumed some small simple number as the tangent of an arc, and then the tangent of the double arc be continually taken, until a tangent be found nearly equal to 1 the tangent of  $45^\circ$ ; by taking the tangent answering to the small difference of  $45^\circ$ , and this multiple, there would be had two very small tangents, viz. the tangent first assumed, and the tangent of the difference between  $45^\circ$  and the multiple arc; and that, therefore, the lengths of the arcs corresponding to these two tangents being calculated, and the arc belonging to the tangent first assumed being so often doubled as the multiple directs, the result increased or diminished by the other arc, would be the arc of  $45^\circ$  according as the multiple arc should be below or above it.

Having thus thought of his method, by a few trials he was lucky enough to find a number (and perhaps the only one) proper for this purpose, viz. knowing that the tangent of  $\frac{1}{4}$  of  $45^\circ$  is nearly  $= \frac{1}{5}$ , he assumed  $\frac{1}{5}$  as the tangent of an arc: Then, since if  $t$  be the tangent of an arc, the tangent of the double arc will be  $\frac{2t}{1-t^2}$ , the radius being 1; the tang. of an arc double to that of which  $\frac{1}{5}$  is the tang. will be  $\frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{2}{4.8} = \frac{5}{12}$ , and the tangent of the double of this last is

$\frac{\frac{10}{12}}{1-\frac{25}{144}} = \frac{10 \times 12}{144-25} = \frac{120}{119}$ ; which, being very nearly equal to 1, shews that the arc which is equal to 4 times the first, is very near  $45^\circ$ . Then, since the tangent of the difference between  $45^\circ$  and an arc whose tangent is  $T$ , is  $\frac{T-1}{T+1}$ , we shall have the tangent of the difference between

$45^\circ$  and the arc whose tangent is  $\frac{120}{119} = \frac{\frac{120}{119}-1}{\frac{120}{119}+1} = \frac{120-119}{120+119} = \frac{1}{239}$ .  
Now,



## EXAMPLE II.

What is the diameter of the circle whose circumference is 355?

$$355 \div 3.1416 = 112.9997 + \text{the diameter} = 113 \text{ nearly.}$$

Ex-

Now, by calculating, from the general series, the arcs whose tangents are  $\frac{1}{5}$  and  $\frac{1}{239}$ , which may be very quickly done, by reason of the smallness and simplicity of the numbers, and taking the latter arc from 4 times the former, the remainder will be the arc of  $45^\circ$ . And this is Mr Machin's famous quadrature of the circle, by means of which he found the circumference of a circle, whose diameter is 1, to be 3.1415926535, 8979323846, 2643383279, 5028841971, 6939937510, 5820974944, 5923078164, 0628620899, 8628034825, 3421170679 +, true to above 100 places of figures.

Or, by substituting the above numbers in the general series, &c. we get the series  $\frac{16}{5} - \frac{4}{239} - \frac{1}{3} \cdot \frac{16}{5^3} - \frac{4}{239^3} + \frac{1}{5} \cdot \frac{16}{5^5} - \frac{4}{239^5}$  &c. equal to the semicircumference whose radius is 1, or the whole circumference whose diameter is 1. And this is the series published by Mr JONES, and which he acknowledges to have received from Mr MACHIN.

If instead of  $t$  in the series  $t \times : 1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6}$  &c. be substituted its value  $\frac{rr}{\pi}$ , we shall have  $\frac{rr}{\pi} \times : 1 - \frac{r^2}{3\pi^2} + \frac{r^4}{5\pi^4} - \frac{r^6}{7\pi^6}$  &c. for the length of the arc whose co-tangent is  $\pi$  and radius  $r$ .

Or, if we write  $\frac{rs}{c}$  instead of  $t$ , we shall have  $\frac{rs}{c} \times : 1 - \frac{s^2}{3c^2} + \frac{s^4}{5c^4} - \frac{s^6}{7c^6}$  &c. for the length of the arc whose sine is  $s$  and cosine  $c$ .

And, farther, by writing  $rr - ss$  for  $cc$  in this last series, we obtain  $s \times : 1 + \frac{s^2}{2.3r^2} + \frac{3s^4}{2.4.5r^4} + \frac{3.5s^6}{2.4.6.7r^6} + \frac{3.5.7s^8}{2.4.6.8.9r^8}$  &c. for the arc whose sine is  $s$ .

Again, by substituting  $\sqrt{2rv - vv}$ , the value of  $s$ , instead of it in the last series, we get  $\sqrt{2rv} \times : 1 + \frac{v}{2.3.2r} + \frac{3v^2}{2.4.5.2r^2} + \frac{3.5v^3}{2.4.6.7.2^3r^3}$  &c. for the length of the arc whose versed sine is  $v$ .—Or, by writing  $d$ , the diameter, instead of  $2r$ , the value of the arc will be expressed by  $\sqrt{dv} \times : 1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3}$  &c.

Much



## EXAMPLE III.

What is the circumference of the earth, supposing it to be perfectly round, and that its diameter is 7958 miles? — — Anf. 25000.8528 miles.

## PROBLEM VI.

*To find the length of any arc of a circle.*

## RULE I.

\* Multiply continually together the radius, the number of degrees in the given arc, and the number .01745329, and the product will be the length of the arc.

Z

Ex-

Much after the same manner the arc may be expressed by the secants, co-secants, or co-sines. And if for any of the letters, in the general series, be substituted any of their particular values, as was done for the tangent, we might, by those means, obtain several different particular forms of infinite series, whose sums would be expressed by means of the circumferences of any given circles: Thus, if, in the last series but one,  $s$  be taken  $= r$ , or the arc be supposed to be a quadrant, we shall obtain the infinite series  $r + \frac{r^3}{2.3r^2} + \frac{3r^5}{2.4.5r^4} + \frac{3.5r^7}{2.4.6.7r^6}$

&c. or  $r \times 1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7}$  &c. whose sum will be equal to  $\frac{1}{4}$

of the circumference of the circle whose radius is  $r$ .

And, if from this series be taken the former series expressed by  $s$ , there will result  $\frac{r-s}{1} + \frac{1}{2.3} \cdot \frac{r^3-s^3}{r^2} + \frac{3}{2.4.5} \cdot \frac{r^5-s^5}{r^4} + \frac{3.5}{2.4.6.7} \cdot \frac{r^7-s^7}{r^6}$  &c. for the length of the arc whose cosine is  $s$ .

## \* DEMONSTRATION.

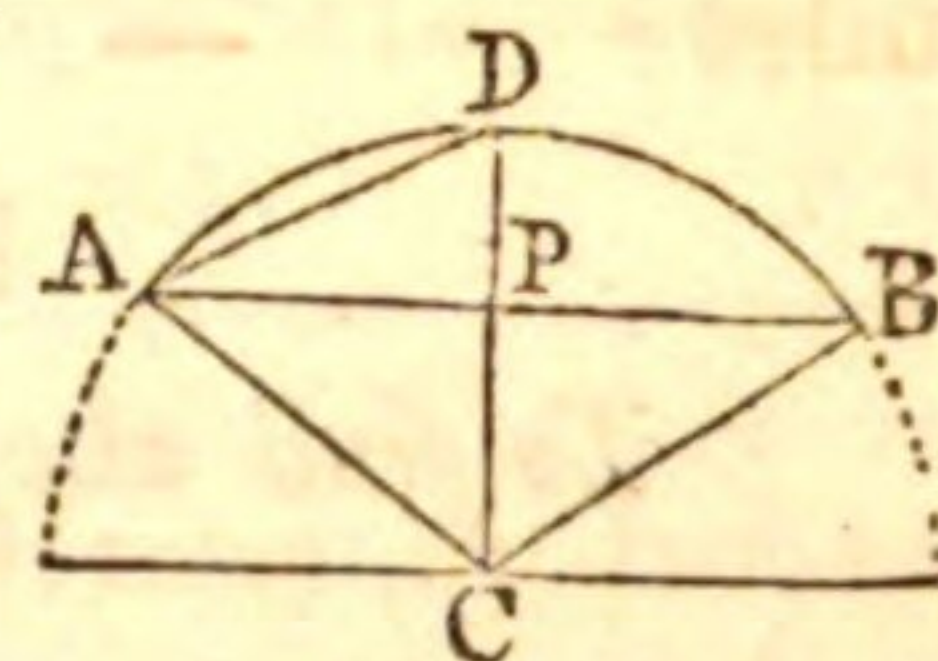
For, when the radius is 1, half the circumference is 3.14159265 &c. and, therefore,  $\frac{3.14159265 \text{ \&c.}}{180 \text{ degrees}} = .0174532925199 \text{ \&c.}$  is the length of an arc of 1 degree; hence  $r \times .01745 \text{ \&c.} =$  the length of 1 degree to the rad.  $r$ , and, consequently,  $.01745 \text{ \&c.} \times r \times$  number of degrees in any arc, will be the length of that arc. Q.E.D.



## EXAMPLE.

Required the length of the arc  $ADB$ , whose chord  $AB$  is 6, the radius being 9.

By trigonometry,  $9 = AC : 3 = AP :: 1 = \text{rad. of the tables} : \frac{3}{9} = .3333333$  the fine of the  $\angle ACP$ , or of the arc  $AD$  to the radius 1; and the degrees, in the table of fines, answering to this fine are  $19.4712206$ , the double of which is  $38.9424412$  the degrees in the whole arc  $ADB$ .



Then, by the rule,  $38.9424412 \times .01745329 \times 9 = 6.117063$  the length of the arc required.

## RULE II.

\* If  $d$  denote the diameter of the circle, and  $v$  the versed sine of half the arc; the arc will be expressed by  $2\sqrt{dv} \times 1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3} \&c.$  or by  $2d\sqrt{q} + \frac{q}{2.3} A + \frac{3.3q}{4.5} B + \frac{5.5q}{6.7} C \&c.$  putting  $q = \frac{v}{d}$ , and  $A, B, C, \&c.$  for the first, second, third,  $\&c.$  terms.

## EXAMPLE.

Let there be taken here the same example as before.

Then  $PC = \sqrt{CA^2 - AP^2} = \sqrt{9^2 - 3^2} = \sqrt{72} = 6\sqrt{2}$ , and  $PD = DC - CP = 9 - 6\sqrt{2} = .51471862 = v$ ; also  $.51471862 \div 18 = .02859548 = \frac{v}{d} = q$ .

Hence

---

\* This rule is proved in page 88.



$$\text{Hence } A = 2d\sqrt{q} = 6.087672$$

$$B = \frac{q}{2.3} A = 29013$$

$$C = \frac{3.3q}{4.5} B = 373$$

$$D = \frac{5.5q}{6.7} C = 6$$

the sum  $6.117064 =$  the arc  $ABD$   
as before, nearly.

## R U L E III.

If  $r$  be the radius, and  $s$  the sine or half chord  $AP$  of the arc, then  $2s \times 1 + \frac{s^2}{3.2r^2} + \frac{3s^4}{5.2.4r^4} + \frac{3.5s^6}{7.2.4.6r^6} \&c.$  will be the arc  $ADB$ .\*

## E X A M P L E.

Taking the same example as in the first rule,

$$2s = A = 6.000000$$

$$\frac{s^2}{2.3rr} A = B = 0.111111$$

$$\frac{3.3s^2}{4.5rr} B = C = 5555$$

$$\frac{5.5s^2}{6.7rr} C = D = 367$$

$$\frac{7.7s^2}{8.9rr} D = E = 28$$

$$\frac{9.9s^2}{10.11rr} E = F = 2$$

the sum  $6.117063$  is the arc  $ADB$   
required.

## R U L E IV.

From 8 times the chord of half the arc subtract the chord of the whole arc, and  $\frac{1}{3}$  of the remainder will be the length of the arc, nearly.

That

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\* This likewise is proved in page 88.



That is,  $\frac{8AD-AB}{3} = ADB$  nearly.\*

#### EXAMPLE.

Taking, again, the same example; we shall have, as in the example to rule 2,  $PD=9-6\sqrt{2}$ , and hence  $DA=\sqrt{AP^2+PD^2}=\sqrt{3^2+9-6\sqrt{2}}=3\sqrt{18-12\sqrt{2}}=3.043836$ .

Then, by the rule,  $\frac{3.043836 \times 8-6}{3} = 6.116896$  the length of the arc, nearly the same as before.

#### RULE V.

† Divide 3 times the versed sine of the half arc by the difference between the said versed sine and 3 times the

#### \* DEMONSTRATION.

For since, by the last rule, the arc  $AD = A = s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4} \&c.$

And  $c =$  the chord  $AD = \sqrt{ss+xx} = \sqrt{s^2+r-CP}^2 =$

$\sqrt{s^2+r-\sqrt{rr-ss}}^2 = s + \frac{s^3}{8r^2} + \frac{7s^5}{128r^4} \&c.$  Therefore  $A-c = \frac{s^3}{24r^2}$

$+ \frac{13s^5}{640r^4} \&c.$  But  $\frac{c-s}{3} = \frac{s^3}{24r^2} + \frac{7s^5}{384r^4} \&c.$  Wherefore  $A-c - \frac{c-s}{3} =$

$A - \frac{4c-s}{3} = \frac{s^5}{480r^4} \&c.$  And  $2A = \frac{8c-2s}{3}$  nearly, = the arc  $ADB$ , as in the rule. And this is Huygens's theorem.

#### † DEMONSTRATION.

By the second rule half the arc is  $\sqrt{dvx}:1 + \frac{v}{2.3d} + \frac{3vv}{2.4.5dd} \&c.$

Let there be assumed an expression which, when expanded into a series, shall be of the same form with the series above, and affected with indeterminate coefficients; then from the comparison of so many of the first terms as there are indeterminate quantities assumed, will be found the values of the said quantities in given numbers; which being written for them in the primary assumed expression, will make that expression an approximate value of the arc.

Thus,



the diameter; multiply the square root of the quotient by 2 times the diameter, and the product will be the arc nearly.

That is,  $4CD \sqrt{\frac{3DP}{6CD-DP}} = ADB$  nearly.

## E X A M P L E.

Taking, still, the same example; we find, as in the example to the last rule,  $DP = 9 - 6\sqrt{2} = .5147186$ .

Then, by the rule,  $36 \sqrt{\frac{.5147186 \times 3}{54 - .5147186}} = 36 \times .1699137 = 6.116893$  the arc nearly the same as before.

## R U L E VI.

† Divide 5 times the versed sine of the half arc by the difference between 5 times the diameter and 3 times  
2 A times

Thus, assume  $d \sqrt{\frac{v}{gd-bv}}$ ; this in a series is  $\sqrt{dv} \times: \frac{1}{\sqrt{g}} + \frac{bv}{2dg\sqrt{g}} + \frac{3bbvv}{2.4ddgg\sqrt{g}} \&c.$  the two first terms of which compared with those of the series above, give these two equations,  $\frac{1}{\sqrt{g}} = 1$ , and  $\frac{b}{g\sqrt{g}} = \frac{1}{3}$ ; hence  $g = 1$ , and  $b = \frac{1}{3}$ ; and therefore the assumed expression  $d \sqrt{\frac{v}{gd-bv}}$  becomes  $d \sqrt{\frac{v}{d - \frac{1}{3}v}} = d \sqrt{\frac{3v}{3d-v}} =$  half the arc nearly; the double of which will be the whole arc as in rule 5.

† Again, to find another, still nearer, approximation, let there be introduced another indeterminate quantity; thus, suppose  $d \sqrt{\frac{v}{gd-bv}} + k\sqrt{dv} = \sqrt{dv} \times: k + \frac{1}{\sqrt{g}} + \frac{bv}{2dg\sqrt{g}} + \frac{3bbvv}{2.4ddgg\sqrt{g}} =$  the semi-arc; then, comparing the first three terms with those of the given series, we have these three equations,  $k + \frac{1}{\sqrt{g}} = 1$ ,  $\frac{b}{g\sqrt{g}} = \frac{1}{3}$ , and  $\frac{bb}{gg\sqrt{g}} = \frac{1}{3}$ ;



times the said versed sine; multiply the square-root of the quotient by 5 times the diameter; and to the product add 4 times the root of the product of the said versed sine and diameter; and  $\frac{2}{3}$  of the sum will be the arc extremely near.

That is, if  $d$  be the diameter and  $v$  the versed sine  $PD$ , then  $5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv} \times \frac{2}{3}$  will be the arc  $ADB$  extremely near.

#### EXAMPLE.

Taking, here, the same example as in the former rules, we shall have  $5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv} \times \frac{2}{3} = 6.11706396$  = the arc very nearly, it being true to the sixth or seventh place of decimals.

#### PROBLEM VII.

*To find the area of a circle.*

#### \* RULE I.

Multiply half the circumference by half the diameter, and the product will be the area.

*Note.* This rule will, also, serve for any sector of a circle.

#### RULE

$= \frac{1}{2}$ ; whence  $g = \frac{81}{25}$ ,  $h = \frac{241}{125}$ , and  $k = \frac{4}{5}$ ; and by writing these values in the assumed quantity, we obtain  $\frac{5}{9} d\sqrt{\frac{v}{d-\frac{3}{5}v}} + \frac{4}{9}\sqrt{dv} = \frac{1}{9}$   $\times 5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv}$  for half the arc, the double of which will be rule 6.

#### \* DEMONSTRATION of RULE I.

For, a circle may be considered as a polygon of an infinite number of sides, the circumference being the perimeter and the radius the perpendicular.

*Otherwise*



## R U L E II.

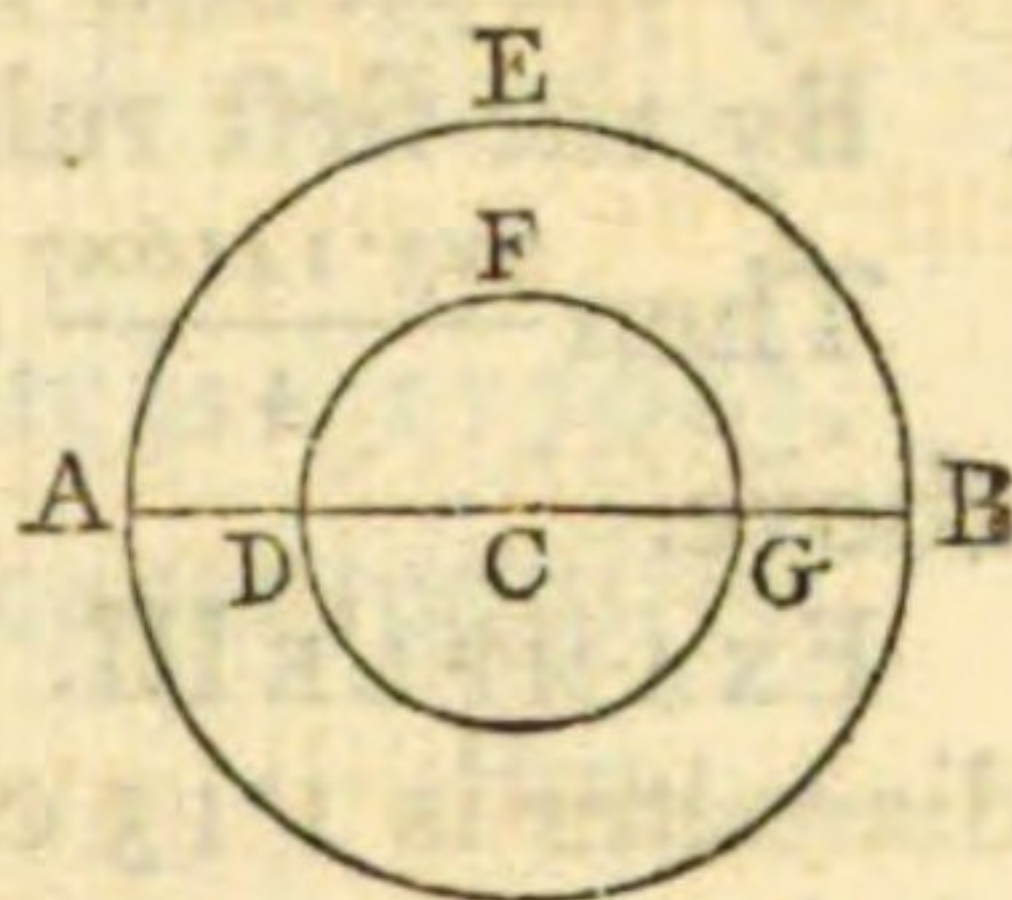
Multiply the square of the diameter by  $\cdot 7854$ , and the product will be the area.

Ex-

*Otherwise.*

Put  $r =$  the radius  $AC$ ,  $c =$  the whole circumference  $AEBA$ , or any part of it, and  $x =$  the radius  $CD$  of a circle continually expanded. Then  $\frac{c}{r}xx$  will express the fluxion of the whole circle or sector whose circumference is  $c$ ; and, consequently  $\frac{cxx}{2r} =$  the

area  $CDF$ ; and  $\frac{1}{2}cr =$  the area  $CAE$  of the whole circle or sector accordingly. *Q.E.D.*



## DEMONSTRATION of RULE II.

And to prove the second rule, let it be first observed, that all circles, as being similar figures, are as the squares of their diameters or circumferences: which, also, appears thus; by rule I, circles are as the rectangles of their diameters and circumferences, but the circumference is as the diameter, by the notes to the last problem; therefore the circles are as the squares of either of them. Then, by rule I, the area of a circle whose diameter is 1, is  $\frac{1 \times 3 \cdot 14159 \text{ \&c.}}{2 \times 2 = 4} = \cdot 78539$  &c. whence  $1^2 : d^2$  (the square of any diameter)  $:: \cdot 78539 \text{ \&c.} : \cdot 78539 \text{ \&c.} \times d^2$  the circle whose diameter is  $d$ . *Q.E.D.*

## COROLLARY.

If  $D$  be the diameter,  $C$  the circumference, and  $A$  the area of any circle; and  $p = 3 \cdot 14159 \text{ \&c.}$  from this prob. and prob. 5, we may deduce these equations:

$$1. D = \frac{C}{p} = \frac{4A}{C} = 2\sqrt{\frac{A}{p}}.$$

$$2. C = pD = \frac{4A}{D} = 2\sqrt{pA}.$$

$$3. A = \frac{pD^2}{4} = \frac{C^2}{4p} = \frac{DC}{4}.$$

$$4. p = \frac{C}{D} = \frac{4A}{DD} = \frac{CC}{4A}.$$



EXAMPLE I. What is the area of a circle whose diameter is 1, and circumference 3.1416?

$$\frac{3.1416 \times 1}{4} = .7854 \text{ the area.}$$

EXAMPLE II. What is the area of a circle whose diameter is 7?

By the second rule.  $7 \times 7 \times .7854 = 38.4846$  the area.

By the first rule.  $7 \times 3.1416 =$  the circumference.

Then  $\frac{7 \times 3.1416 \times 7}{4} = 7 \times 7 \times .7854$  the area, the same as above.

EXAMPLE III. What is the area of the circle whose diameter is 1.13 chains, and circumference 3.550008 chains? — Ans. 1.00287726 square chains.

EXAMPLE IV. How many square yards are in a circle whose diameter is  $3\frac{1}{2}$  feet? — Ans. 1.06901  $\frac{1}{2}$ .

EXAMPLE V. How many square feet are in a circle whose circumference is 10.9956 yards?

Ans. 86.59035.

#### PROBLEM VIII.

*To find the area of any sector of a circle.*

##### RULE I.

Multiply the radius or half the diameter by half the arc of the sector, found by prob. 6, and the product will be the area, as in the whole circle.\*

E x-

---

\* The demonstration of this rule is contained in that of the last problem.

Putting  $r =$  the radius of a circle,  $d =$  the diameter,  $A =$  the area of a sector of it,  $a =$  the length of the arc of the sector,  $b =$  the degrees in  $\frac{1}{2} a$ ,  $s =$  half the chord of the arc  $a$ , or the sine of  $\frac{1}{2} a$ , and

$v =$



## EXAMPLE I.

What is the area of a sector whose radius is 10 and arc 20?

By the rule,  $\frac{20 \times 10}{2} = 100$  the area.

## EXAMPLE II.

To find the area of the sector whose radius is 9 and the chord of whose arc is 6.

By the last problem the length of the arc is 6.117063.

Therefore  $\frac{6.117063 \times 9}{2} = 27.5267835$  the area required.

2 B

RULE

$v$  = the versed sine of  $\frac{1}{2} a$ ; then, by multiplying the radius by half the arc, as found by the several rules of the last problem, we shall obtain the following values of the area of the sector.

$$I. A = \frac{1}{2} ar = .01745329 br.$$

$$II. A = r \sqrt{dv} \times: 1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3} \&c.$$

$$III. A = rs \times: 1 + \frac{s^2}{2.3r^2} + \frac{3s^4}{2.4.5r^4} + \frac{3.5s^6}{2.4.6.7r^6} \&c.$$

$$IV. A = \frac{4\sqrt{ss+vv}-s}{3} r = \frac{4\sqrt{2rv}-s}{3} r = \frac{4\sqrt{ss+vv}-s}{3} \times \frac{ss+vv}{2v},$$

nearly.

$$V. A = rd \sqrt{\frac{3v}{3d-v}}, \text{ nearly.}$$

$$VI. A = \frac{r}{9} \times 5d \sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv}, \text{ nearly.}$$

It is evident that the area of the sector might be expressed in several other different manners; such as by the tangent, cosine, &c. of its semi-arc; but the forms above given are the most useful ones.



## R U L E II.

If it be known what proportion it bears to the whole circle, or what part it is of it; Take the like proportional part of the area of the whole circle for that of the sector.\*

*Note.* A sector bears the same proportion to the whole circle as doth the degrees in its arc or angle to 360, or as its arc to the whole circumference.

## E X A M P L E I.

What is the area of a semicircle, and of a quadrant, the radius being 10?

First,  $.7854 \times 10 \times 10 = 78.54 =$  the whole circle.

Then  $78.54 \div 2 = 39.27 =$  the semicircle.

And  $78.54 \div 4 = 19.635 =$  the quadrant.

## E X A M P L E II.

To find the area of a sector whose arc or angle contains 18 degrees, to a radius of 3 feet?

First,  $.7854 \times 3 \times 3 = 7.0686 =$  the whole circle.

Then  $360 : 18 :: 7.0686 : \frac{7.0686 \times 18}{360} = \frac{7.0686}{20} = .35343$   
the area of the sector required.

## E X A M P L E III.

To find the area of a sector whose radius is 9 and the chord of whose arc is 6.

By the example to rule I of prob. 6, the degrees in the arc were found to be  $38.9424412$ . But  $.785398 \times 9 \times 9 \times 4 =$  the area of the circle.

Therefore

---

\* This rule is too evident to need any formal proof.



Therefore  $360 : 38.9424412 :: .785398 \times 9 \times 9 \times 4 :$   
 $\frac{38.9424412 \times .785398 \times 9 \times 9 \times 4}{360} = 38.9424412 \times .0785398 \times 9 =$   
 $27.52678388$  the area of the sector required.

## P R O B L E M IX.

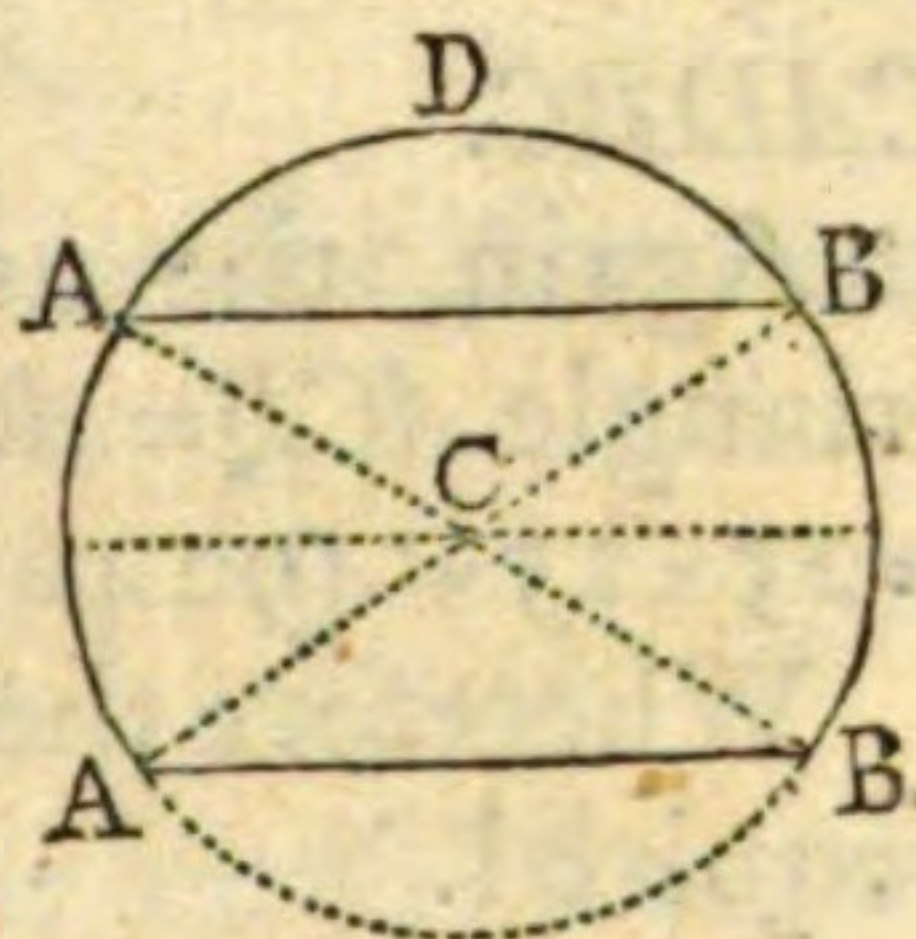
*To find the area of the segment of a circle.*

## R U L E I.

By one of the rules of the last problem, find the area of the sector  $ACB$  having the same arc with the segment  $ADB$  required.

Find also the area of the triangle  $ABC$  formed by the chord of the segment and the two radiuses of the sector.

Then the sum or difference of these areas will be that of the segment, according as it is greater or less than a semicircle.



## E X A M P L E I.

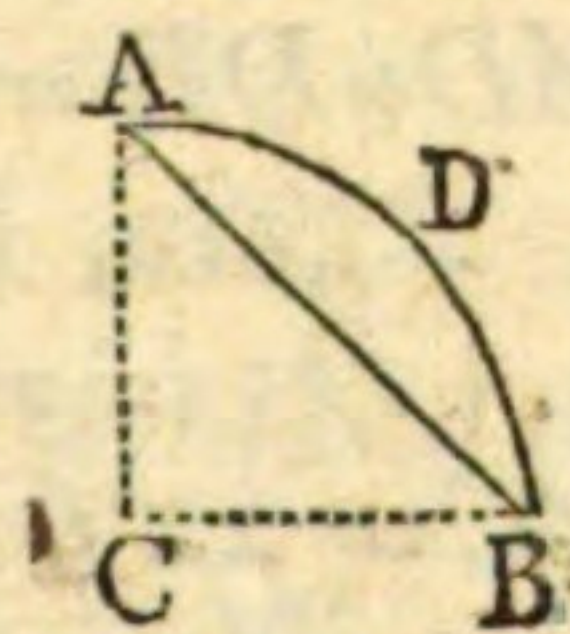
What is the area of a segment whose arc is a quadrant or contains 90 degrees, the diameter being 18 feet?

First,  $.7854 \times 18 \times 18$  is the whole circle.

And  $\frac{.7854 \times 18 \times 18}{4} = .7854 \times 9 \times 9 =$  the quadrant or sector  $CADB$ .

But  $\frac{AC \times CB}{2} = \frac{9 \times 9}{2} = .5 \times 9 \times 9 =$  the triangle  $ACB$ .

Therefore  $.7854 \times 9 \times 9 - .5 \times 9 \times 9 = .2854 \times 9 \times 9 = 23.1174$  = the area of the segment required.



Ex-

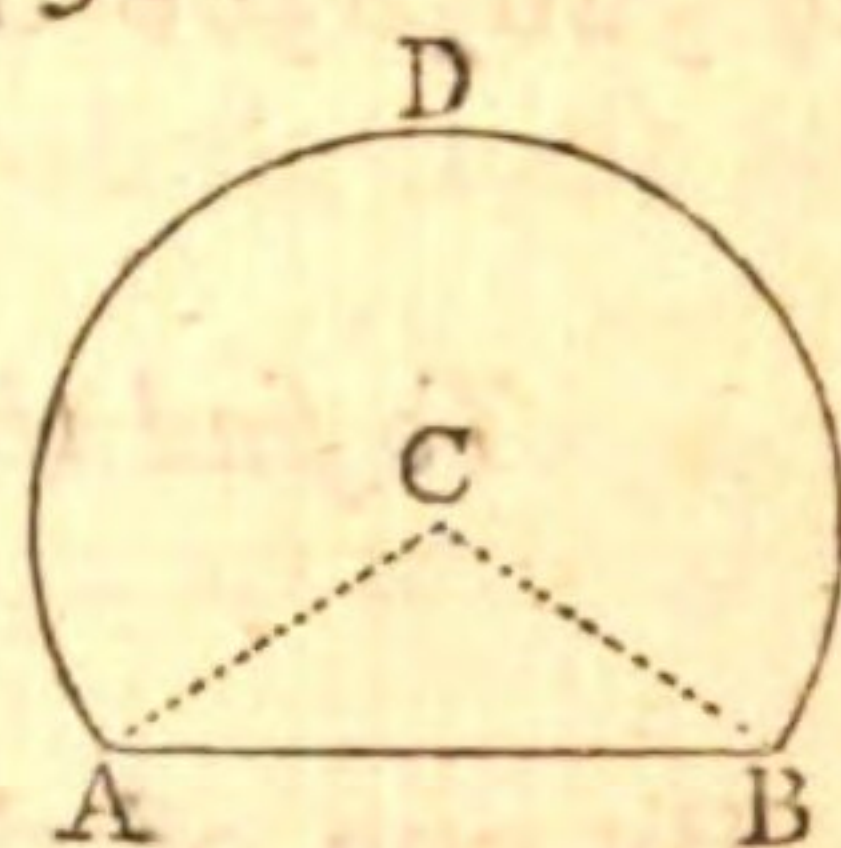


## EXAMPLE II.

What is the area of a segment of a circle whose arc contains  $280^\circ$ , the diameter being 50?

First,  $.7854 \times 50 \times 50 = 1963.5 =$  the whole circle.

And  $360:280::1963.5:\frac{1963.5 \times 280}{360} = \frac{1963.5 \times 7}{9} = 1527.16666 =$  the sector  $CADBC$ .



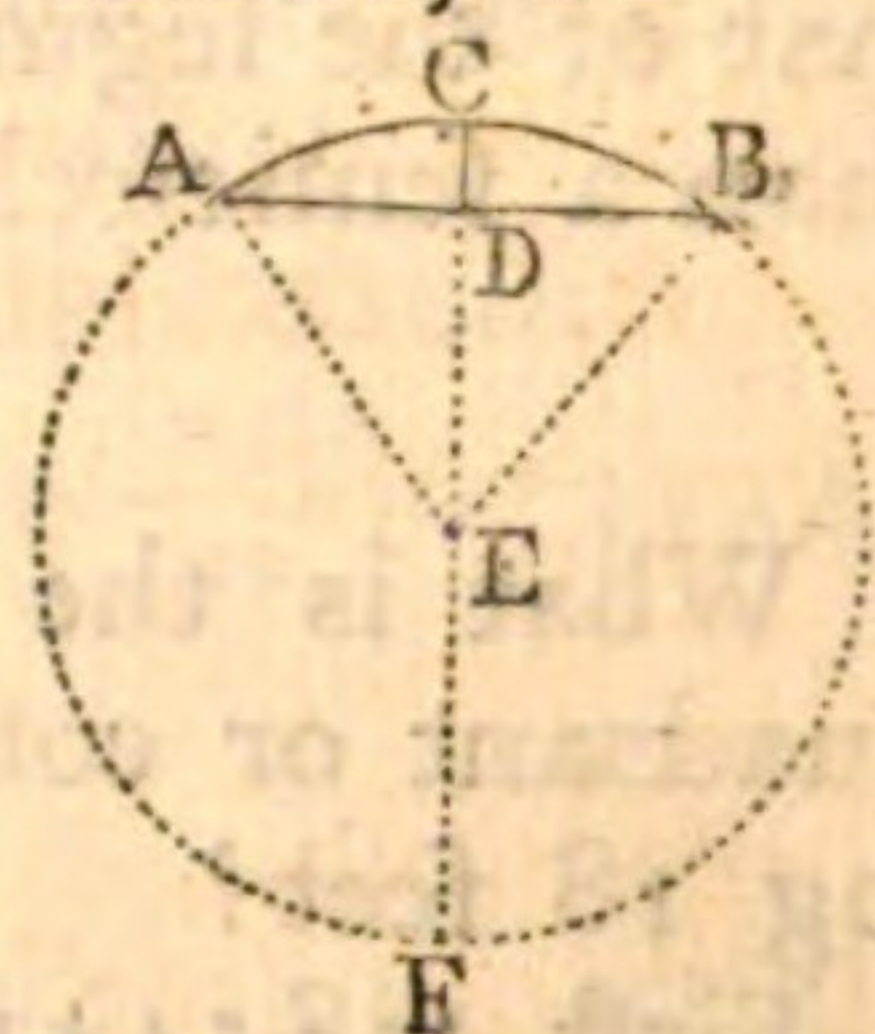
Again, the angle  $ACB = 360 - 280 = 80^\circ$ . And the triangle  $ACB = AC \times CB \times \frac{1}{2} \text{ fine } \angle ACB = 25 \times 25 \times \frac{1}{2} \text{ s. of } 80^\circ = 25 \times 25 \times .4924039 = 307.7524375$ .

Wherefore the sum  $1834.9191 =$  the segment  $ADBA$  required.

## EXAMPLE III.

Required the area of the segment whose chord is 20, and height, or versed sine of half its arc, 2.

Since  $AD^2 = DC \times DE$ , therefore  $DF = \frac{AD^2}{DC} = \frac{100}{2} = 50$ . And the diameter  $CF = FD + DC = 50 + 2 = 52$ . Also  $DE = EC - CD = \frac{1}{2}CF - CD = 26 - 2 = 24$ .



Therefore the triangle  $ABE = AD \times DE = 10 \times 24 = 240$ .

Again, by trigonometry,  $26 = AE:10 = AD::1 = \text{s. } \angle D:\frac{1}{2}^\circ = \frac{1}{13} = .38461538 = \text{s. } \angle AED$ , or of the arc  $AC = 22.6198614$  degrees, the double of which or  $45.2397228$  are the degrees in the arc  $ACB$ .

But  $.785398 \times 52 \times 52 =$  the whole circle.

Therefore  $360:45.2397228::.785398 \times 52 \times 52:\frac{45.2397228 \times .785398 \times 13 \times 26}{45} = 266.8786995 =$  the sector  $AEB$ .

Whence



Whence  $266.8786995 - 240 = 26.8786995 =$  the segment  $ACBA$  required.

*Otherwise.*

Having found the triangle  $= 240$ , as above; we shall have, by rule 6 of the last problem, the sector  $=$

$$\frac{5 \times 52}{4} \sqrt{\frac{5 \times 2}{5 \times 52 - 3 \times 2}} + \sqrt{52 \times 2 \times \frac{2 \times 52}{9}} = 65 \sqrt{\frac{5}{127}} + 2 \sqrt{26} \times \frac{104}{9}$$

$$= \frac{65}{127} \sqrt{635} + 2 \sqrt{26} \times \frac{104}{9} = 266.87868.$$

Then the difference is  $26.87868 =$  the segment nearly the same as before.

### R U L E II.

Let  $d$  be the diameter, and  $v$  the versed sine, or the height of the segment;

Then  $2v \sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2} - \frac{v^3}{72d^3} \&c.$  or  $2v \sqrt{dv}$   
 $\times \frac{2}{3} - \frac{3v}{5.2d} A - \frac{5v}{7.4d} B - \frac{7.3v}{9.6d} C - \frac{9.5v}{11.8d} D \&c. =$  the segment;  
 $A, B, C, \&c.$  being the first, second, third,  $\&c.$  terms.\*  
2 C Ex.

#### \* DEMONSTRATION.

By the last problem, the value of the sector  $CADB$  (fig. in page 99) is  $\frac{1}{2} d \sqrt{dv} \times 1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3} \&c.$  But  $\pm \frac{1}{2} d \mp v =$  the altitude of the triangle  $ABC$ , and its base  $AB = 2 \sqrt{dv - vv} = 2 \sqrt{dv} \times 1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6d^3} \&c.$  and, therefore, the area of the triangle  $ABC = \pm \frac{1}{2} d \mp v \times \sqrt{dv} \times 1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6d^3} \&c.$  which being added to, or subtracted from the sector, gives  $2v \sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{4.7d^2} - \frac{3v^3}{4.6.9d^3} - \frac{3.5v^4}{4.6.8.11d^4} \&c.$  for the value of the segment.  
*Other*



## EXAMPLE.

Take, here, the last example, in which  $d=52$  and  $v=2$ ; then  $\frac{v}{d}=\frac{1}{26}$ ; and the process will be thus:

$$+A = \frac{4v\sqrt{dv}}{3} = \frac{8}{3}\sqrt{104} = +27.1947707$$

$$-B = \frac{3v}{5.2d}A = \frac{3}{10.26}A = -3.137858$$

$$-C = \frac{5v}{7.4d}B = \frac{5}{7.4.26}B = -0.0021551$$

$$-D = \frac{7.3v}{9.6d}C = \frac{7}{3.6.26}C = -0.0000322$$

$$-E = \frac{9.5v}{11.8d}D = \frac{9.5}{11.8.26}D = -0.0000007$$

the sum of the negative terms  $-3.159738$   
taken from the first term leaves  $26.8787969 =$  the  
area required.

## RULE

*Otherwise.*

Putting  $y =$  the semi-chord of the segment, and the other letters as before. We have the fluxion of the segment  $y\dot{v} = \dot{v}\sqrt{dv - vv} = \dot{v}\sqrt{dv} \times 1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6d^3}$  &c. the fluent of which gives  $v\sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{4.7d^2} - \frac{3v^3}{4.6.9d^3}$  &c. for the value of half the segment, as before. Q.E.D.

## COROLLARY I.

Let the chord of half the arc of the segment be denoted by  $c$ ; then  $d = \frac{cc}{v}$ , by which exterminating  $d$  out of the above series, gives  $cv \times \frac{2}{3} - \frac{v^2}{5c^2} - \frac{v^4}{4.7c^4} - \frac{3v^6}{4.6.9c^6} - \frac{3.5v^8}{4.6.8.11c^8}$  &c. for the value of the semi-segment.

## COROLLARY II.

Let the supplemental versed sine be denoted by  $V$ , viz.  $V = d - v$ ; then  $d = V + v$ , by which exterminating  $d$  out of the same series, will



## R U L E III.

If  $v$  denote the versed sine, as before, and  $V$  the versed sine of the supplement, or  $V = d - v$ ; Then  $\frac{4}{3}v\sqrt{vV} + \frac{5}{5V}A - \frac{1v}{7V}B + \frac{3v}{9V}C - \frac{5v}{11V}D$  &c. will be = the segment; where  $A, B, C$ , &c. denote the preceding terms.

## E X A M P L E.

Taking, again, the same example; we have  $v = 2$ , and  $V = 52 - 2 = 50$ . Hence  $\frac{v}{V} = \frac{2}{50} = .04$ , and

$$A =$$

will produce  $2v\sqrt{vV}$  &c.  $\frac{1}{1.1.3} + \frac{v}{1.3.5V} - \frac{v^2}{3.5.7V^2} + \frac{v^3}{5.7.9V^3} - \frac{v^4}{7.9.11V^4}$  &c. for the value of the said semi-segment. And this series, which is rule 3, converges very quickly in small segments.

## C O R O L L A R Y III.

By supposing the several indeterminate quantities in these series to be expressed by any given numbers, we may obtain an endless variety of infinite series whose sums will be given. And, in particular, if the diameter be 1, and supposing the versed sines to be equal to the radius or the diameter, we shall obtain the following series, in which  $n$  denotes  $.785398$  &c. the area of the whole circle whose diameter is 1, or  $\frac{1}{4}$  of the circumference of the same circle.

$$\text{I. } n = \sqrt{2} \times \frac{2}{3} - \frac{1}{5.2} - \frac{1}{4.7.2^2} - \frac{1.3}{4.6.9.2^3} - \frac{1.3.5}{4.6.8.11.2^4} \text{ \&c.}$$

$$\text{II. } n = 2 \times \frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5}{4.6.8.11} \text{ \&c.}$$

$$\text{III. } n = \frac{1}{1.1.3} + \frac{1}{1.3.5} - \frac{1}{3.5.7} + \frac{1}{5.7.9} - \frac{1}{7.9.11} \text{ \&c.}$$

Several other values of  $n$  may be assigned from the corollaries in pages 29 and 30, and from pages 85, &c.



$$A = \frac{4}{3} v \sqrt{vV} = + 26.66666666\frac{2}{3}$$

$$B = \frac{v}{5V} A = + 21333333\frac{1}{3}$$

$$C = \frac{v}{7V} B = - 121904\frac{1}{2}\frac{6}{4}$$

$$D = \frac{3v}{9V} C = + 1625\frac{3}{2}\frac{1}{4}$$

$$E = \frac{5v}{11V} D = - 29\frac{1}{2}\frac{2}{1}$$

$$F = \frac{7v}{13V} E = + \frac{1}{2}\frac{3}{1}$$

sum of the affir. terms + 26.88001626

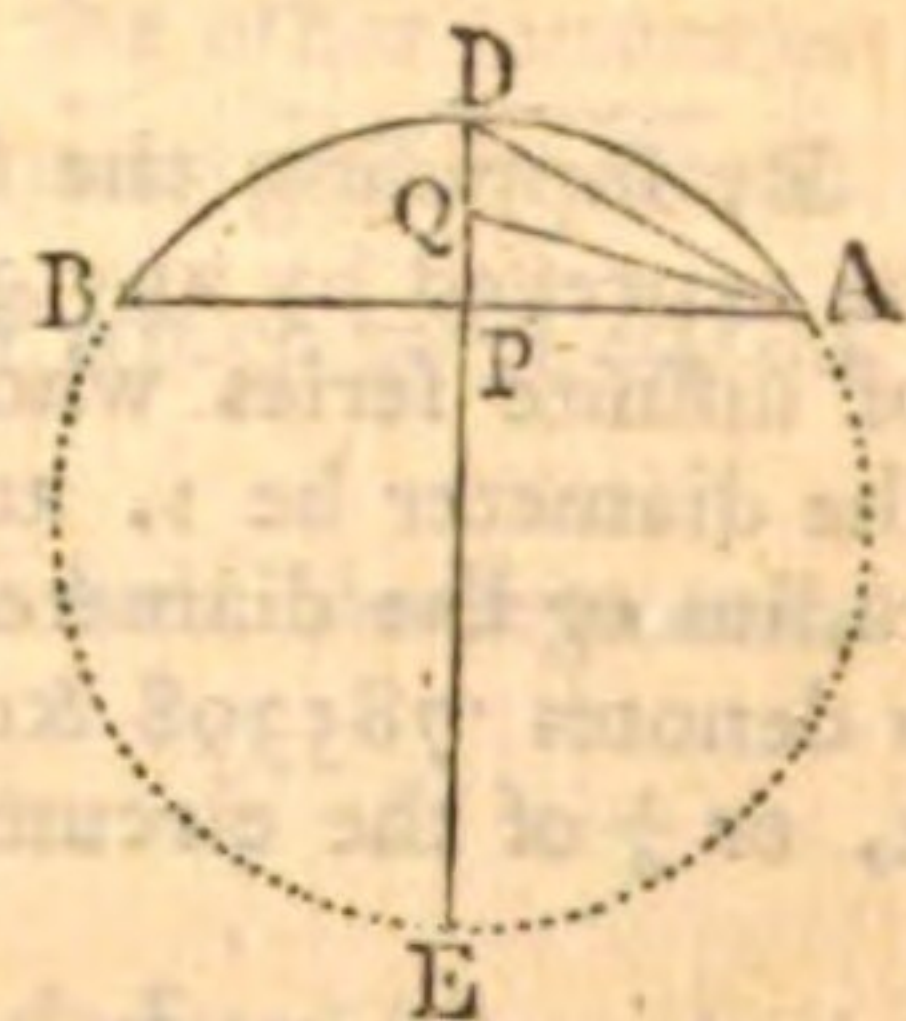
sum of the negat. terms - 0.00121934 $\frac{1}{3}$

the difference is 26.87879691 $\frac{2}{3}$  = the area required.

#### R U L E IV.

To the chord of the whole arc add  $\frac{4}{3}$  of the chord of half the arc, multiply the sum into the versed sine or height of the segment, and  $\frac{4}{15}$  of the product will be the area of the segment nearly.

That is,  $C + \frac{4}{3}c \times \frac{4}{15}v$ , or  $C + \frac{4}{3}\sqrt{dv}$   
 $\times \frac{4}{15}v$ , or  $C + \frac{4}{3}\sqrt{\frac{1}{4}C^2 + v^2} \times \frac{4}{15}v$ , or  
 $\sqrt{dv} - vv + \frac{2}{3}\sqrt{dv} \times \frac{4}{3}v =$  the area  
 nearly; where  $d = DE$  the diameter,  
 $C = BA$  the chord of the whole  
 arc,  $c = AD$  the chord of the half  
 arc, and  $v = DP$  the height of the segment.\*



Ex-

#### \* DEMONSTRATION.

The segment is  $= 2v\sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v}{28d}$  &c. by the last rule.

Suppose



## EXAMPLE.

Taking, still, the same example, in which  $C=20$ , and  $v=2$ .

Then  $C + \frac{4}{3}\sqrt{\frac{1}{4}CC + vv} \times \frac{4}{15}v = 26.877908 =$  the area nearly the same as before.

## RULE V.

Draw  $AQ$  [see the last figure] so that  $DP:PQ::5:\sqrt{10}$ , then the segment  $ADBA$  will be nearly equal to  $\frac{4}{3}DP \times AQ = \frac{4}{3}v \sqrt{dv - \frac{3}{5}vv} = \frac{4}{3}v \sqrt{cc - \frac{3}{5}vv} = \frac{4}{3}v \sqrt{\frac{1}{4}CC + \frac{2}{5}vv}$ . Where  $d$  = the diameter,  $v=DP$ ,  $c=DA$ , and  $C=AB$ , as in the last rule.\*

2 D

Ex-

Suppose it  $= 2DP \times m \times PA + n \times AD = 2v \times m \sqrt{dv - vv} + n \sqrt{dv} = 2v \sqrt{dv} \times m \sqrt{1 - \frac{v}{d}} + n = 2v \sqrt{dv} \times \left( \frac{m}{1} - \frac{mv}{2d} - \frac{mv^2}{8d^2} \right) \&c.$

Then let the coefficients of the corresponding terms be equated, and we shall have  $m+n = \frac{2}{3}$ , and  $\frac{m}{2} = \frac{1}{5}$ ; whence  $m = \frac{2}{5}$ , and  $n = \frac{2}{3} - m = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$ .

Then, by substituting these values of  $m$  and  $n$  in the assumed quantity, we shall have  $2DP \times m \times PA + n \times AD = 2DP \times \frac{2}{5}PA + \frac{4}{15}AD = \frac{4}{15}DP \times 2PA + \frac{4}{15}AD$ , which is the rule. Or it is  $= \frac{4}{15}DP \times 3PA + 2AD$ . And this is one of SIR I. NEWTON'S rules published by JONES, WALLIS, and COLSON.

## \* DEMONSTRATION of RULE V.

By rule 2 the segment is  $= 2v \sqrt{dv} \times \left( \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2} \right) \&c.$

Suppose it equal to  $2v \sqrt{mdv - nvv} = 2v \sqrt{dv} \times \sqrt{m - \frac{nv}{d}} = 2v \sqrt{dv} \times \left( m^{\frac{1}{2}} - \frac{nv}{2m^{\frac{1}{2}}d} - \frac{n^2v^2}{8m^{\frac{3}{2}}d^2} \right) \&c.$  Then, by equating the coefficients, we have  $m^{\frac{1}{2}} = \frac{2}{3}$ , and  $\frac{n}{2m^{\frac{1}{2}}} = \frac{1}{5}$ ; whence  $m = \frac{4}{9}$ , and  $n = \frac{4}{15}$ .

Then, by substituting these values, we get  $2v \sqrt{mdv - nvv} = 2v \sqrt{\frac{4}{9}dv - \frac{4}{15}vv} = \frac{4}{3}v \sqrt{dv - \frac{3}{5}vv}$ , as in the rule. COROL.



## EXAMPLE.

Taking, again, the same example;

We have  $\frac{4}{3}v\sqrt{\frac{1}{4}CC+\frac{2}{3}vv}=\frac{8}{3}\sqrt{101\frac{1}{3}}=26.87915$ =the area nearly.

## RULE VI.

If  $AQ$  bisect  $DP$ , the segment  $ADBA$  will be very nearly equal to  $4AQ+AD \times \frac{4}{15}DP = 2\sqrt{CC+vv+c} \times \frac{4}{15}v = 2\sqrt{4dv-3vv} + \sqrt{dv} \times \frac{4}{15}v = 2\sqrt{CC+vv} + \sqrt{\frac{1}{4}CC+vv} \times \frac{4}{15}v$ ; where the letters denote the same quantities as in the last rule.\* [See the last figure.]

EX-

## COROLLARY.

But, now, to find what line the quantity  $\sqrt{dv-\frac{3}{4}vv}$  represents; I consider that it is greater than  $AP$  which is  $=\sqrt{dv-vv}$ , and therefore I suppose it to be some line  $AQ$  meeting  $DP$  between  $D$  and  $P$ .

Then  $PQ = \sqrt{AQ^2 - AP^2} = \sqrt{dv - \frac{3}{4}vv - dv + vv} = v\sqrt{\frac{2}{5}} = \frac{v\sqrt{10}}{5}$ .

Wherefore, if there be made  $DP:PQ::5:\sqrt{10}$ , and  $AQ$  be drawn; the segment will be nearly  $\frac{4}{3}DP \times AQ$ . And this is another of SIR I. NEWTON'S Rules.

## \* DEMONSTRATION of RULE VI.

For suppose the segment  $2v\sqrt{dv} \times \frac{2}{3} - \frac{v}{5d}$  &c. to be  $= 2PD \times$

$$m \times AQ + n \times AD = 2v \times m \sqrt{dv - \frac{3}{4}vv} + n \sqrt{dv} = 2v \sqrt{dv} \times m \sqrt{1 - \frac{3v}{4d}} + n \\ = 2v \sqrt{dv} \times \left( \frac{m}{1} - \frac{3mv}{8d} \right) \text{ \&c.}$$

Then, equating the coefficients, we get  $m+n=\frac{2}{3}$ , and  $\frac{3m}{8}=\frac{1}{5}$ ; whence  $m=\frac{8}{15}$ , and  $n=\frac{2}{15}$ .

Therefore  $2PD \times m \times AQ + n \times AD = 2PD \times \frac{8}{15}AQ + \frac{2}{15}AD = \frac{4}{15}PD \times 4AQ + AD$ , as in the rule. And this likewise is a rule given by SIR I. NEWTON.



## EXAMPLE.

Taking, still, the same example.

We shall have  $2\sqrt{CC+vv} + \sqrt{\frac{1}{4}CC+vv} \times \frac{4}{15}v = 26.8786888 =$  the area very nearly.

## RULE VII.

From the square of the chord of half the arc subtract  $\frac{5}{7}$  of the square of the versed sine of the said half arc, to 7 times the root of the difference add  $\frac{4}{3}$  of the said chord, multiply the sum by the said versed sine, and  $\frac{4}{25}$  of the product will be the area extremely near.

That is,  $7\sqrt{cc-\frac{5}{7}vv} + \frac{4}{3}c \times \frac{4}{25}v = 7\sqrt{dv-\frac{5}{7}vv} + \frac{4}{3}\sqrt{dv} \times \frac{4}{25}v = 7\sqrt{\frac{1}{4}CC+\frac{2}{7}vv} + \frac{4}{3}\sqrt{\frac{1}{4}CC+vv} \times \frac{4}{25}v =$  the area extremely near: where the letters are the same as in the last rules.\*

Ex-

## \* DEMONSTRATION.

The segment is  $= 2v\sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2} \&c.$

Suppose it  $= 2v \times m\sqrt{dv-nvv} + p\sqrt{dv} = 2v\sqrt{dv} \times p + m\sqrt{1-\frac{nv}{d}}$   
 $= 2v\sqrt{dv} \times \frac{p}{+m} - \frac{mnv}{2d} - \frac{mn^2v^2}{8d^2} \&c.$

Then, equating the like coefficients, we have  $m+p = \frac{2}{3}$ ,  $\frac{mn}{2} = \frac{1}{5}$ ,  
 and  $\frac{mn^2}{8} = \frac{1}{28}$ ; whence  $n = \frac{5}{7}$ ,  $m = \frac{14}{25}$ , and  $p = \frac{8}{75}$ .

Wherefore  $2v \times m\sqrt{dv-nvv} + p\sqrt{dv} = \frac{4}{25}v \times 7\sqrt{dv-\frac{5}{7}vv} + \frac{4}{3}\sqrt{dv}$ , as in the rule.

## COROLLARY.

Suppose  $\sqrt{dv-\frac{5}{7}vv} = A\mathcal{Q}$  [see last fig.]; then  $P\mathcal{Q} = \sqrt{\mathcal{Q}^2 - AP^2}$   
 $= \sqrt{dv-\frac{5}{7}vv-dv+vv} = v\sqrt{\frac{2}{7}} = \frac{v\sqrt{14}}{7}.$

Wherefore if  $A\mathcal{Q}$  divide  $DP$  in  $\mathcal{Q}$  so that  $7:\sqrt{14}::DP:P\mathcal{Q}$ ; then  $7\mathcal{Q}A + \frac{4}{3}AD \times \frac{4}{25}DP$  will be nearly equal to the segment  $ADBA$ . And this rule is not only new and very simple, but also the most exact of any approximation that I know of for this purpose.



## EXAMPLE.

Taking, again, the same example as before, where  $C=20$ , and  $v=2$ ;

We have  $7\sqrt{\frac{1}{4}CC} + \frac{2}{7}vv + \frac{4}{3}\sqrt{\frac{1}{4}CC} + vv \times \frac{4}{25}v = 26.8787998 = \text{the area very exact.}$

## RULE VIII.

1. Divide the versed sine or height by the diameter.

2. Find the quotient in the column of versed sines (marked *V. S.*) in the table of circular segments at the end of the book.

3. Multiply the number immediately on the right of the versed sine, in the table, by the square of the diameter, and the product will be the area.\*

*Note.*

## \* DEMONSTRATION.

This rule is founded on this property, That segments whose versed sines are as the diameters, will be to each other as the squares of the diameters, which is thus proved.

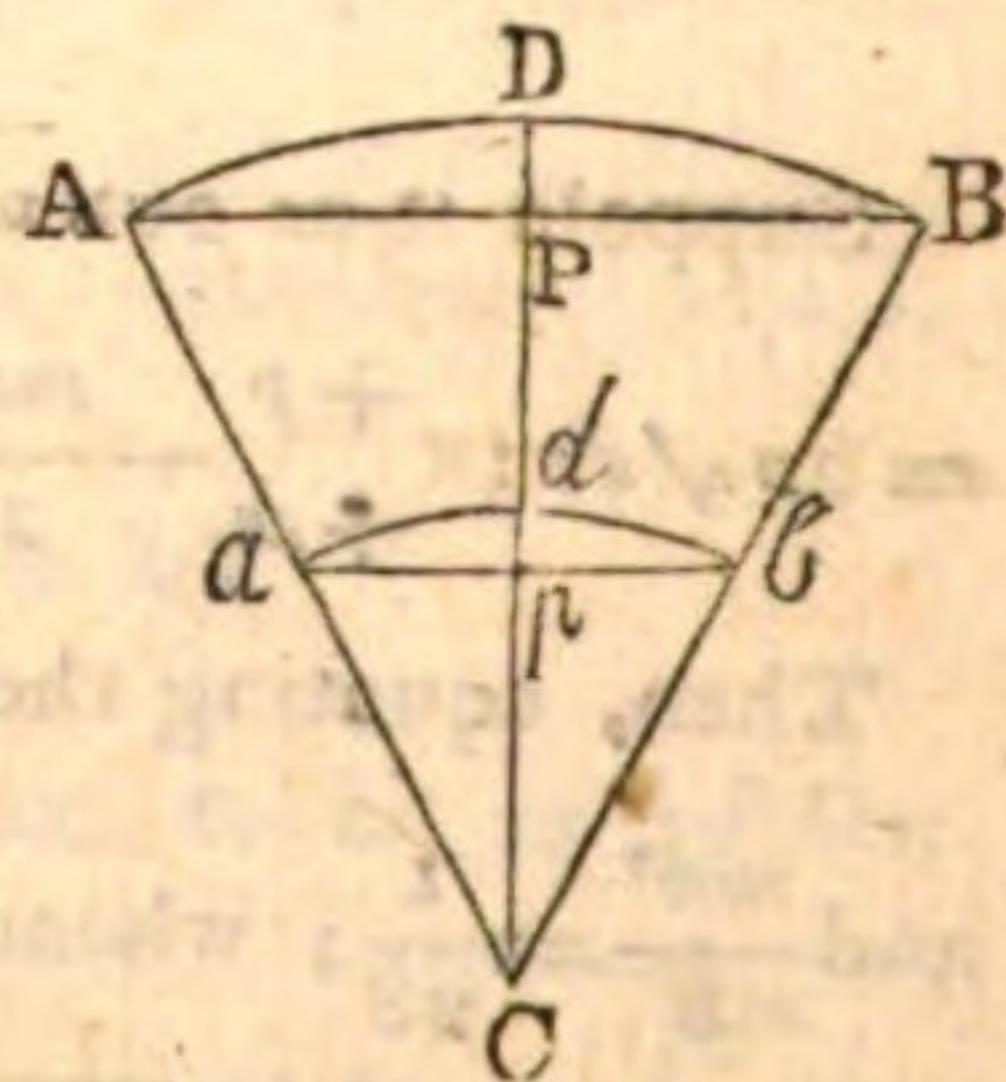
Let  $ADBA$ ,  $adba$ , be two similar segments cut off from the similar sectors  $ADBCA$ ,  $adbCa$ , by the chords  $AB$ ,  $ab$ . And draw  $CD$  to bisect them.

Then, by similar triangles,  $CA : Ca :: CA - DP (CP) : Ca - dp (Cp) :: DP : dp$ . That is, similar segments have their versed sines as their radiuses or diameters.

Again, since similar sectors are as the squares of the diameters as well as similar triangles as the squares of their like sides, we shall have  $CA^2 : Ca^2 :: \text{sector } CADB : \text{sector } Cadb :: \triangle CAB : \triangle Cab :: \text{segment } ADBA (\text{sector } CADB - \triangle CAB) : \text{segment } adba (\text{sector } Cadb - \triangle Cab)$ .

Now, putting  $d = \text{any diameter}$ , and  $v = \text{versed sine}$ . We shall have  $d : v :: 1 (\text{diameter in the table}) : \frac{v}{d} = \text{the versed sine of a similar segment in the table, whose area let be called } d$ .

Then  $1^2 : dd :: a : add = \text{the area of the segment whose height is } v \text{ and diameter } d, \text{ as in the rule.}$





*Note.* When the quotient arising from the versed sine divided by the height hath a remainder or fraction after the fourth place of decimals, having taken the area opposite to the said four first figures, subtract it from the next following area, multiply the remainder by the said fraction, and add the product to the first area; and the sum will be the area for the whole quotient.—This method is used when accuracy is required; but for common use the fraction may be entirely omitted.

## E X A M P L E.

If the diameter be 52 and versed sine 2; what is the area?

First  $\frac{2}{52} = \frac{1}{26} = .0384\frac{2}{13}$  the tabular versed sine.

Then against  $.0384$  stands  $.00991672$ , and the difference between this area and the next  $.00995517$  is  $.00003845$ , which multiplied by  $\frac{2}{13}$  produces  $.00002366$ , which added to  $.00991672$  gives  $.00994038$  = the area belonging to the versed sine  $.0384\frac{2}{13}$ .

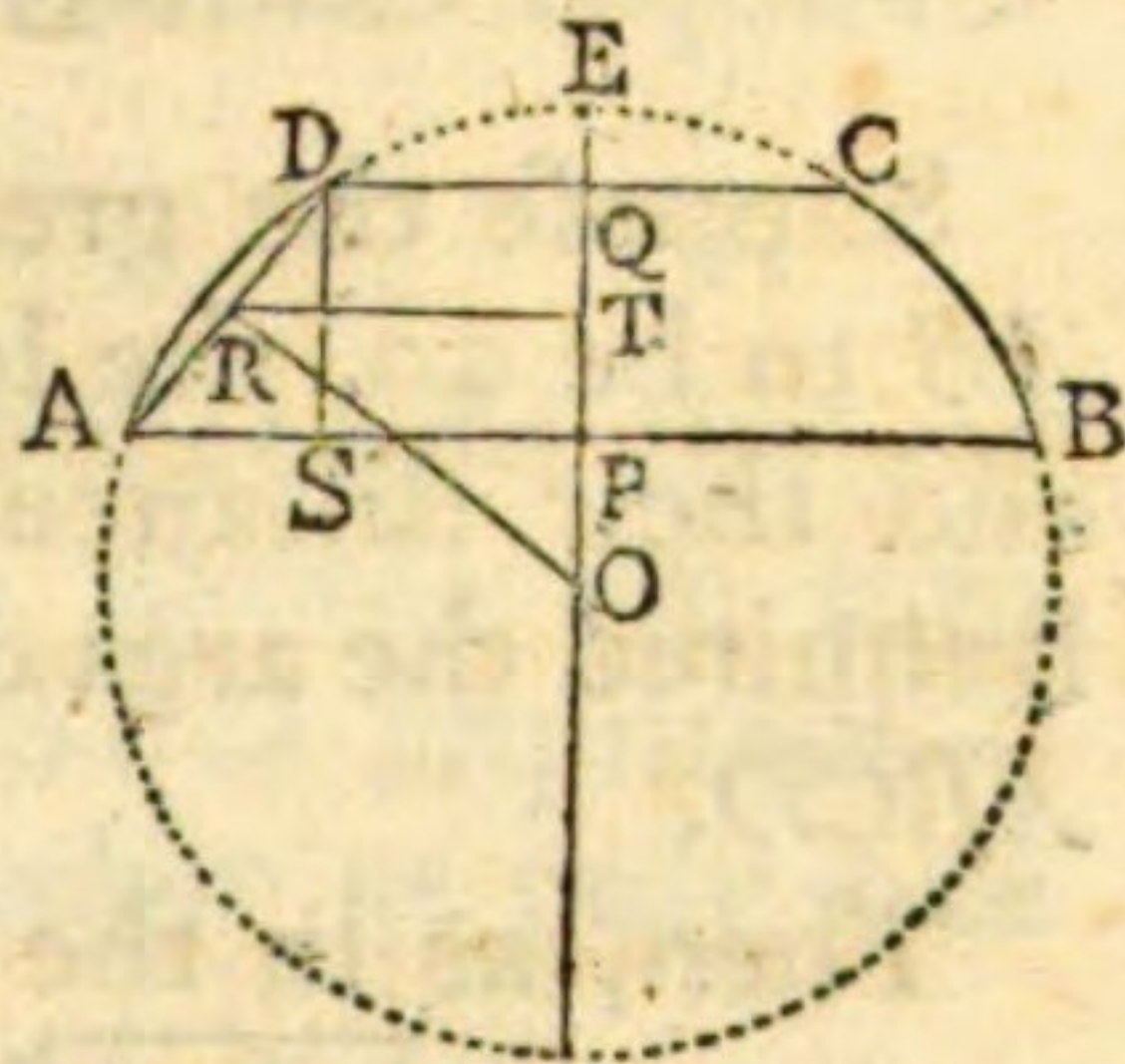
Wherefore  $.00994038 \times 52 \times 52 = 26.87878752$  = the area, nearly the same as before.

## P R O B L E M X.

To find the area of the zone or space *ABCD A* included between two parallel chords *AB*, *CD*, and the two arcs *AD*, *CB*.

## R U L E I.

Find the area of each segment *ABEA*, *DCED*, and their difference will be that of the zone required.



## E X A M P L E I.

Suppose the greater chord to be 96, the less 60, and the distance between them = 26; required the area.

2 E

Draw



Draw  $AD$ , and let  $DS$  be  $\perp AP$ ,  $OR \perp AD$ , and  $RT \perp PQ$ ; then  $QT = TP$ , because  $AR = RD$ ; and  $TR = \frac{AP + DQ}{2}$ . But, in the similar triangles  $DAS$ ,  $ROT$ , it will be  $DS : SA :: RT : TO = \frac{RT \times SA}{DS} = \frac{PA + DQ \times PA - DQ}{2PQ} = \frac{78 \times 18}{52} = 27$ . And hence  $OP = OT - TP = 27 - 13 = 14$ .

Then  $EO = OA = \sqrt{AP^2 + PO^2} = \sqrt{48^2 + 14^2} = 50$  = the radius.

Whence  $\begin{cases} PE = EO - OP = 50 - 14 = 36 \\ QE = EO - OQ = 50 - 40 = 10 \end{cases}$  the two versed sines.

And  $\begin{cases} \frac{36}{50} = .72 \\ \frac{10}{50} = .2 \end{cases}$  = the two tab. versed sines. Against

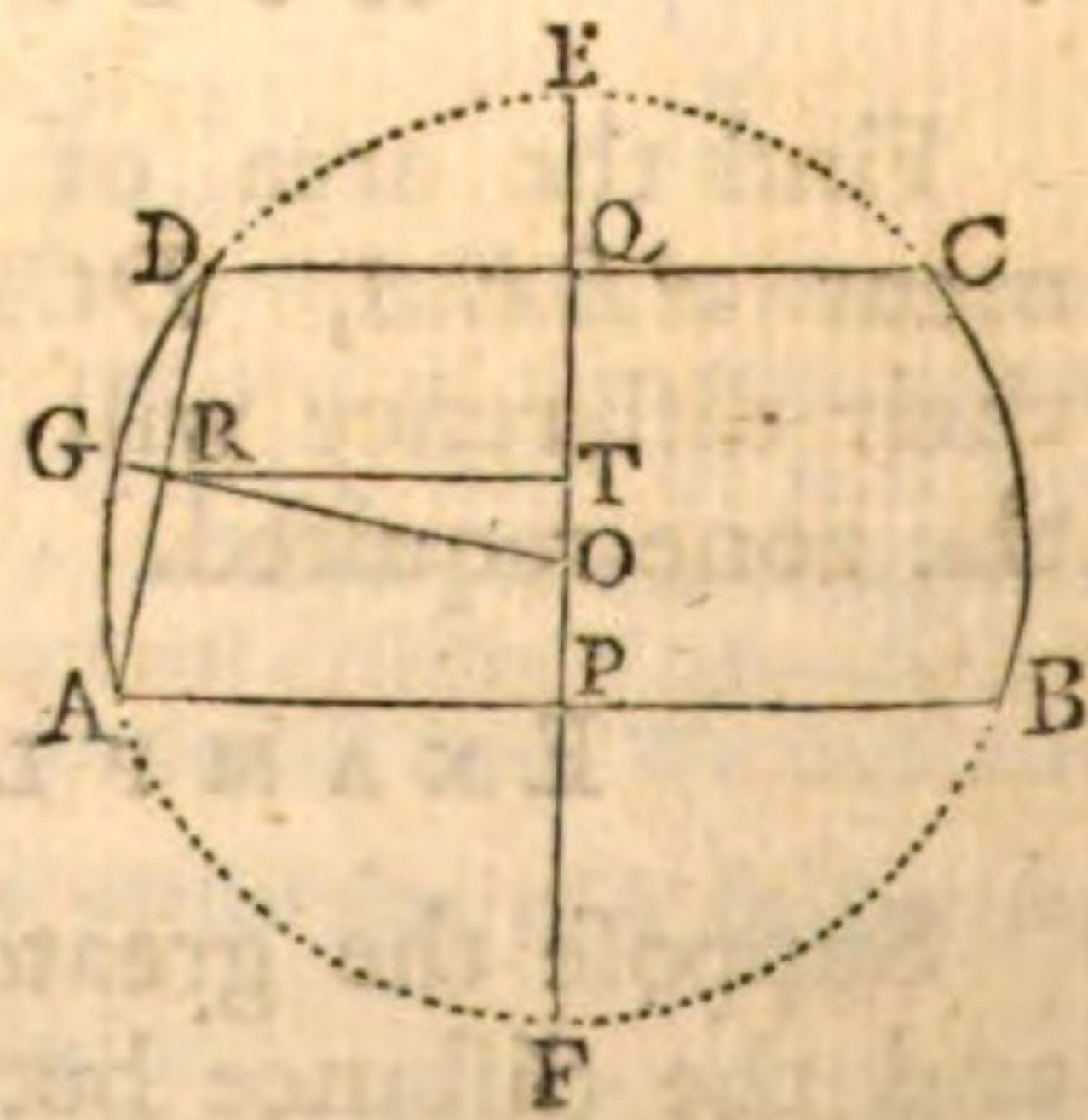
which, in the table, stand the areas  $\begin{cases} .25455055, \\ .04087528. \end{cases}$

Then  $\begin{cases} .25455055 \times 100 \times 100 = 2545.5055 \\ .04087528 \times 100 \times 100 = 408.7528 \end{cases}$   
their difference  $2136.7527$  = the area of the zone required.

#### EXAMPLE II.

Suppose the greater chord  $AB$  to be 20, the less  $DC$  15, and their distance  $PQ$   $17\frac{1}{2}$ . Required the area of the zone  $ABCD$ .

Then, as in the last example,  $TO = \frac{PA + DQ \times PA - DQ}{2PQ} = \frac{35 \times 5}{2 \times 2 \times 35} = \frac{5}{4}$ . Which being less



than



than  $TP$  ( $8\frac{1}{4}$ ) half the distance of the chords, shews that the center falls within the zone; and then it is evident that the whole circle diminished by the sum of the two segments  $AFB$ ,  $DEC$ , will give the zone.

Now,  $OP = OT - TO = 8\frac{3}{4} - \frac{1}{4} = 7\frac{1}{2}$ . Hence  $AO = \sqrt{OP^2 + PA^2} = \sqrt{\left(\frac{15}{2}\right)^2 + 10^2} = \frac{25}{2} =$  the radius; therefore the diameter is 25. And  $OQ = QT + TO = 8\frac{3}{4} + 1\frac{1}{4} = 10$ .

Whence  $\begin{cases} QE = EO - OQ = 12\frac{1}{2} - 10 = 2\frac{1}{2}, \\ PF = FO - OP = 12\frac{1}{2} - 7\frac{1}{2} = 5. \end{cases}$

Then  $\begin{cases} \frac{2\frac{1}{2}}{25} = \frac{5}{50} = \frac{1}{10} = .1 \\ \frac{5}{25} = \frac{1}{5} = .2 \end{cases}$  the tab. vers. fines.

Against which stand the areas  $\begin{cases} .04087528, \\ .11182380. \end{cases}$

Then  $\begin{cases} .04087528 \times 25 \times 25 = \text{the segment } DEC, \\ .11182380 \times 25 \times 25 = \text{the segment } AFB, \end{cases}$   
whose sum is  $.15269908 \times 25^2$ .

But  $.78539816 \times 25 \times 25 =$  the whole circle.

Therefore  $25^2 \times .78539816 - .15269908 = 25^2 \times .63269908 = 395.436925 =$  the zone  $ABCD$  required.

#### R U L E II.

To the area of the trapezoid  $APQD$  add that of the small segment  $AGDRA$ , and double the sum will be the area of the zone  $ABCD$  required.— And this rule is easier than the other as only one segment is to be found.

#### E X A M P L E.

Suppose the same things as in the last example.

Then,



Then, drawing the radius  $ORG$ , we shall have  $OR$   
 $= \sqrt{RT^2 + TO^2} = \sqrt{\left(\frac{DQ + PA}{2}\right)^2 + TO^2} = \sqrt{\left(\frac{35}{4}\right)^2 + \left(\frac{5}{4}\right)^2} =$   
 $\frac{1}{4}\sqrt{7^2 + 1^2} = \frac{1}{4}\sqrt{50} = \frac{5}{4}\sqrt{2} = 8.838834765$ . Hence  
 $RG = GO - OR = 12.5 - 8.838834765 = 3.661165$  the  
 versed sine.

Therefore  $\frac{3.661165}{25} = .1464466 =$  the tabular versed  
 sine; the tabular area for which is  $.07134954$ .

Then  $.07134954 \times 25 \times 25 = 44.5934625 =$  the seg-  
 ment  $AGDRA$ .

But  $\frac{DQ + PA}{2} \times PQ = \frac{35}{4} \times \frac{35}{2} = \frac{1225}{8} = 153.125 =$  the  
 trapezoid  $APQD$ .

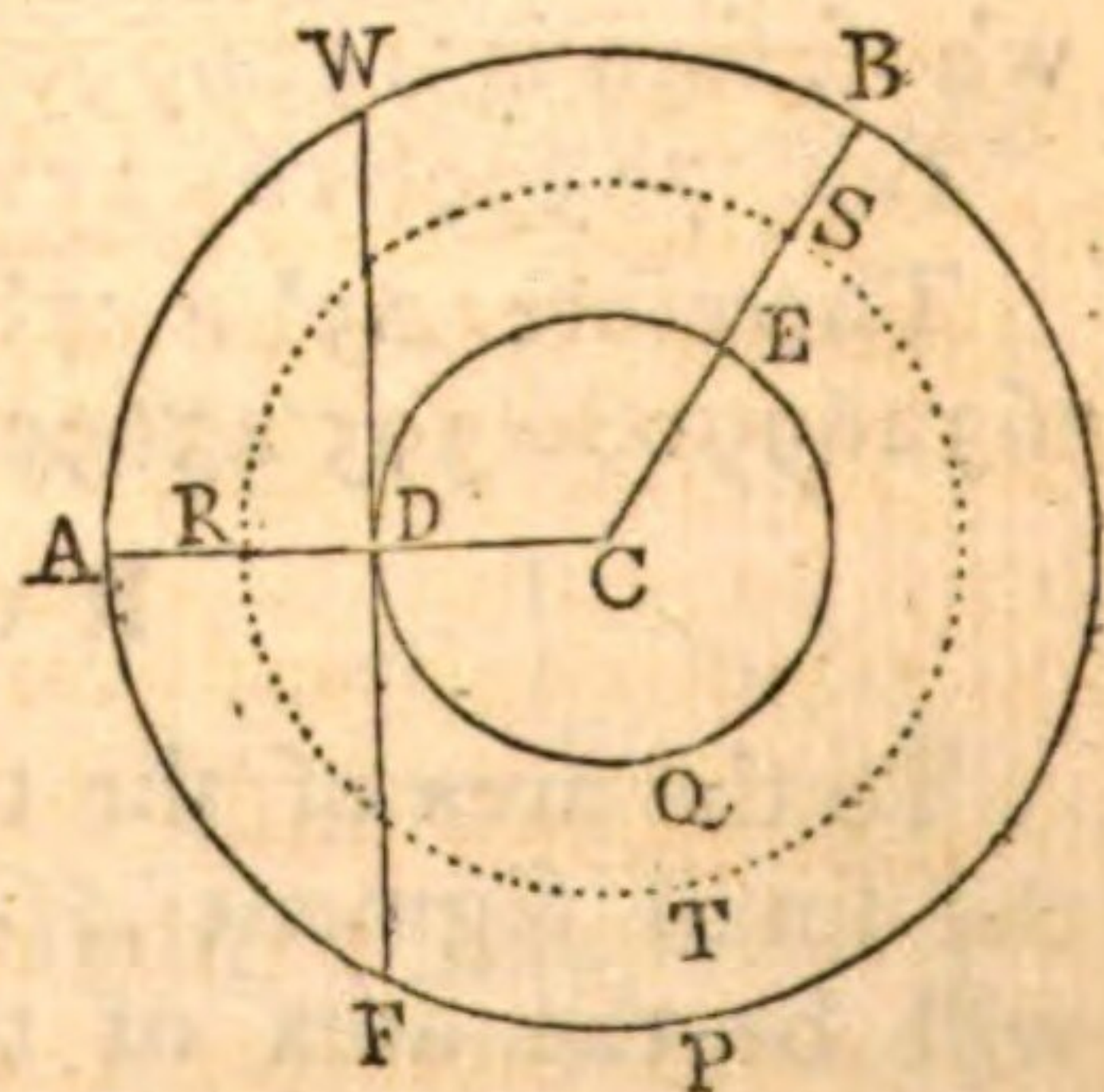
Therefore  $153.125 + 44.5934625 = 197.7184625 =$   
 the semi-segment  $APQDGA$ . The double of which is  
 $395.436925 =$  the zone  $ABCD$ , the same as before.

#### PROBLEM XI.

To find the area of the ring  
 included between the circum-  
 ferences  $ABPA$ ,  $DEQD$  of  
 two concentric circles.

#### RULE I.

Multiply the sum of the  
 diameters by the difference  
 of the diameters, and the  
 product drawn into  $.7854$  will be the area of the  
 ring required.\*



Ex-

#### \* DEMONSTRATION.

\* For, since the ring is evidently equal to the difference of the  
 two circles, if the diameters be called  $D$ ,  $d$ , and  $.785398 \&c. = a$ ;  
 the ring will be  $= aD^2 - ad^2 = a \times \overline{D+d} \times \overline{D-d}$ . Q.E.D. Co-



## EXAMPLE.

If the diameters of two concentric circles be 10 and 6, what will be the area of the ring included between their circumferences?

$\cdot 7854 \times 10 + 6 \times 10 - 6 = \cdot 7854 \times 16 \times 4 = 50 \cdot 2656 =$  the area required.

## RULE II.

Multiply half the sum of the circumferences by half the difference of the diameters, and the product will be the area.\*

## EXAMPLE.

Taking the same example as before, we shall have

First  $3 \cdot 1416 \times \left\{ \begin{array}{l} 10 = 31 \cdot 416 \\ 6 = 18 \cdot 8496 \end{array} \right\} =$  the circumferences.

Then  $\frac{31 \cdot 416 + 18 \cdot 8496}{2} \times \frac{10 - 6}{2} = \frac{50 \cdot 2656}{2} \times \frac{4}{2} = 50 \cdot 2656 =$  the area the same as before.

2 F

Note.

## COROLLARY I.

Whence, if there be made  $DW$  perpendicular to the radius  $CDA$ , the ring will be equal to the circle whose radius is  $DW$ ; or equal to the circle whose diameter is  $FW$  a tangent to the less circle, and terminated by the greater.

## COROLLARY II.

Or the ring is equal to an ellipse whose axes are  $D+d$  and  $D-d$ . As will appear by comparing the above rule with the rule for an ellipse to be hereafter given.

## \* DEMONSTRATION of RULE II.

For the circumferences  $C, c$ , are equal to  $4aD, 4ad$ ; therefore  $a \times \overline{D+d} = \frac{C+c}{4}$ ; which substituted in the last rule, gives  $a \times \overline{D+d} \times \overline{D-d} = \frac{C+c}{4} \times \overline{D-d} = \frac{C+c}{2} \times \frac{D-d}{2}$ , as in the rule. Q.E.D.

And it is evident that the same rule will serve for a part of the ring cut off by two radiuses, using the lengths of the included curves for  $C$  and  $c$ .



*Note.* This rule will serve, also, for any part  $ABEDA$  of the ring included between the parts  $AD$ ,  $BE$ , of two radiuses, by using half the sum of their intercepted arcs for half the sum of the circumferences.

Thus, if the length of the arc  $AB$  be 15; then, by reason of the similarity of the arcs  $AB$ ,  $DE$ , it will be  $10:6::15:\frac{15 \times 6}{10} = 3 \times 3 = 9$  = the arc  $DE$ .

Whence  $\frac{15+9}{2} \times \frac{10-6}{2} = 12 \times 2 = 24$  = the area of the part  $ABEDA$ .

### R U L E III.

Multiply the perpendicular breadth of the ring, that is, the difference of the radiuses, by the circumference  $RST$  (or part  $RS$  for the part  $ABEDA$ ) having the same center with, and equally distant from the bounding arcs.\*

### E X A M P L E.

Taking, still, the same example; it will be

First  $DA = AC - CD = 5 - 3 = 2$  = the distance of the circumferences.

And  $RC = CD + DR = CD + \frac{1}{2}DA = 3 + 1 = 4$  = the radius of the middle arc.

Hence  $3.1416 \times 8 = 25.1328$  = the middle circumference.

Wherefore  $25.1328 \times 2 = 50.2656$  = the area the same as before.

And for the part  $ABEDA$ , we have  $AC:CR::AB:SR = \frac{AB \times RC}{CA} = \frac{15 \times 4}{5} = 3 \times 4 = 12$ .

Then  $AD \times RS = 2 \times 12 = 24$  = the area.

P R O-

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### \* DEMONSTRATION.

For this circumference, being equally distant from the other two, will be equal to half their sum. Wherefore &c.



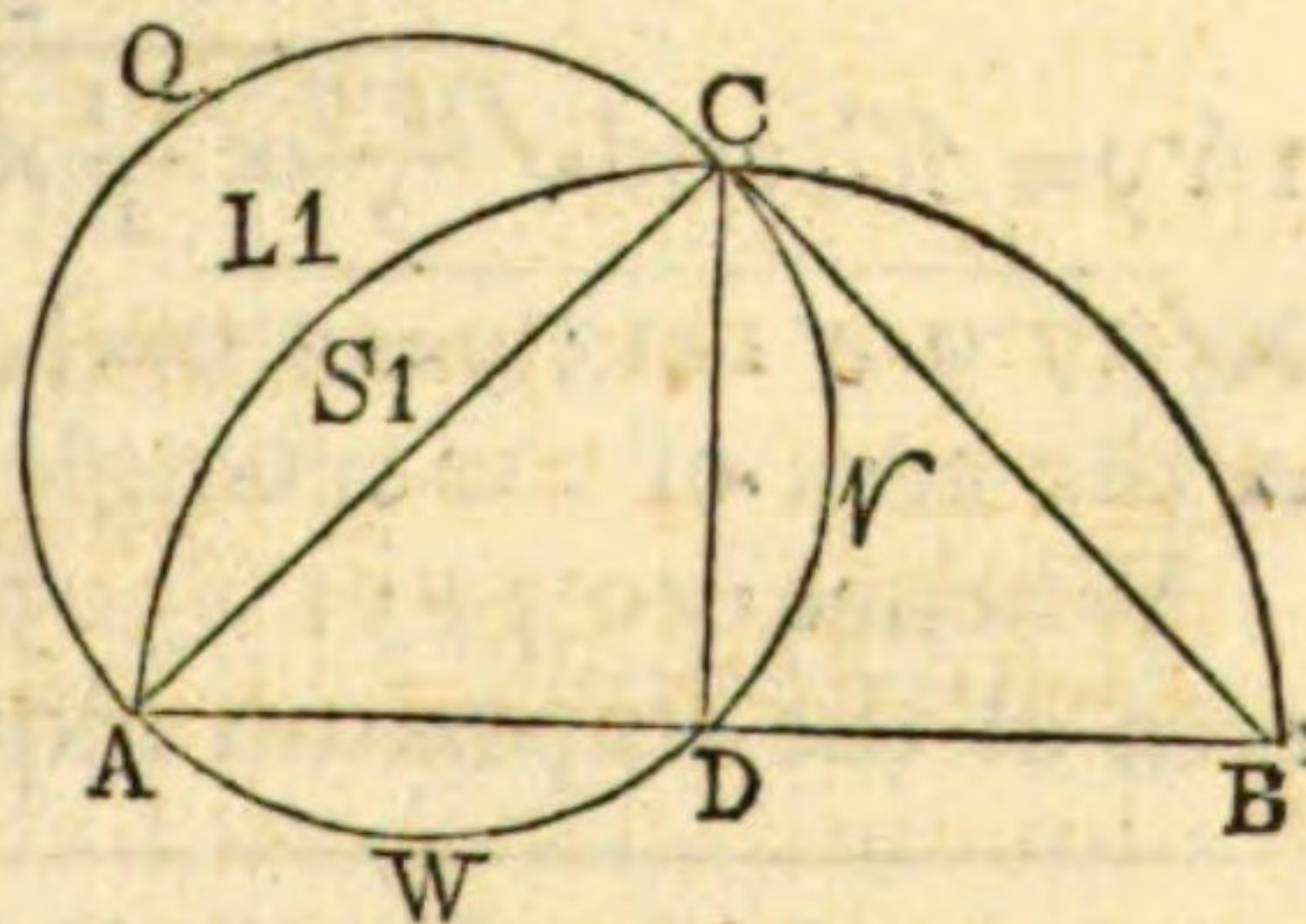
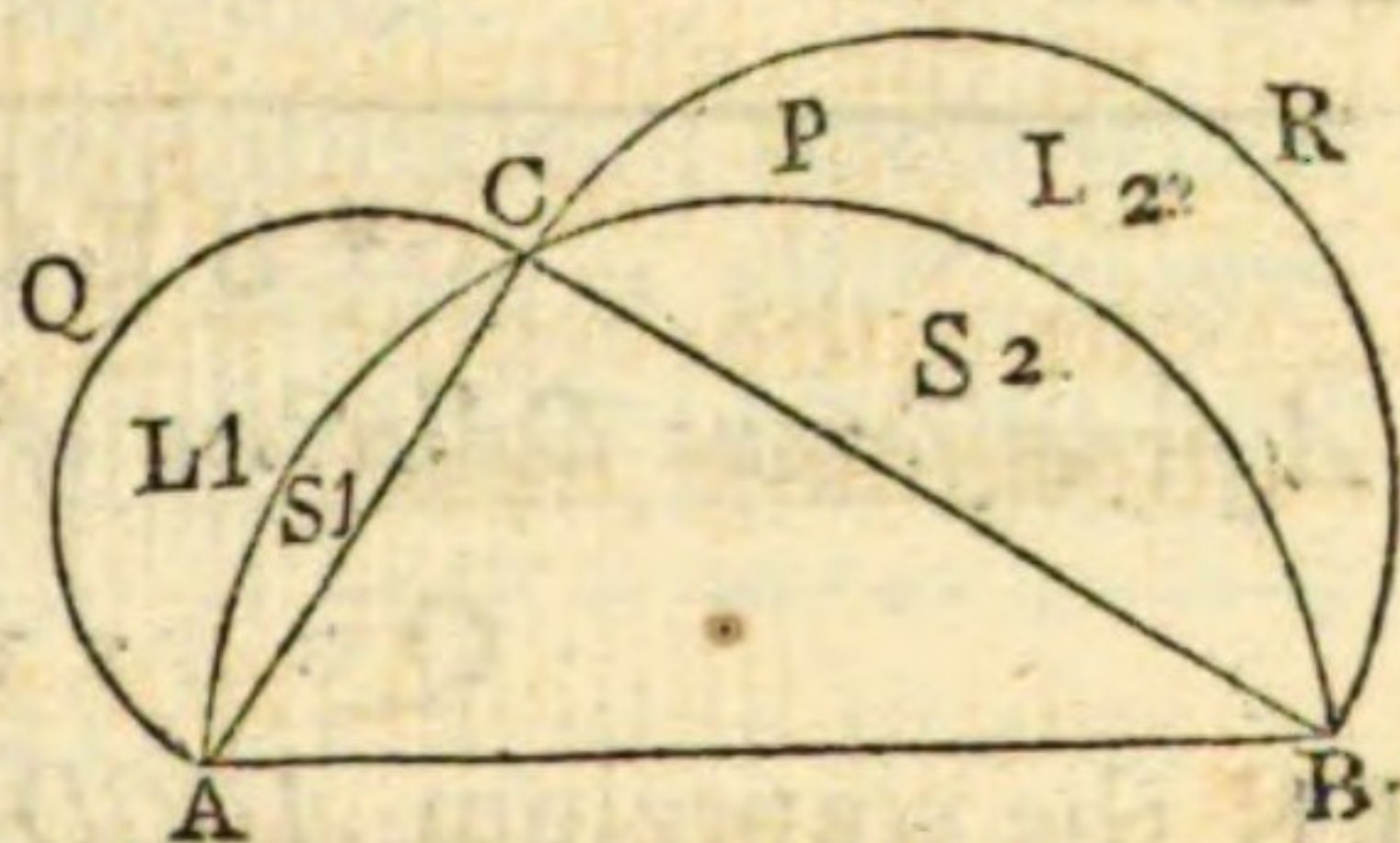
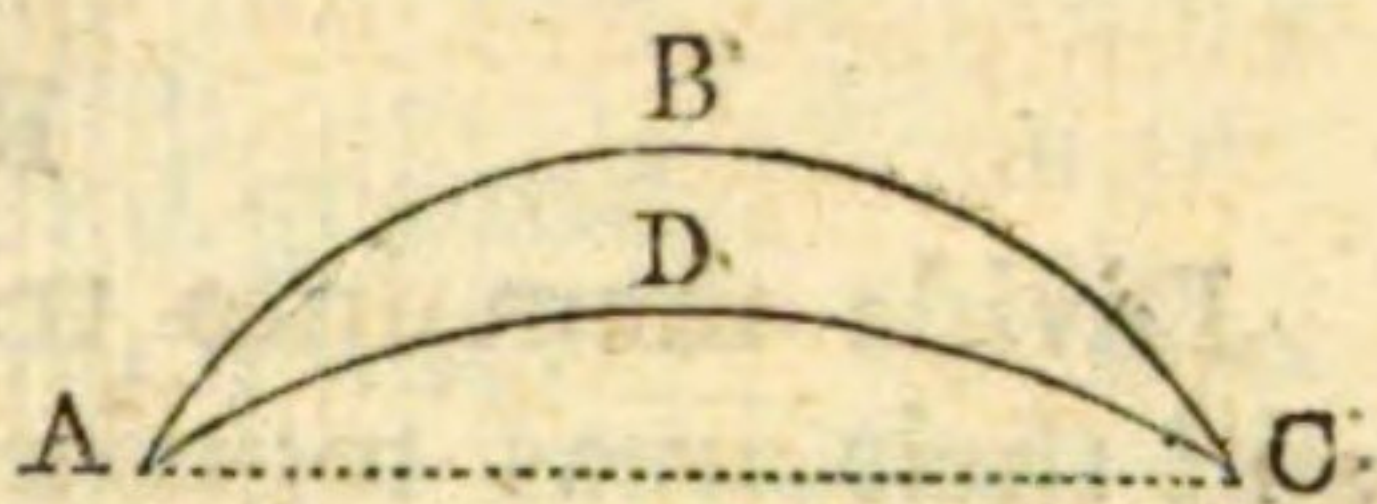
## PROBLEM XII.

To find the areas of lunes.

Lunes, *lunule*, or little-moons, are spaces included between the intersecting arcs of two excentric circles, as the lune  $ABCD$ ; which is, evidently, equal to the difference of the segments  $ABCA$ ,  $ADCA$ .

If  $ABC$  be a triangle right-angled at  $C$ , and if semicircles be described on the three sides as diameters; Then the triangle  $T$  ( $ABC$ ) will be equal to the sum of the two lunes  $L_1$ ,  $L_2$ .—For the greatest semicircle is equal to the sum of the other two; from the greatest semicircle take the segments  $S_1$ ,  $S_2$ , and there will remain the  $\Delta T$ ; from the two less semicircles take, also, the two segments  $S_1$ ,  $S_2$ , and there will remain the two lunes  $L_1$ ,  $L_2$ ; wherefore  $T = L_1 + L_2$ .

Whence, if the two sides  $AC$ ,  $CB$ , of the triangle be equal to each other; the two lunes will, also, be equal, and each lune  $L_1$  equal to the  $\Delta ACD$ ; and therefore the segment  $S_1$  = semicircle  $AQCA - \Delta ACD = 2$  times the seg.  $AWDA = 2$  times the seg.  $DVCD$ .





## PROBLEM XIII.

To find the areas of irregular polygonal figures, such as most pieces of land generally are.

## RULE.

Divide them into triangles and trapeziums, and add their areas together. But for the best methods of doing this, I must refer the reader to the latter part of this work, in which surveying is particularly treated of.

## \* SECT. II.

*A promiscuous Collection of Questions concerning Areas.*

## QUESTION I.

IN the trapezium  $ABCD$  are given  $AB = 6\frac{1}{2}$ ,  $BC = 15\frac{3}{4}$ ,  $CD = 12$ , and  $DA = 9$ , also  $B$  a right angle; to find the area of the trapezium.

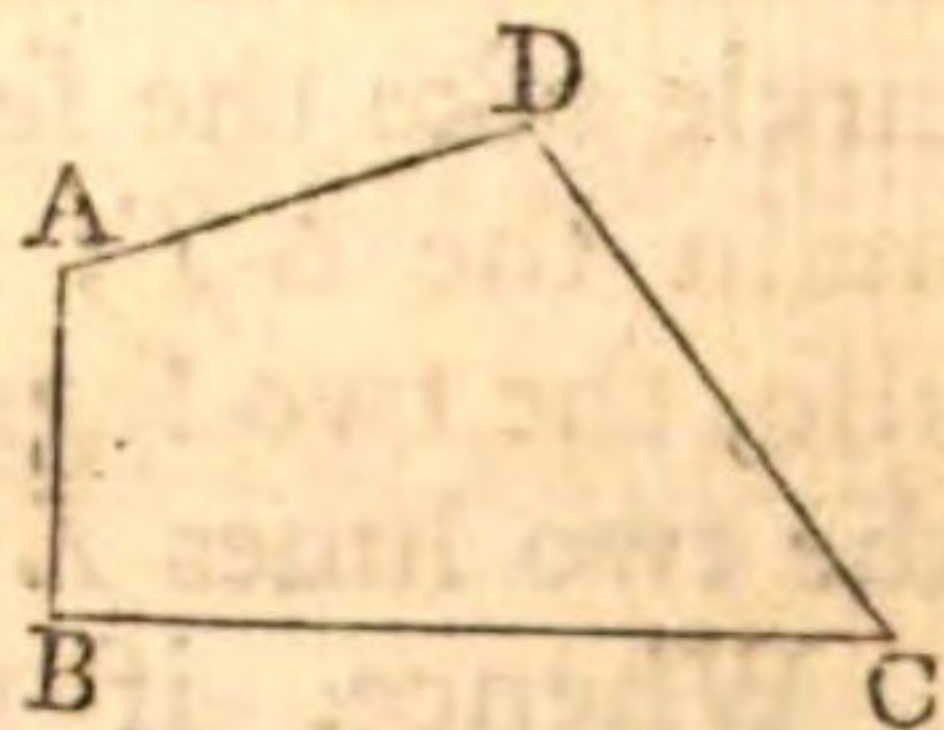
First  $\frac{6\frac{1}{2} \times 15\frac{3}{4}}{2} = \frac{13}{4} \times \frac{78}{5} = \frac{1014}{20} = 50.7$   
 = the area of the triangle  $ABC$ .

But  $\sqrt{AB^2 + BC^2} = \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{78}{5}\right)^2} =$   
 $16.9 = AC$ ; and  $\sqrt{\frac{37.9}{2} \times \frac{4.1}{2} \times \frac{13.9}{2} \times \frac{19.9}{2}} =$

$\frac{1}{4} \sqrt{37.9 \times 4.1 \times 13.9 \times 19.9} = \frac{1}{4} \sqrt{42982.4279} = 51.8305098$   
 = the area of the triangle  $ADC$ .

Whence  $50.7 + 51.8305098 = 102.5305098 =$  the area of the trapezium required.

QUES-



\* Some of the questions in this section are taken from other books; but the methods of solution are, generally, different from those used in the books from which they were taken, they being there mostly solved by an analytical process. And I have constructed those of which the constructions do not appear to be self-evident.



## QUESTION II.

One morning in may, I went to survey  
 As soon as bright sol I espi'd;  
 I measur'd around a four-corner'd ground,  
 I'th' margin's the length of each side:  
 The angle at *B* together with *D*,  
 An hundred and 80 degrees,  
 The meadow's content is all that I want;  
 Assist me, kind sirs, if you please.

Chains

$AB = 15.6$

$BC = 13.2$

$CD = 10.0$

$DA = 26.0$

Since a trapezium can be inscribed in a circle when the sum of two of its opposite angles is  $180^\circ$ ; therefore, by rule 5 of prob. 3 of sect. 1. ( $32.4$  being half the perimeter)  $\sqrt{32.4 - 15.6 \times 32.4 - 13.2 \times 32.4 - 10 \times 32.4 - 26} = \sqrt{16.8 \times 19.2 \times 22.4 \times 6.4} = 215.04$  square chains =  $21.504$  acres =  $21$  a.  $2$  r.  $0.64$  perches = the area required.

## QUESTION III.

In the pentangular field *ABCDE* are given  $AB = 14$ ,  $BC = 7$ ,  $CD = 10$ ,  $DE = 12$ ,  $EA = 5$ , and the diagonal  $AC = 17$  chains, also *E* a right angle; required the area.

First  $AD = \sqrt{DE^2 + EA^2} =$

$\sqrt{12^2 + 5^2} = 13.$

Then  $AE \times \frac{1}{2} ED = 5 \times 6 = 30 =$   
 the area of the triangle *ADE*.

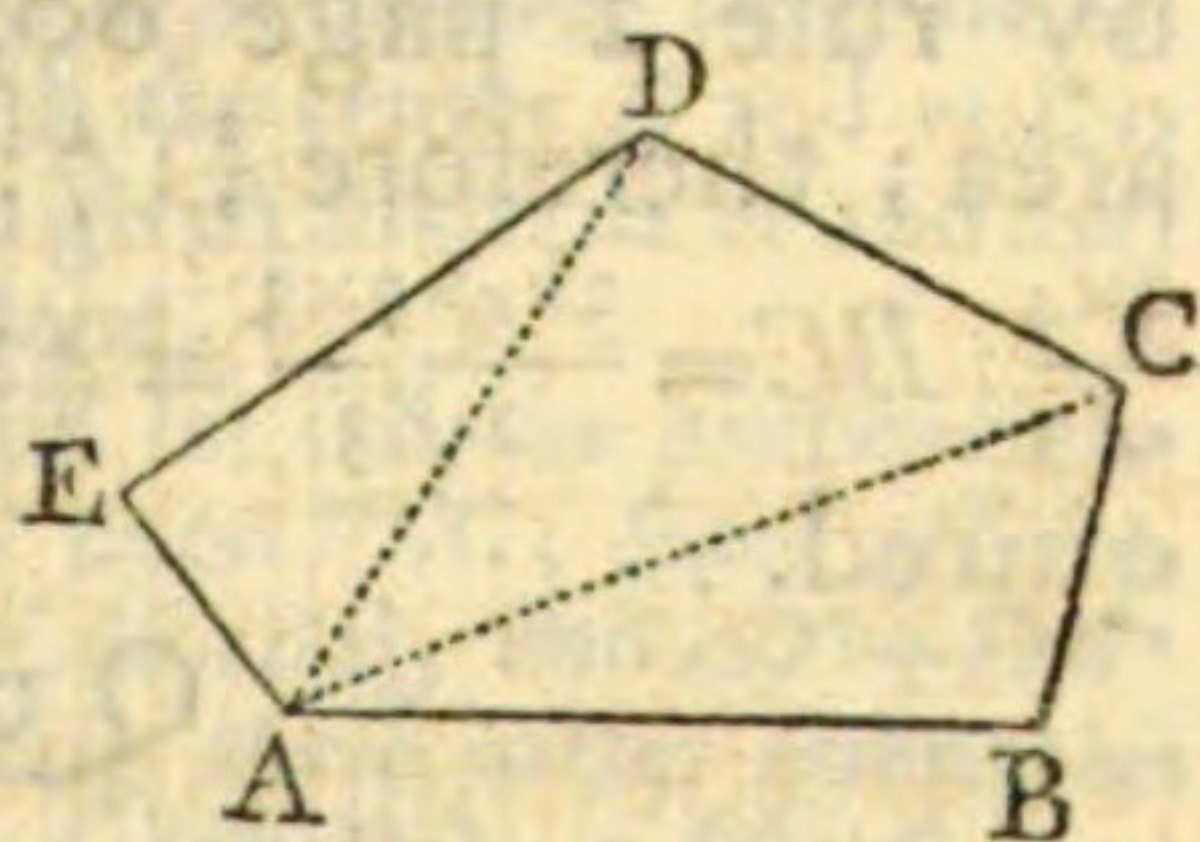
And  $\sqrt{20 \times 3 \times 7 \times 10} = 10\sqrt{42} = 64.80740698 = \Delta ADC.$

Also  $\sqrt{19 \times 2 \times 5 \times 12} = 2\sqrt{570} = 47.74934554 = \Delta ABC.$

Wherefore the sum of the three is  $142.55675252$  square chains =  $14$  a.  $1.0227$  r. = the area required.

2 G

QUES-

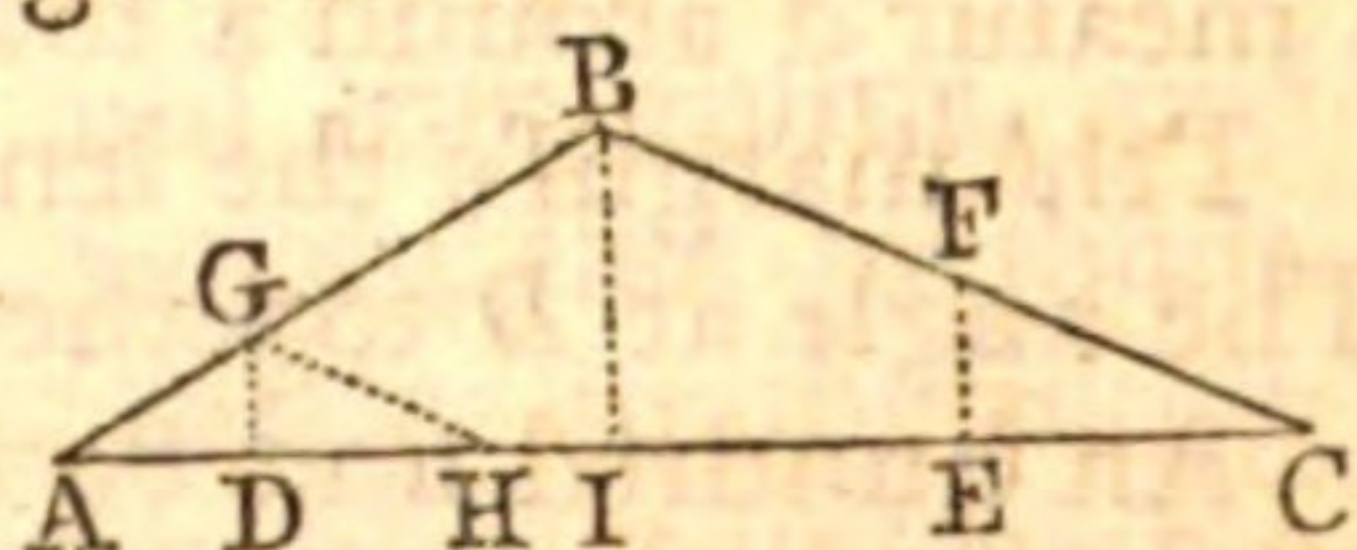




## QUESTION IV.

Given the base  $AC=32$ ,  $AD=5$ ,  $EC=9$ , the perpendicular  $EF=4$ , and the perpendicular  $DG=3$ ; required the area of the triangle  $ABC$ .

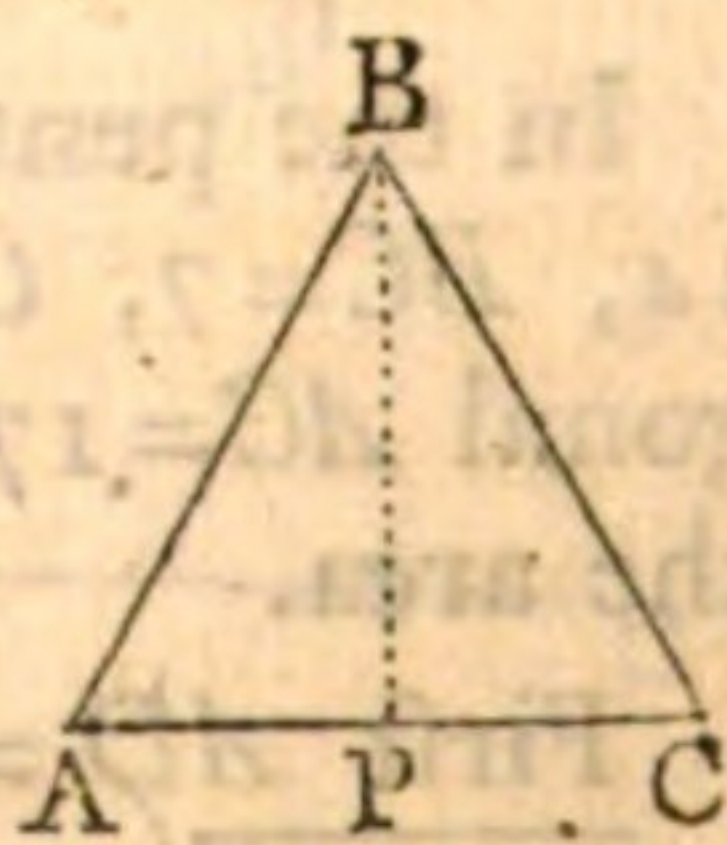
Draw  $GH$  parallel to  $BC$ ; then, by the similar triangles  $GDH$ ,  $FEC$ ,  $FE=4$ :  $EC=9::GD=3:DH=\frac{27}{4}$ , and hence  $AH=\frac{47}{4}$ ; again, from the similar triangles  $AGH$ ,  $ABC$ ,  $AH=\frac{47}{4}$ :  $AC=32::DG=3$ : the perpendicular  $IB=\frac{384}{47}$ . Hence  $\frac{1}{2}AC \times IB = 16 \times \frac{384}{47} = 130\frac{34}{47} = 130.7234\frac{2}{47}$  = the area of the triangle required.



## QUESTION V.

What is the side of that equilateral triangle, whose area cost as much paving at 8d. a foot, as the palli-fading the three sides did at a Guinea a yard?

The sides are 7s. a foot, and the area  $\frac{2}{3}$ s. a foot. And that the produces may be equal, the quantities must be inverfly as the prices; but, by rule 2 page 80,  $\frac{1}{4}BC^2\sqrt{3}$  = the area; therefore  $\frac{2}{3}:7::3BC:\frac{1}{4}BC^2\sqrt{3}::\frac{3 \times 4}{\sqrt{3}}:BC = \frac{3 \times 4 \times 7 \times 3}{2\sqrt{3}} = 42\sqrt{3} = 72.7461339$  = the side required.



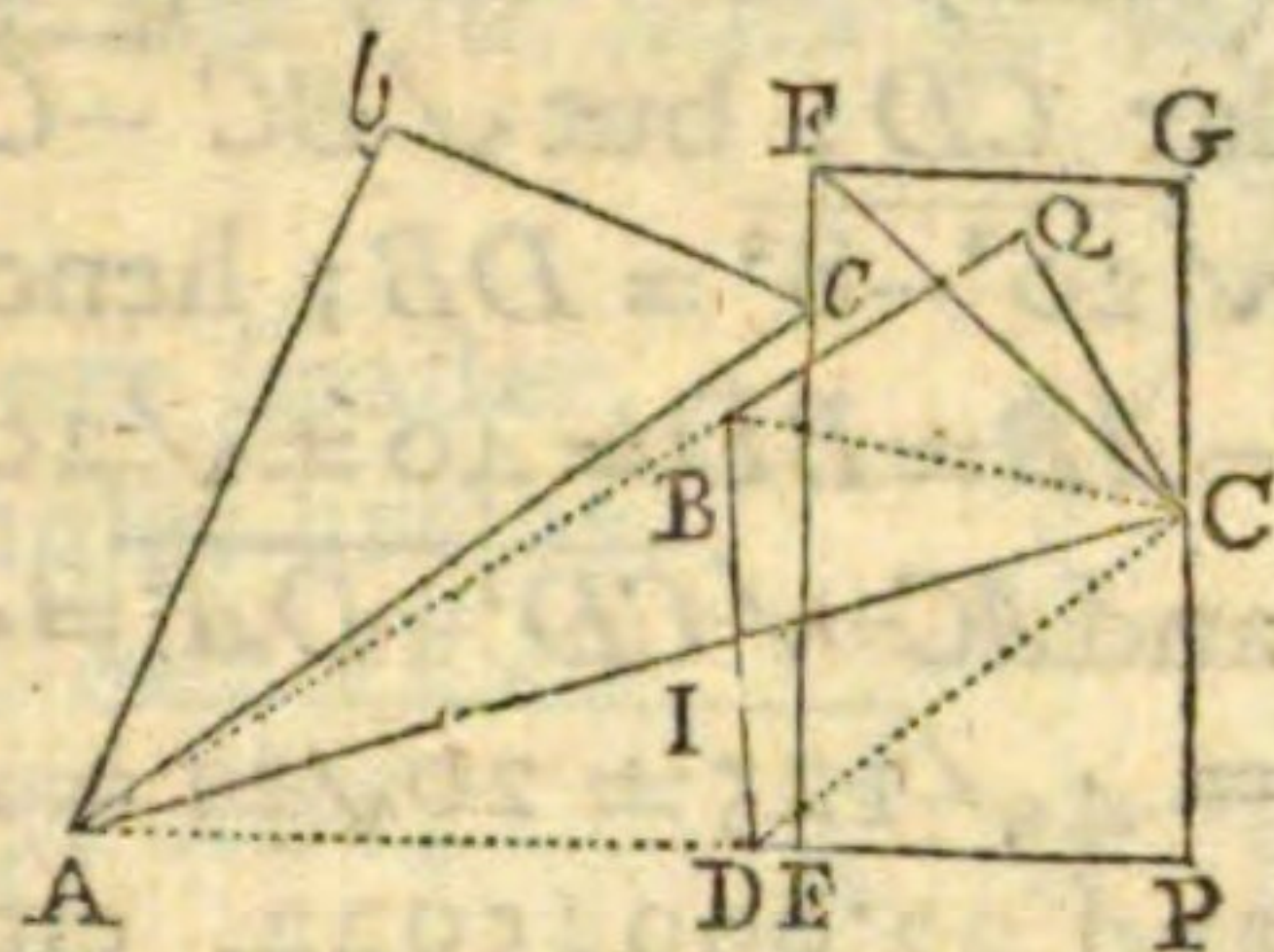
## QUESTION VI.

Surveying a field, I found the four sides thereof to be 10, 9, 7, and 6 chains, in a successive order: I likewise, at the two extremes of the longest side, took the bearings of the opposite angles, which were N. E. by



by *E.* and *N.W.* Hence the content of the field is required.

From the given bearings of the angles *C* and *D* from *A* and *B*, it appears that the diagonals *AC*, *BD*, make the angle *AID* at their intersection of 7 points or  $78\frac{3}{4}$  deg. wherefore, by rule 3 of problem 3 of section 1, we have  $10^2 + 7^2 - 9^2 + 6^2 \times \frac{1}{4} \text{ tang. } 78\frac{3}{4}^\circ = 32 \times \frac{1}{4} \times 5.0273395 = 8 \times 5.0273395 = 40.218716$  square chains = 4a. 3.499456 perches = the content required.\*



### QUESTION VII.

Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land.

Now

#### \* CONSTRUCTION.

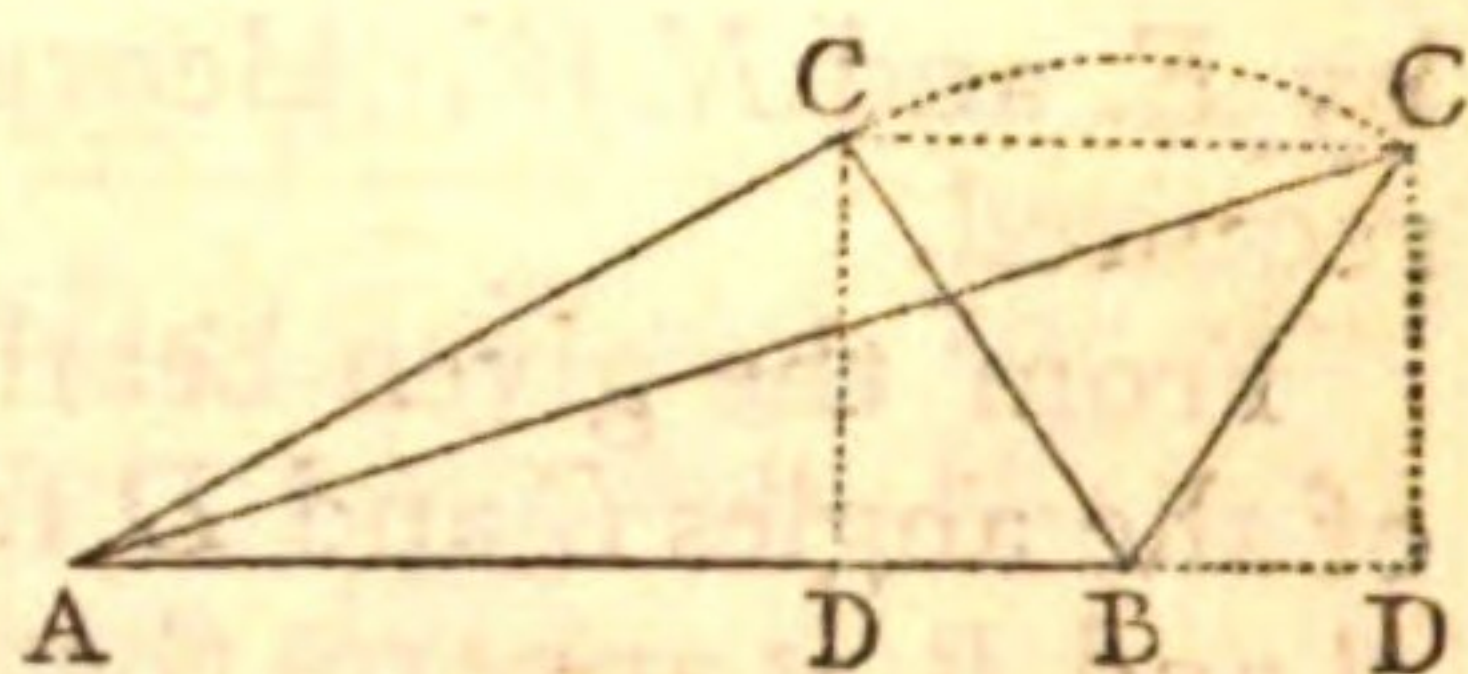
With two of the given lines *Ab* (10) and *bc* (6) make a right angle *b*; and with the other two complete the trapezium *AbcD*: In the perpendicular *Ec*, produced, take *EF* = 8.937492 = the double of the area (40.218716) divided by *AD* (9); with the center *F* and radius equal to  $(6\frac{2}{3})$  a fourth proportional to *DA*, *Ab*, *bc*, describe an arc meeting, in *C*, another arc described with the center *D* and radius *Dc*; then join *D*, *C*, and with the other two given lines (= *cb* and *bA*) complete the trapezium *ABCD*, and it is done.

For, having drawn *cA*, *AC*, and *CF*, and let fall the perpendiculars *CP* upon *AD*, *CQ* upon *AB*, and *FG* upon *PCG*; since  $AD^2 + DC^2 + 2AD \times DP = AC^2 = CB^2 + BA^2 + 2AB \times BQ$ , and  $cD^2 + DA^2 + 2AD \times DE = (Ac^2 =) cb^2 + bA^2$ , by taking these latter quantities from the former, it follows that  $2AD \times EP = (2AD \times DP - 2AD \times DE =) 2AB \times BQ$ , and consequently  $BQ : FG (EP) :: DA : AB :: BC : CF$  (by the construction); whence the triangles *BQC*, *CGF*, are similar, and therefore  $QC : CG :: BC : CF :: (by the construction) DA : AB$ ; and hence  $QC \times AB = AD \times CG$ ; and, by adding  $CP \times DA$  to each, we have  $CP \times DA + CQ \times AB (= \text{twice the area } ABCD) = CP \times DA + AD \times CG = EF \times DA = (\text{by the construction}) \text{ twice the given area.}$



Now the area  $160 \div 20$   
 $(\frac{1}{2}AB) = 8 =$  the perpendicular  $CD$ ; but  $\sqrt{BC^2 - CD^2} =$   
 $\sqrt{20^2 - 8^2} = DB$ ; hence  $AD$   
 $= AB \pm BD = 40 \pm \sqrt{20^2 - 8^2}$ ,

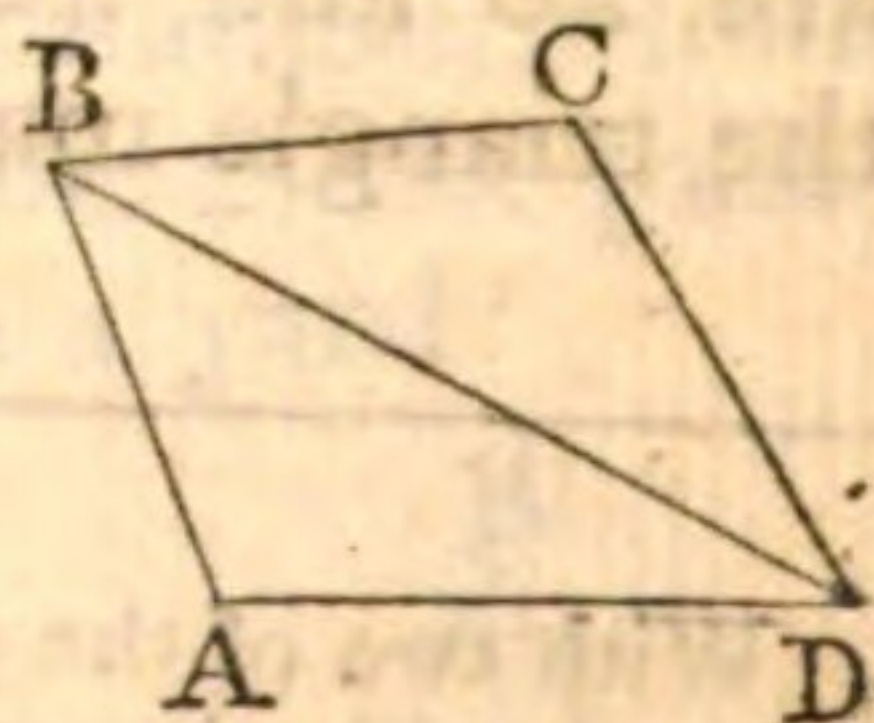
and  $AC = \sqrt{CD^2 + DA^2} = \sqrt{8^2 + 40^2 \pm 20^2 - 8^2} \pm 80\sqrt{20^2 - 8^2}$   
 $= 4\sqrt{125} \pm 20\sqrt{21} = 4 \times \sqrt{105} \pm 2\sqrt{5} = 58.87634686$   
 and  $23.09925922$ , either of which may be the side  
 required.



### QUESTION VIII.

Given the four sides  $AB = 9$ ,  $BC = 10$ ,  $CD = 11$ ,  
 and  $DA = 12$ , and the angle  $BDC = 30^\circ$ , formed by  
 the diagonal  $BD$  and the side  $DC$ ; required the area  
 of the trapezium.

First, by trigonometry,  $BC : CD ::$   
 $\frac{1}{2}(s. \angle BDC) : \frac{DC}{2CB} = \frac{11}{20} = .55 =$  the sine  
 of the angle  $DBC = 33^\circ 22'$ ; whence  
 $180^\circ - 30^\circ + 33^\circ 22' = 180^\circ - 63^\circ 22' =$   
 $116^\circ 38' = \angle C$ ; and  $s. \angle BDC : s. \angle C ::$   
 $CB : BD = 10 \times .8938936 \times 2 = 17.877872$ .



Then  $BC \times CD \times \frac{1}{2} s. \angle C = 10 \times 11 \times .4469468 =$   
 $49.164148 =$  the area of the triangle  $BCD$ .

And  $\sqrt{19.438936 \times 10.438936 \times 7.438936 \times 1.561064}$   
 $= 48.54337 =$  the area of the triangle  $ABD$ .

Wherefore  $49.164148 + 48.54337 = 97.707518 =$   
 the area of the whole trapezium.

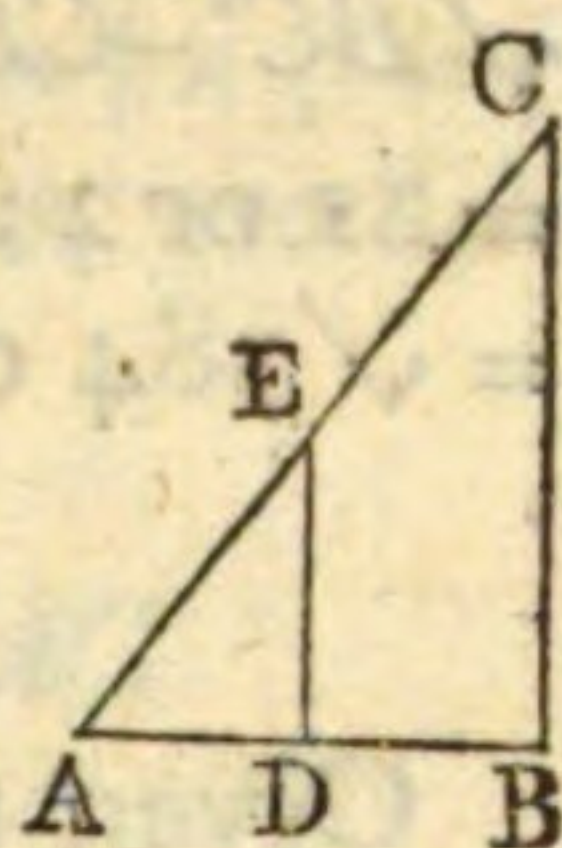
### QUESTION IX.

If from the right-angled triangle  $ABC$ , whose base  
 is 12, and perpendicular 16 feet, be cut off, by a line  
 $DE$



$DE$  parallel to the perpendicular, a triangle whose area is 24 square feet; what are the sides of this triangle?

The triangles  $ABC$ ,  $ADE$ , are similar, and the areas of similar triangles are as the squares of their like sides; but  $12 \times 8 = 96$  is the area of the triangle  $ABC$ ,



$$\therefore \sqrt{96} : \sqrt{24} :: \left\{ \begin{array}{l} 16 : 16\sqrt{\frac{24}{96}} = 16\sqrt{\frac{1}{4}} = 16 \times \frac{1}{2} = 8 = ED \\ 12 : 12\sqrt{\frac{24}{96}} = 12\sqrt{\frac{1}{4}} = 12 \times \frac{1}{2} = 6 = DA \end{array} \right\},$$

and  $AE = \sqrt{AD^2 + DE^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10.$

### QUESTION X.

If from a triangle whose sides are 13, 14, 15, be cut off by a line drawn parallel to the longest side, an area of 24; what are the lengths of the sides including that area?

First,  $\frac{13+14+15}{2} = \frac{42}{2} = 21 =$  half the sum of the sides, from which each side being taken, leaves 6, 7, 8; therefore  $\sqrt{21 \times 6 \times 7 \times 8} = \sqrt{3^2 \times 7^2 \times 4^2} = 3 \times 4 \times 7 = 84 =$  the area of the given triangle. Then, as in the last,

$$\sqrt{84} : \sqrt{24} :: \left\{ \begin{array}{l} 13 : 13\sqrt{\frac{24}{84}} = \frac{13}{7}\sqrt{14} \\ 14 : 14\sqrt{\frac{24}{84}} = 2\sqrt{14} \\ 15 : 15\sqrt{\frac{24}{84}} = \frac{15}{7}\sqrt{14} \end{array} \right\} = \text{the three sides inclosing the area cut off.}$$

### QUESTION XI.

If two sides of a triangle, whose area is  $\sqrt{10800} = 60\sqrt{3}$ , be 20 and 12, what is the third side?

From the end of one of the given sides let fall the perpendicular  $CD$  upon the other side produced.

2 H

Then



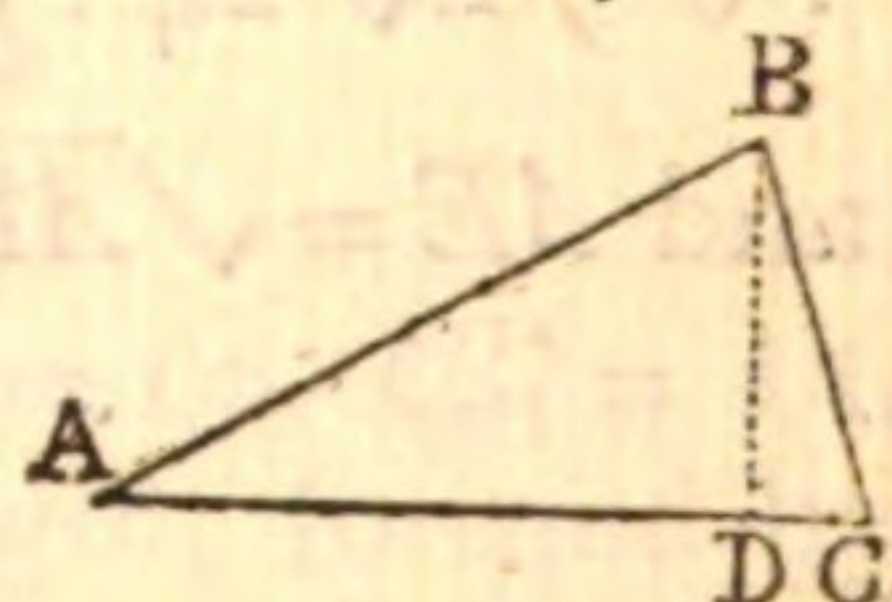
Then (fig. to quest. 7)  $CD = \frac{60\sqrt{3}}{\frac{1}{2} \times 12} = 10\sqrt{3}$ , and  $DB = \sqrt{BC^2 - CD^2} = \sqrt{20^2 - 3 \times 10^2} = 10$ ; hence  $DA = AB \pm BD = 22$  or  $2$ ; and  $CA = \sqrt{AD^2 + DC^2} = \sqrt{22^2 \text{ or } 2^2 + 3 \times 10^2} = \sqrt{784} \text{ or } \sqrt{304} = 28 \text{ or } 4\sqrt{19}$ .

### QUESTION XII.

Given the area = 144, the base  $AC = 24$ , and one angle  $BAC = 30^\circ$ , at the base; to find the sides  $AB$ ,  $BC$ .

First,  $144 \div 12 (\frac{1}{2}AC) = 12 =$  the perpendicular  $BD$ .

Then, as  $\frac{1}{2} = s. \angle A 30^\circ : 1 = s. \angle D :: 12 = DB : 24 = BA$ .



But  $DA = \sqrt{AB^2 - BD^2} = \sqrt{24^2 - 12^2}$ , and  $DC = CA - AD = 24 - \sqrt{24^2 - 12^2}$ .

Wherefore  $BC = \sqrt{CD^2 + DB^2} = 24\sqrt{2 - \sqrt{3}} = 12 \times \sqrt{6 - \sqrt{2}} = 12.423314184$ .

Or, since  $AB$  is  $= AC$ , the  $\angle C$  will be  $=$  the  $\angle B = \frac{180 - 30}{2} = \frac{150}{2} = 75^\circ$ ; hence  $\angle DBC = 15^\circ$ ; and therefore as radius  $= 1 : \sec. 15^\circ = 1.0352762 :: BD = 12 : 12 \times 1.0352762 = 12.4233144 = BC$ .

*Note.* It is pretty evident, that this triangle will be constructed by drawing  $AB$  to make with the given base  $AC$  an angle of  $30^\circ$ , and meeting with a line drawn parallel to and at the distance of the perpendicular  $BD$  from  $AC$ , in  $B$  the vertex of the triangle.

### QUESTION XIII.

Given the area 1012, and ratio of the sides of a triangle, viz.  $AC$  to  $CB$ , as 4 to 3, and  $CB$  to  $BA$  as 2 to 3; to find the sides.

It is evident that the sides  $AC$ ,  $CB$ ,  $BA$ , are in the ratio of the three numbers 8, 6, 9, respectively, and therefore



therefore a triangle whose sides are 8, 6, 9, will be similar to the triangle proposed; but similar triangles are as the squares of their like sides, and  $\sqrt{\frac{23}{2} \times \frac{1}{2} \times \frac{7}{2} \times \frac{1}{2}} = \frac{1}{4}\sqrt{8855}$  = the area of the triangle whose sides are 8, 6, 9; therefore  $\frac{1}{2}\sqrt[4]{8855}:\sqrt{1012}::$

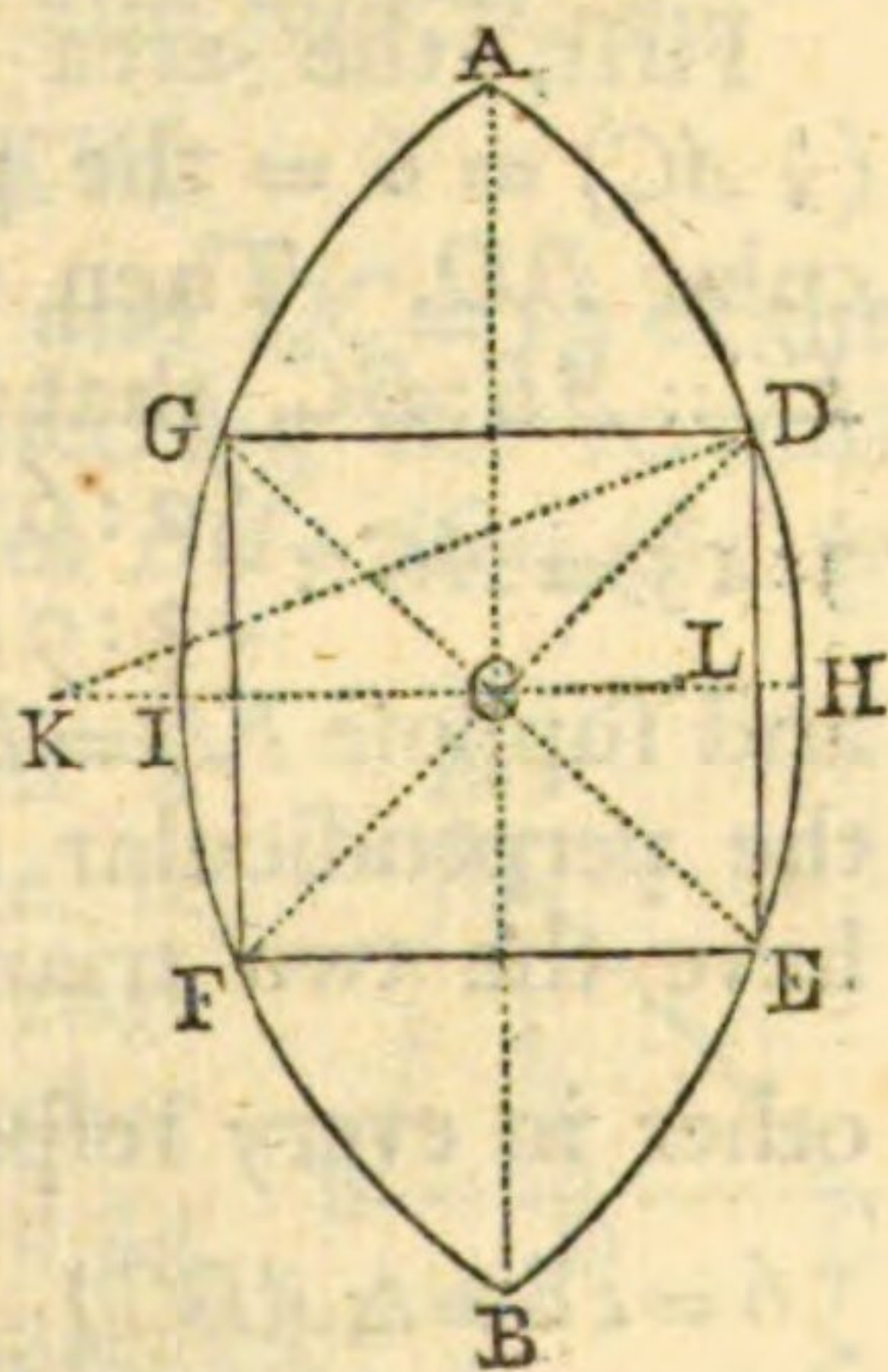
$$\begin{cases} 8:32\sqrt[4]{\frac{23 \times 11}{35}} = \frac{32}{35}\sqrt[4]{10847375} = 52.47024 = AC. \\ 6:24\sqrt[4]{\frac{23 \times 11}{35}} = \frac{24}{35}\sqrt[4]{10847375} = 39.35268 = CB. \\ 9:36\sqrt[4]{\frac{23 \times 11}{35}} = \frac{36}{35}\sqrt[4]{10847375} = 59.02902 = BA. \end{cases}$$

## QUESTION XIV.

The distance of the centers of two circles, whose diameters are each 50, being given equal to 30; it is required to find the area of the space inclosed by their circumferences, and that of a square inscribed in the said space?

*Construction.*

Having described the circumferences including the common space *AHBI*, *IH* being the common part of both the diameters, bisect *IH* in *C*, and through *C* draw *FCD*, *GCE*, making each half a right angle with *IH*, and meeting the arcs in *D*, *E*, *F*, *G*, the four angular points of the square.

*Cal-*



## Calculation.

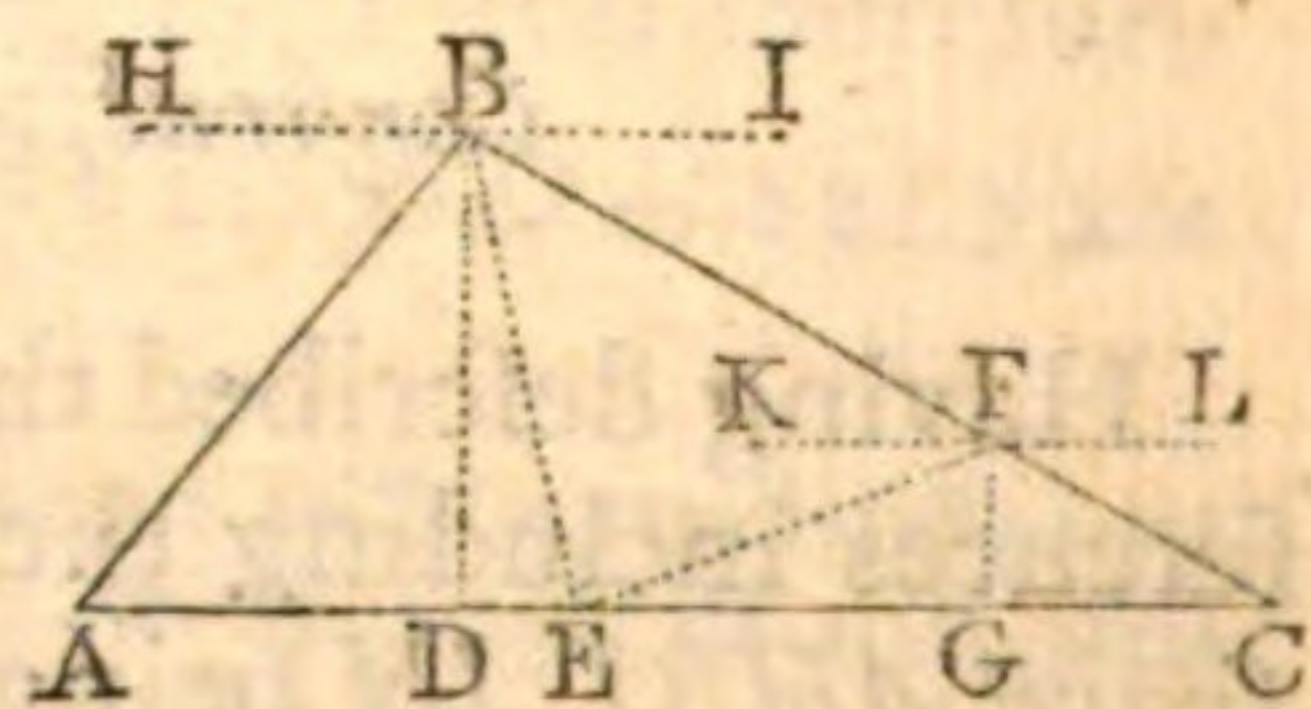
If  $K$  be the center of one of the circles, then  $HK-KC = 25-15=10=CH=CI$  the height of each segment; and  $\frac{CH}{2HK} = \frac{10}{50} = .2$  the tab.versed sine; against which, in the table of circular segments, stands the area  $\cdot 11182380$ ; therefore  $\cdot 1118238 \times 50 \times 50 \times 2 = 559 \cdot 119$  = the two segments or common space  $AHBIA$ .

Again, drawing  $KD$ , we shall have as  $DK=25$ :  $KC=15::s. \angle C=135^\circ$  or  $45^\circ$ :  $s. \angle CDK=25^\circ 6\frac{1}{4}'$ ; hence  $180^\circ-135^\circ-25^\circ 6\frac{1}{4}' = 180^\circ-160^\circ 6\frac{1}{4}' = 19^\circ 53\frac{3}{4}' = \angle K$ ; then as radius= $s. \angle L$ :  $s. \angle K=19^\circ 53\frac{3}{4}'::KD=25:DL=8 \cdot 553925$ , the double of which is  $DE=17 \cdot 10785$  = the side of the square. And therefore  $17 \cdot 10785 \times 17 \cdot 10785 = 292 \cdot 67853$  = the area of the square.

## QUESTION XV.

Given the base  $AC=15$ , the area 45, and the ratio of  $AB$  to  $BC$  as 2 to 3; to find the sides  $AB$ ,  $BC$ , of the triangle?

First, the area  $45 \div 7\frac{1}{2}$  ( $\frac{1}{2} AC$ ) = 6 = the perpendicular  $BD$ . Then take  $AE:EC::AB:BC$ , that is  $5=2+3:15=AC::\begin{cases} 2:6=AE \\ 3:9=EC \end{cases}$ ;



and suppose  $FB=BA$ , as also the lines  $BE$ ,  $EF$ , and the perpendicular  $FG$  to be drawn; and we shall have the two triangles  $ABE$ ,  $EBF$ , equal to each other in every respect; but  $\frac{1}{2} BD \times \begin{cases} AE \\ EC \end{cases} = 3 \times \begin{cases} 6=18=\triangle ABC \\ 9=27=\triangle EBC \end{cases}$ ; and the  $\triangle EBC-\triangle BEF$  ( $\triangle ABE$ )

=



$= 27 - 18 = 9 = \triangle EFC$ ; hence  $9 (\triangle EFC) \div 4\frac{1}{2} (\frac{1}{2} EC)$   
 $= 2 =$  the perpendicular  $FG$ ; but  $GE = \sqrt{EF^2 - FG^2}$   
 $= \sqrt{6^2 - 2^2} = \sqrt{32}$ , and  $GC = CE - EG = 9 - \sqrt{32}$ ; whence,  
 by similar triangles,  $FG : GC :: BD : DC = 27 - 3\sqrt{32}$ ,  
 and, consequently,  $DA = AC - CD = 3\sqrt{32} - 12$ ; where-  
 fore, lastly,  $AB = \sqrt{BD^2 + DA^2} = \sqrt{6^2 + 3\sqrt{32} - 12|^2} =$   
 $3\sqrt{2^2 + \sqrt{32} - 4|^2} = 6\sqrt{13 - 8\sqrt{2}} = 7.79146$ , and, as  $2 :$   
 $3 :: AB : BC = 9\sqrt{13 - 8\sqrt{2}} = 11.68719$ .

From the foregoing solution the following construction is evident.

Having drawn  $HI$  parallel to  $AC$ , and at such distance from it as is expressed by the perpendicular  $BD$ , found from the given area and base, and taken  $AE$  to  $EC$  in the given proportion of  $AB$  to  $BC$ ; with the center  $E$  and radius  $EA$  describe an arc meeting in  $F$ , with a line  $KL$  drawn parallel to  $AC$  and at such distance from it as is denoted by a third proportional to  $CE$ ,  $CE - EA$ , and  $BD$ ; then through  $F$  draw  $CB$  meeting with  $HI$  in  $B$ , and join  $B, A$ .

#### QUESTION XVI.

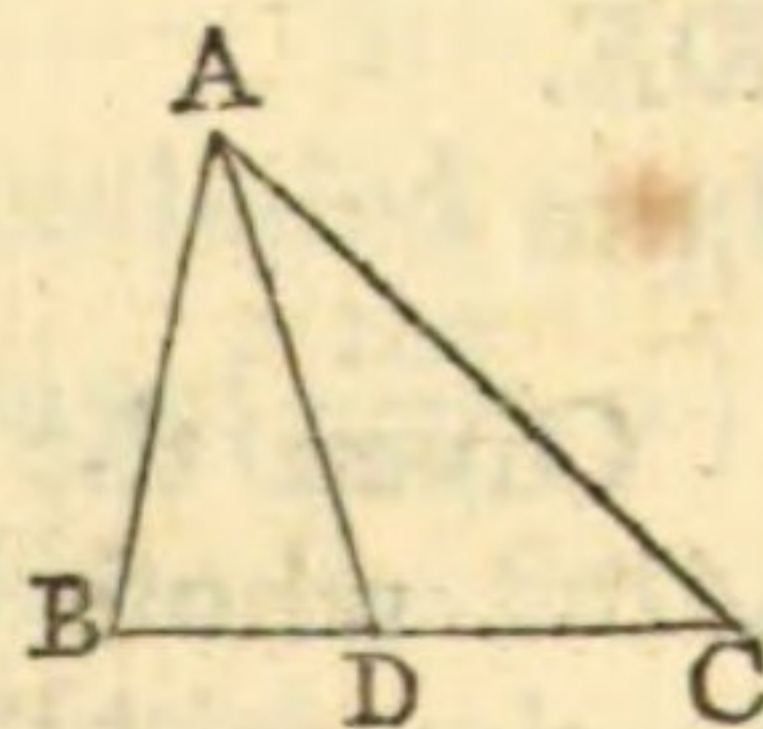
Given the segments  $BD = 10$ , and  $DC = 14$  of the base  $BC$  made by a line  $AD$  bisecting the vertical angle  $A$ , and the sum of the sides  $BA + AC = 48$ ; to find the area.

If a line bisect any angle of a triangle and cut the opposite side, it is known that the segments of that side are to each other as the other sides adjacent to them. Wherefore  $24$

$= BC : 48 = BA + AC :: \begin{cases} 10 = BD : 20 = BA, \\ 14 = DC : 28 = CA; \end{cases}$  hence the

2 I

area



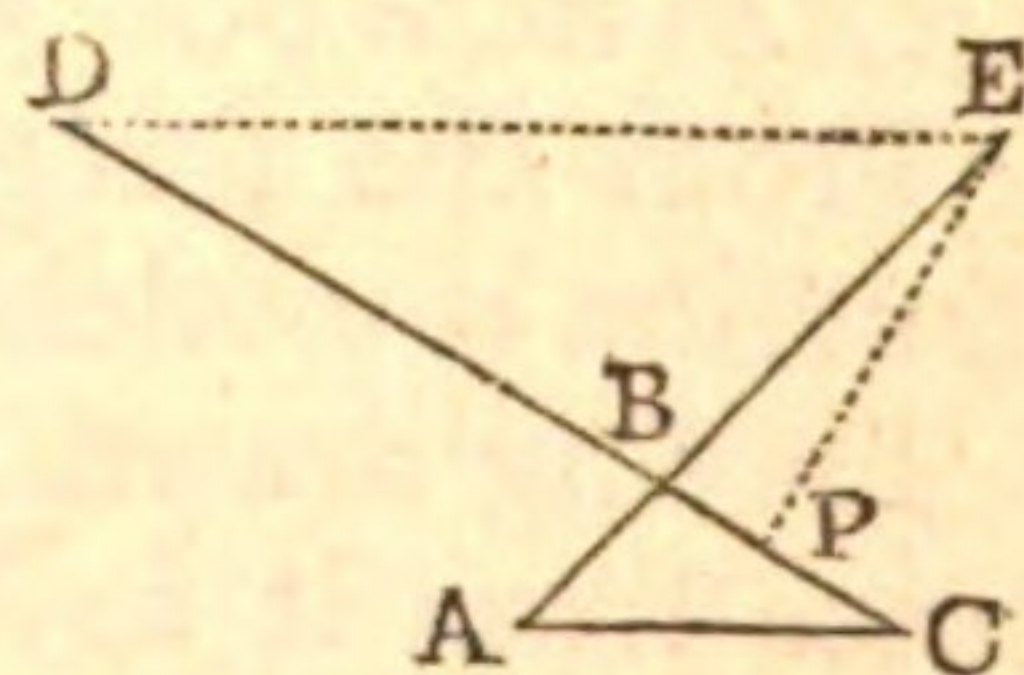


area is  $\sqrt{36 \times 8 \times 12 \times 16} = 6 \times 2 \times 2 \times 4 \sqrt{2 \times 3} = 96\sqrt{6} = 235.15100154$ .

### QUESTION XVII.

Having given the continuations of the two sides of the vertical angle of a given triangle, to find the area of the triangle formed by the said continuations and a line connecting their extremities.

That is, Given  $AB=2$ ,  $BC=3$ ,  $CA=4$ ; and  $AE=7$ ,  $CD=10$ , or  $BE=5$ ,  $BD=7$ ; required  $DE$  and the area of the triangle  $DBE$ .



First,  $\sqrt{\frac{2}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2}} = \frac{1}{4} \sqrt{15}$  = the area of the triangle  $ABC$ . But since the vertical angles at  $B$  are equal, and triangles having one angle in each equal, are as the rectangles of the sides about the equal angles; we shall have  $6:35::\frac{1}{4}\sqrt{15}:\frac{3}{8}\sqrt{15} = 16.944305$  = the area of the triangle  $BDE$ .

Then, having drawn  $EP$  perpendicular to  $BC$ ,  $\frac{3}{8}\sqrt{15} \div \frac{7}{2} (\frac{1}{2}BD) = \frac{5}{4}\sqrt{15} = EP$ ; but  $\sqrt{BE^2 - EP^2} = \sqrt{25 - \frac{1}{16} \times 25} = \sqrt{\frac{25}{16}} = \frac{5}{4} = PB$ , and  $PB + BD = 7 + \frac{5}{4} = \frac{33}{4} = DP$ ; hence  $\sqrt{DP^2 + PE^2} = \sqrt{\left(\frac{33}{4}\right)^2 + 15 \times \frac{5}{4}} = \frac{1}{4}\sqrt{33^2 + 15 \times 5^2} = \frac{1}{4}\sqrt{1464} = \frac{1}{2}\sqrt{366} = 9.565563235 = DE$ .

### QUESTION XVIII.

Given the area (100) of the equilateral triangle  $ABC$ , whose base falls on the diameter and vertex in the middle of the arc of a semicircle; required the diameter  $DE$  of the semicircle?

Since







be the center of the circle circumscribing the triangle; therefore, with the center  $E$  and radius  $EB$  or  $EA$ , describe an arc  $ABF$  meeting  $DF$ , drawn parallel to  $AB$ , in  $F$ ; then draw  $AF$ ,  $FB$ , and it is evident that  $ABF$  will be the triangle required.

*Calculation.*

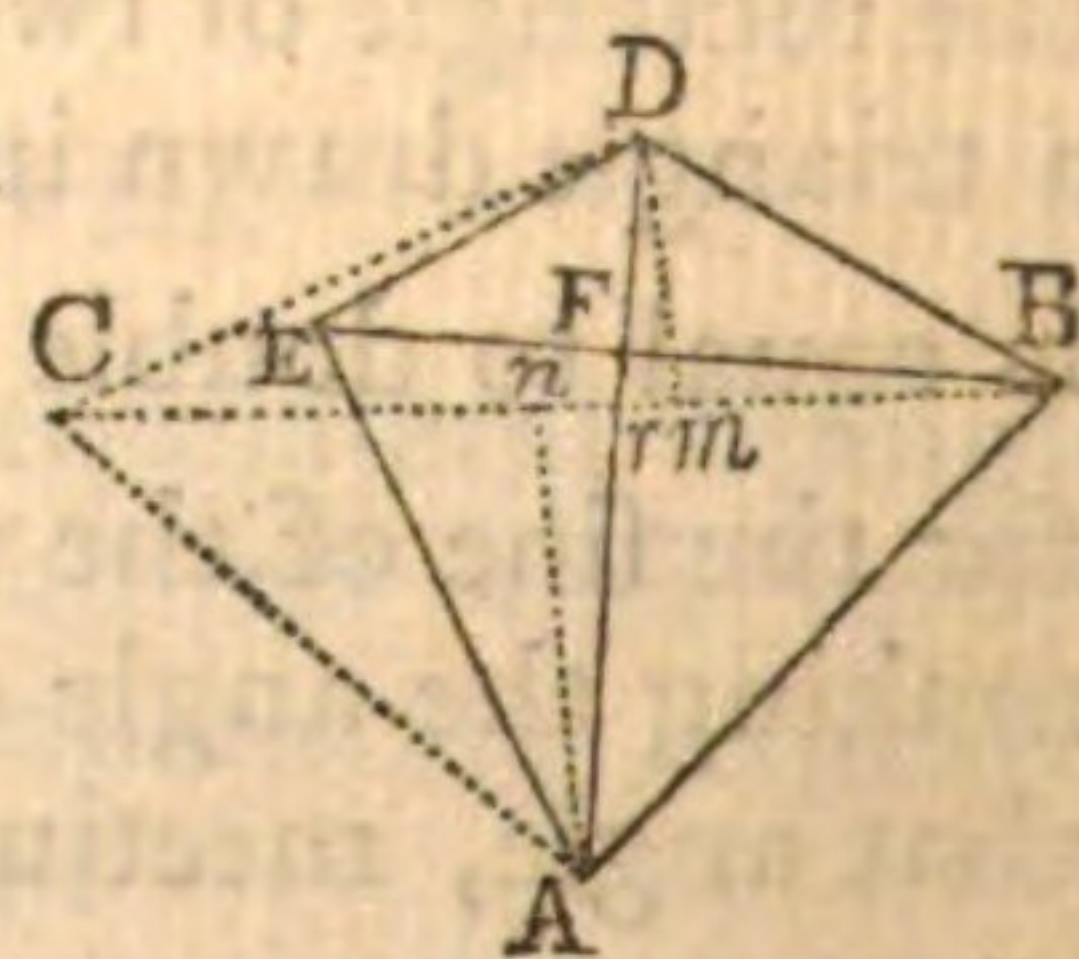
Draw  $FP$  perpendicular to and meeting  $AB$ , produced, in  $P$ . Then, since  $\frac{1}{2}$  is the s. of  $\angle E$ , it will be as  $\frac{1}{2} : 1 = s. \angle C :: 20 = CB : 160 = BE$ ; or thus, by geometry,  $\frac{AF \times FB}{2FP} = \frac{40 \times 40}{10} = 160 = BE$ . Again,  $CE = \sqrt{EB^2 - BC^2} = \sqrt{160^2 - 20^2} = 20\sqrt{63} = 60\sqrt{7}$ ; hence  $DE = EC - CD = 60\sqrt{7} - 5$ , and  $DF = \sqrt{FE^2 - ED^2} = 5\sqrt{15 + 24\sqrt{7}}$ ; but  $BP = DF - CB = 5\sqrt{15 + 24\sqrt{7}} - 20$ , and therefore  $BF = \sqrt{FP^2 + PB^2} = 10\sqrt{8 + 6\sqrt{7} - 2\sqrt{15 + 24\sqrt{7}}} = 24.80833$ , and consequently  $FA = \frac{AB^2}{BF} = \frac{40 \times 40}{24.80833} = 64.49447$ .

QUESTION XX.

The four sides of a field, whose diagonals are equal to each other, are known to be 25, 35, 31, and 19 poles, in a successive order; from whence the content of the field is required?

*Construction.*

Make  $AB$  and  $AC$  perpendicular to one another, and each equal to one of the given sides, as 35; with the centers  $B$ ,  $C$ , and radiuses equal to the two remaining opposite sides 25 and 31, describe two arcs, intersect-



ing



ing in  $D$ , draw  $AD$ , and  $BE$  perpendicular and equal to it; draw  $AE$  and  $DE$ , and  $ABDE$  will be the trapezium.\*

*Calculation.*

Draw  $BC$  meeting  $AD$  in  $r$ , upon which let fall the perpendiculars  $An$ ,  $Dm$ . Then  $BC = \sqrt{BA^2 + AC^2} = 35\sqrt{2}$ , and  $An = nB = \frac{1}{2} BC = \frac{35\sqrt{2}}{2} = \frac{35}{\sqrt{2}}$ ; again,  $mB = \frac{DB^2 + BC^2 - CD^2}{2BC} = \frac{25^2 + 2 \times 35^2 - 31^2}{70\sqrt{2}} = \frac{2114}{70\sqrt{2}} = \frac{151\sqrt{2}}{10}$ , and  $mn = nB - Bm = \frac{35}{2} - \frac{151}{10} \times \sqrt{2} = \frac{12}{5} \sqrt{2}$ ; but  $mD = \sqrt{DB^2 - Bm^2} = \sqrt{25^2 - \left(\frac{151}{10}\right)^2} \times 2 = \frac{1}{10} \sqrt{250^2 - 151^2} \times 2 = \frac{1}{10} \sqrt{16898}$ ; and, by similar triangles,  $An + mD : mn :: An : nr = \frac{420\sqrt{2}}{175 + \sqrt{8449}}$ ; hence  $rA = \sqrt{An^2 + nr^2} = \sqrt{\frac{1}{2} \times 35^2 \times 175 + \sqrt{8449}^2 + 2 \times 420^2} \div 175 + \sqrt{58449}$ ; and, by similar triangles,  $rn : nm :: rA : AD = \sqrt{793 + 7\sqrt{8449}} = BE$ .—Then, because  $AD$  is equal and perpendicular to  $BE$ , by rule 2 page 69 we obtain the area =  $AD \times BE \times \frac{1}{2} \text{ sine } L F = \frac{1}{2} BE^2 = \frac{1}{2} \times 793 + 7\sqrt{8449} = 718.214547 \text{ square poles} = 4 \text{ ac. } 1 \text{ ro. } 38.214547 \text{ perches.}$

2 K

*Otherwise.*

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\* For, by the construction,  $AB$ ,  $BD$  are the two sides 35, 25, and the diagonal  $BE = AD$ ; so that we have only to shew that  $AE$ ,  $ED$  are equal to the other two sides, 31, 19.—Now, by the construction, the angles  $CAD$ ,  $ABE$ , being each the complement of  $BAD$ , are equal to each other; and, by the construction,  $CA$  and  $AD$ ,  $AB$  and  $BE$ , about those equal angles, are also equal; and therefore the remaining sides  $CD$ ,  $AE$  are, likewise, equal, that is  $AE = 31$ .—But, by the property of all  $\Delta$ s,  $BA^2 - AE^2 = BF^2 - FE^2 = BD^2 - DE^2$ ; and, therefore,  $DE = \sqrt{BD^2 + AE^2 - AB^2} = \sqrt{25^2 + 31^2 - 35^2} = 19$ .  
Q.E.D.



*Otherwise.*

Having found  $mB = \frac{151}{10} \sqrt{2}$ ; we shall have as  $DB = 25$ :  $Bm = \frac{151}{10} \sqrt{2} :: 1 : \frac{151}{250} \sqrt{2} = .604 \sqrt{2} = .85418499$  = the cosine of the  $\angle DBm = 31^\circ 20'$ ; but  $mBA = 45^\circ$ , consequently  $\angle DBA = 76^\circ 20'$ ; then  $60 = AB + BD$ :  $10 = AB - BD :: \text{tang. } \frac{\angle BDA + \angle BAD}{2} = 51^\circ 20' : \text{tang. } \frac{\angle BDA - \angle BAD}{2} = 11^\circ 58'$ , hence  $51^\circ 50' - 11^\circ 58' = 39^\circ 52' = \angle BAD$ ; lastly,  $s. \angle BAD : s. \angle DBA :: BD = 25 : DA = 37.897 = BE$ , which is nearly equal to  $\sqrt{793 + 7\sqrt{8449}} = 37.90025$ , the value above found.

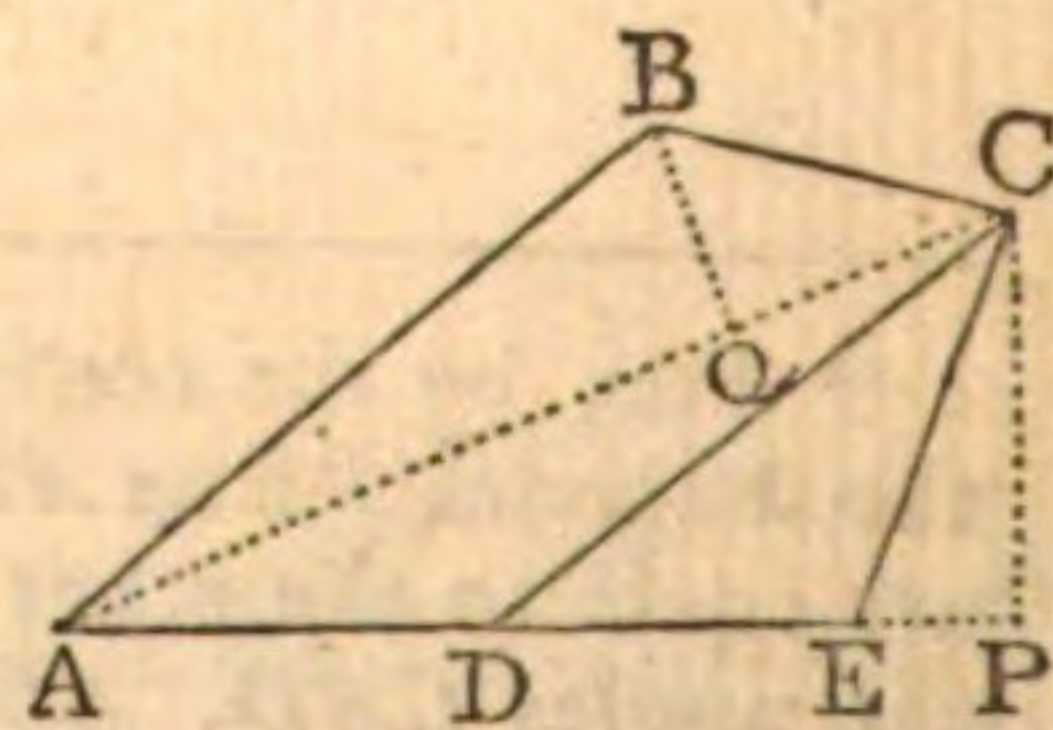
### QUESTION XXI.

The trapezium  $ABCD$  is to be measured; and, because of some obstructions from wood, water, &c. or for want of proper instruments, there can be taken only the following measures: viz.  $AB$  is 58.5 chains,  $BC$  27.3,  $CD$  50, and  $DA$  32 chains; also in  $AD$ , continued, there is measured any distance  $DE$  27.5, and the distance from  $E$  to  $C$  is 32.5 chains; required the area.

Draw the diagonal  $AC$ , and on  $AD$ , produced, let fall the perpendicular  $CP$ .

Because  $DC^2$  is  $= CE^2 + ED^2 + 2DE \times EP$ , we shall have  $EP = \frac{DC^2 - CE^2 - ED^2}{2DE} = \frac{50^2 - 32.5^2 - 27.5^2}{55} = 12.5$ ; but  $PC = \sqrt{CE^2 - EP^2} = \sqrt{32.5^2 - 12.5^2} = 30$ , therefore  $\frac{AD \times PC}{2} = 16 \times 30 = 480$  = the area of the triangle  $ADC$ .

Again,





Again,  $AP = AD + DE + EP = 32 + 27.5 + 12.5 = 72$ ;  
therefore  $CA = \sqrt{AP^2 + PC^2} = \sqrt{72^2 + 30^2} = 78$ .

Also, letting fall the perpendicular  $BQ$ , since  $AB^2 = BC^2 + CA^2 - 2AC \times CQ$ , we shall have  $CQ = \frac{BC^2 + CA^2 - AB^2}{2AC}$   
 $= 21.84$ ; hence  $QB = \sqrt{BC^2 - CQ^2} = 16.38$ ; therefore  
 $\frac{AC \times QB}{2} = 39 \times 16.38 = 638.82 =$  the triangle  $ABC$ .

Whence  $480 + 638.82 = 1118.82$  square chains =  
 $111.882$  acres =  $111$  ac.  $31$ .  $21.12$  perches = the whole  
area required.

### QUESTION XXII.

To find the length of a chord cutting off any  
part, as suppose  $\frac{1}{3}$ , from a circle whose diameter is  
 $289$ .

The area of the whole tabular circle, or of the  
circle whose diameter is  $1$ , is  $.78539816$ ; the  $\frac{1}{3}$  of  
which is  $.26179939$  nearly, which is a segment simi-  
lar to the segment to be cut off. Now, the versed  
sine answering to this area, in the table of circular  
segments, is  $.36753395$ , which taken from the whole  
diameter  $1$  leaves  $.63246605$  for the other part of the  
diameter; then, because the semi-chord is a mean  
proportional between the two segments of the di-  
ameter, we shall have  $2\sqrt{.63246605 \times .36753395} =$   
 $.96426162 =$  the chord of the tabular similar seg-  
ment. Wherefore as  $1 : 289 :: .96426162 : 278.6716$   
 $=$  the chord line required.

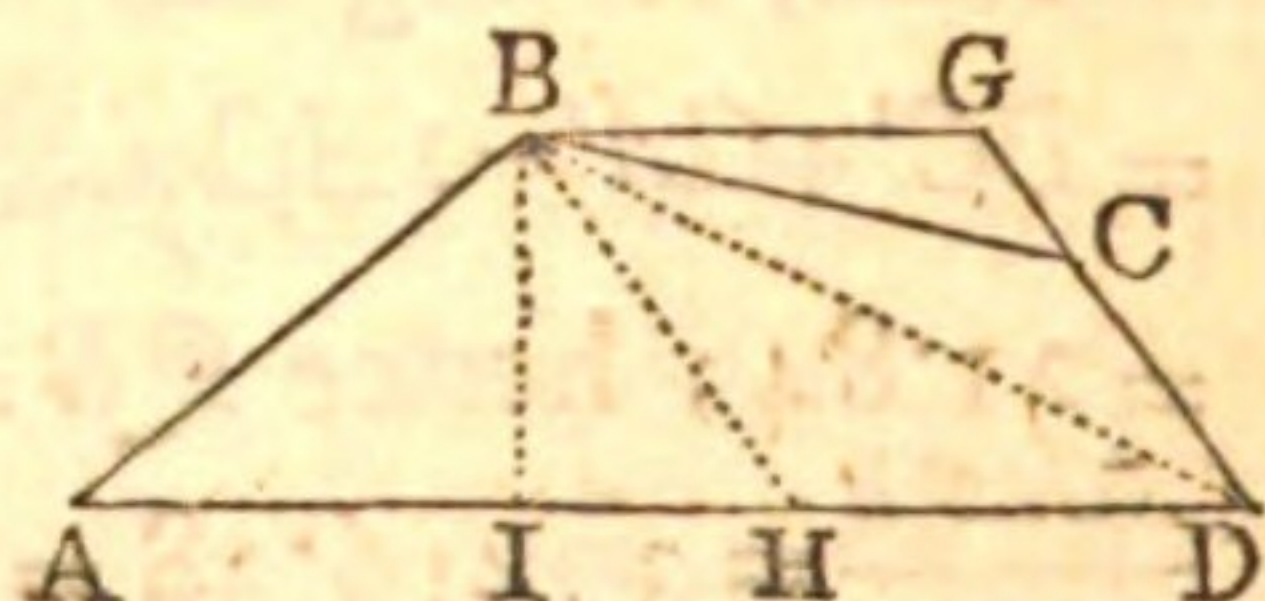
### QUESTION XXIII.

In the unpassable field  $ABCD$  there are measured  
the sides  $AB$ ,  $29$ ;  $AD$ ,  $59$ ; and  $DC$ ,  $15$  chains: also,  
by



by drawing  $BG$  parallel to  $AD$  till it meet with  $DC$ , produced, in  $G$ ,  $BG$  measures 23; and  $GC$ , 10 chains; required the content.

Draw  $BH$  parallel to  $GD$ : then is  $BHDG$  a parallelogram, and consequently  $BH = GD = 25$ , and  $HD = BG = 23$ ; hence  $AH = AD - DH = 36$ .



But, letting fall the perpendicular  $BI$ ,  $HI = \frac{BH^2 + HA^2 - AB^2}{2AH} = 15$ , and  $IB = \sqrt{BH^2 - HI^2} = 20$ . Whence  $\frac{AD + BG}{2} \times IB = 82 \times 10 = 820 =$  the area of the trapezoid  $ABGD$ .

Again, drawing the diagonal  $BD$ ,  $HD \times \frac{1}{2} IB = 23 \times 10 = 230 =$  the triangle  $BHD =$  the triangle  $BGD$ ; but the triangles  $BGD$ ,  $BGC$ , being of the same height, are to each other as their bases  $DG$ ,  $GC$ ; that is,  $DG = 25 : GC = 10 ::$  triangle  $BDG = 230 : \text{triangle } BCG = \frac{2}{5} \times 230 = 2 \times 46 = 92$ .

Whence, by taking the triangle  $BCG$  from the trapezoid  $ABGD$ , we have  $820 - 92 = 728$  square chains  $= 72 \text{ a. } 3 \text{ r. } 8 \text{ perches} =$  the area required.

*Note.* It is evident that this figure will be constructed, by first forming the triangle  $ABH$ , and joining the parallelogram  $BGDH$  to it.

#### QUESTION XXIV.

To find the area of the hexagonal field  $ABCDEF$ , the sides  $BC$ ,  $DE$ ,  $FA$ , being produced to meet in the points  $Z$ ,  $Y$ ,  $X$ ; there are given the following dimensions;  $ZB = 39$ ,  $BC = 118$ ,  $CY = 25$ ,  $YD = 30$ ,  $DE = 100$ ,  $EX = 65$ ,  $XF = 23$ ,  $FA = 109$ , and  $AZ = 37$  chains.

First,



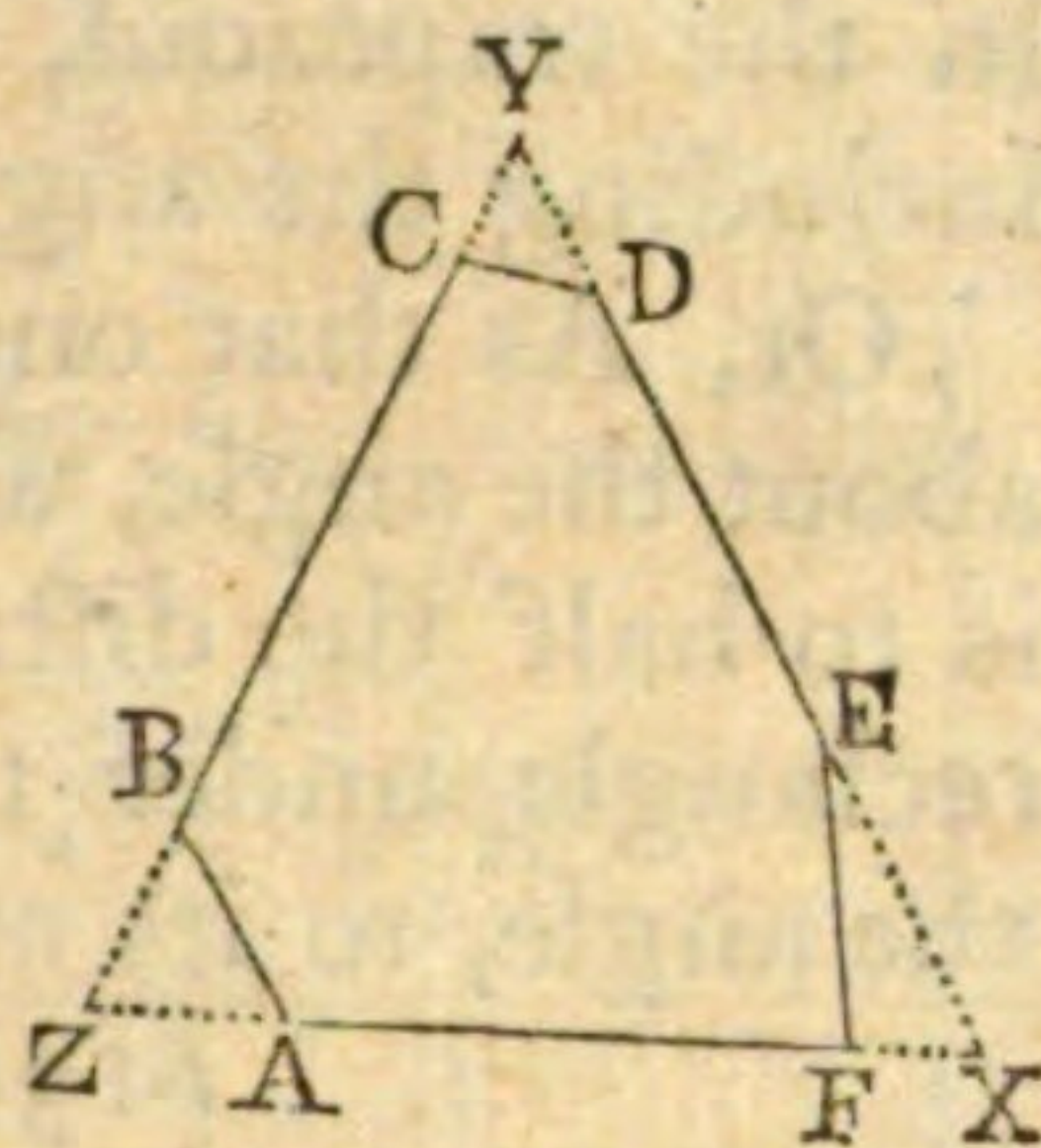
First, by prob. 2 sect. 1, the area of the triangle  $ZYX$  is 14196 square chains.

But, it appears from rule 3 of the same problem that, the areas of triangles, having one angle common, are as the rectangles of the sides including the common angle; wherefore

$$\begin{aligned} XZ \times ZY &= 169 \times 182 : 37 \times 39 :: 14196 : 666 = \Delta AZB, \\ ZY \times YX &= 182 \times 195 : 25 \times 30 :: 14196 : 300 = \Delta CYD, \\ YX \times XZ &= 195 \times 169 : 65 \times 23 :: 14196 : 644 = \Delta EXF, \end{aligned}$$

whose sum 1610 taken

from 14196 the triangle  $ZYX$ , leaves 12586 = the area of the hexagon  $ABCDEF$  required.



### QUESTION XXV.

It is required to find the area of the heptagon  $ABCDEFG$  inscribed in the trapezoid  $HIKL$ , the dimensions as below:

Viz.  $HB=34$ ,  $BC=85$ ,  $CI=17$ ,  $ID=42$ ,  $DK=66$ ,  $KE=$ ,  $EF=91$ ,  $FL=26$ ,  $LG=29$ ,  $GA=160$ , and  $AH=33$ .

First, drawing  $IM$  parallel to  $KL$ , and letting fall the perpendicular  $IN$ ,  $IM$  will be  $=KL$  = 130,  $MH = HL - IK = 114$ ;

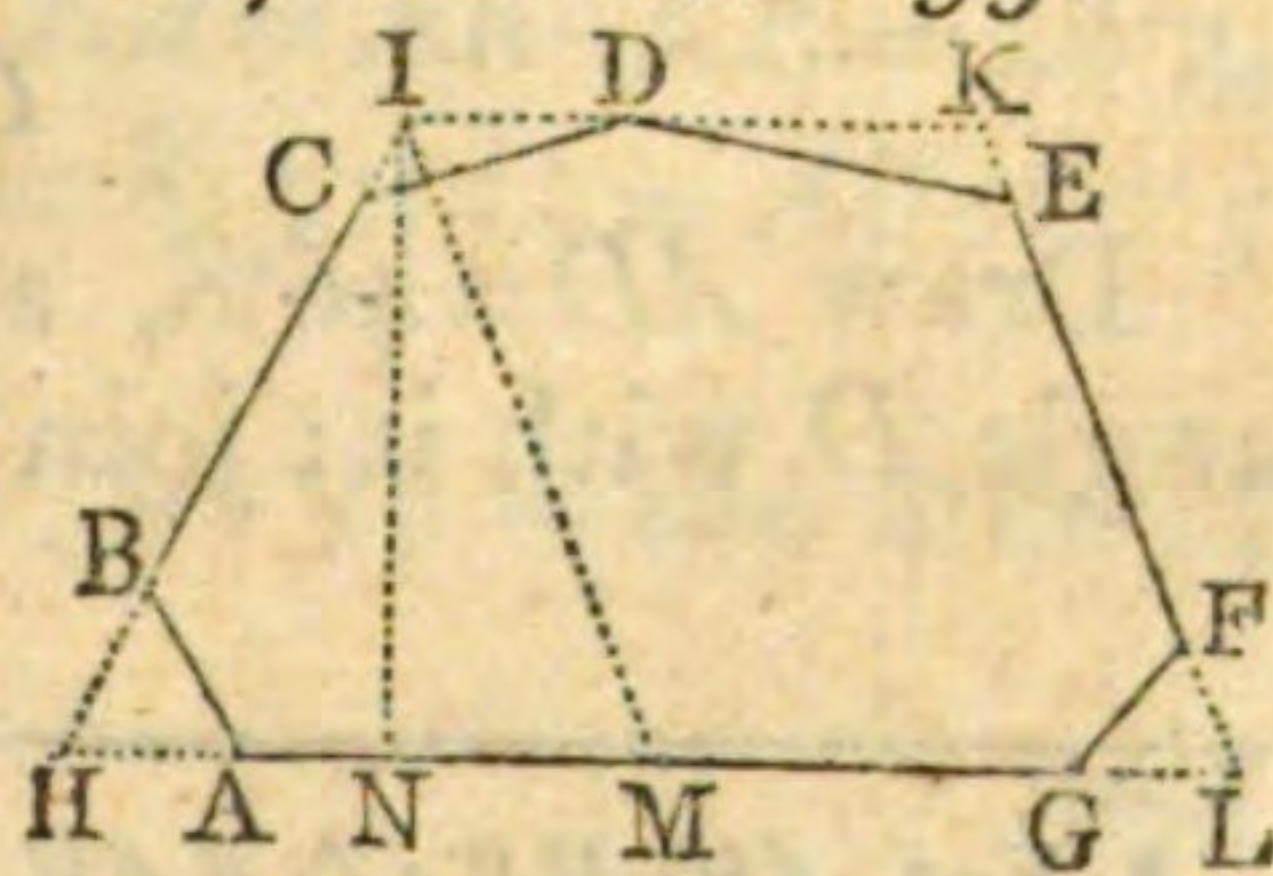
hence  $MN = \frac{IM^2 + MH^2 - HI^2}{2MH} =$

50, and  $NI = \sqrt{IM^2 - MN^2} = 120$ .

But a trapezoid is to a triangle, having each one angle equal, as the rectangle under the sum of the parallel sides and the other side including that angle,

2 L

in





in the trapezoid, is to the rectangle under the sides including the angle, in the triangle.

Or, As that one of the two sides in the trapezoid, about the angle, which is not one of the parallel sides, is to half the distance of the parallel sides, so is the rectangle under the two sides of the triangle about the angle, to its area.\*

$$\text{Wherefore } \begin{cases} 136:60::33\times34:495=\triangle AHB, \\ 136:60::17\times42:315=\triangle CID, \\ 130:60::66\times13:396=\triangle DKE, \\ 130:60::26\times29:348=\triangle FLG, \end{cases}$$

the sum is 1554, which taken from 19800, the area of the trapezoid, leaves 18246 square chains = 1824 a. 2 r. 16 perches.

*Note.* This is constructed as the 23d.

### QUESTION XXVI.

In the quadrangular field  $ABCD$ , if  $AD$  be = 46.8,  $DC$  = 19.5,  $BE$  perpendicular to  $AD$  and = 44.8 chains, and  $D$  a right angle; also if  $AB$  be to  $BC$  as 8 to 5; it is required to find the content of the trapezium  $ABCD$ , and the lengths of the sides  $AB$ ,  $BC$ .

#### *Construction.*

Draw  $AD$  = 46.8, and  $DC$  = 19.5 making a right angle  $D$  with it; join  $A, C$ ; take  $AF$  to  $AC$  as 8 to 13  
(8+5)

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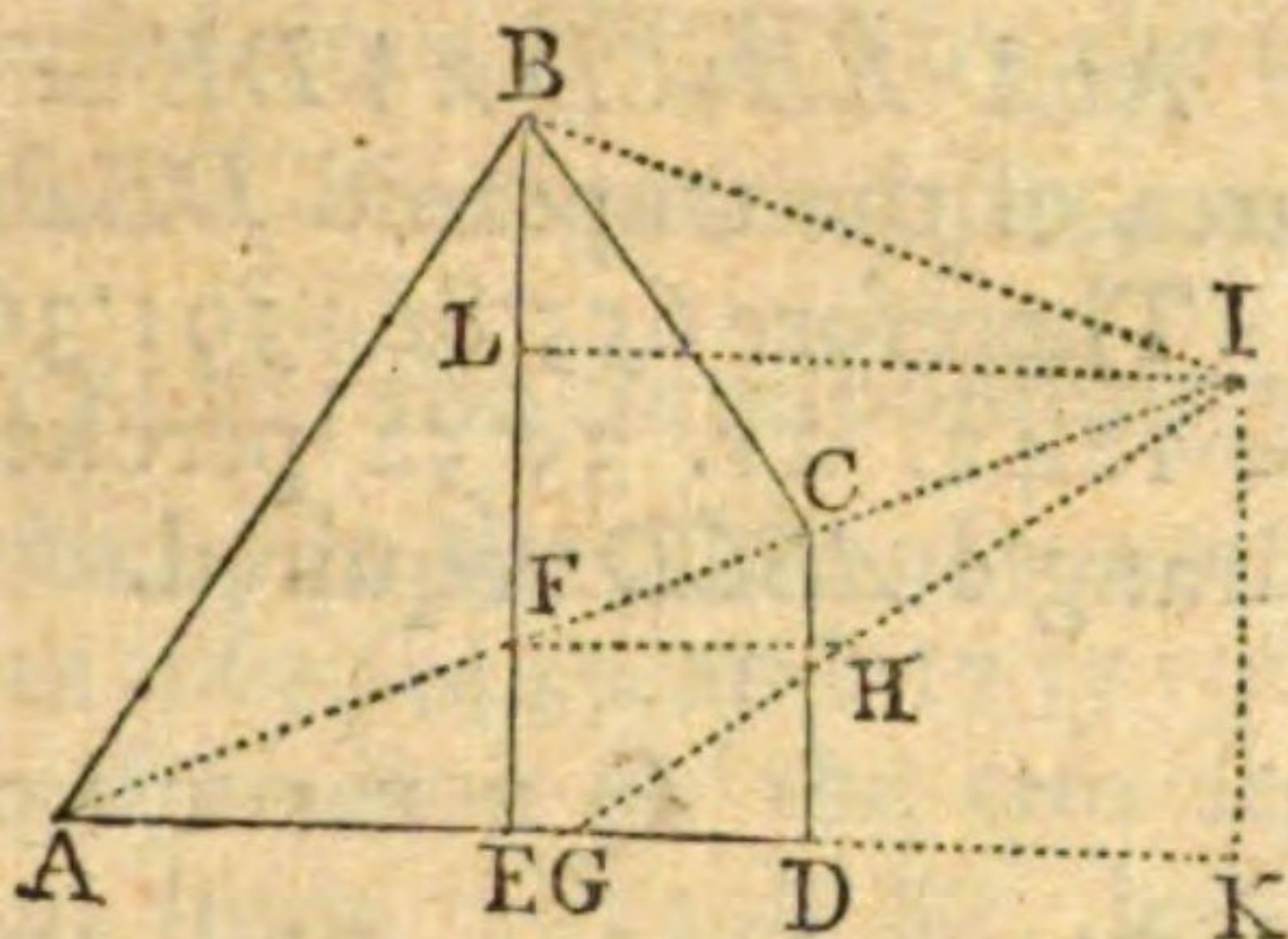
\* For  $AH \times HB : \triangle BAH :: (\text{radius} : \frac{1}{2} s. LH ::) HI : \frac{1}{2} IN ::$  (by equal mult.)  $HI \times \overline{IK+HL} : \frac{1}{2} IN \times \overline{IK+HL} = \text{the trapezium.}$

In the same manner,  $GL \times LF : \triangle GLF :: LK : \frac{1}{2} IN :: LK \times \overline{IK+HL} : \frac{1}{2} IN \times \overline{IK+HL}.$

But the angles  $I$  and  $K$  are the supplements of the angles  $H$  and  $L$ , and have, therefore, the same sines; and, consequently, the rule will be the same for them.



(8+5), or  $AF:FC::8:5$ ; make  $AG = AF$ , and  $FH$  parallel to it and = to  $FC$ ; draw  $GHI$  meeting  $AC$ , produced, in  $I$ ; with the center  $I$  and radius  $IF$ , describe an arc meet-



ing a line drawn parallel to and at the distance of 44.8 (the perpendicular) from  $AD$ , in  $B$  the other point of the figure required.\*

*Calculation.*

First,  $AC = \sqrt{AD^2 + DC^2} = 50.7$ , and, by the construction,  $13:8::50.7 = AC:31.2 = AF$ ; hence  $FC = CA - AF = 19.5$ ; then, by similar triangles,  $11.7 (= AG - FH = AF - FC):31.2 (= AF = AI - IF)::19.5 (= FH = FC):52 = FI = IB$ ; whence  $AI = AF + FI = 83.2$ ; then, by similar triangles again, (demitting  $IK$  perpendicular to  $AD$ , produced)  $50.7 = AC:83.2 = AI::$   
 $\left. \begin{array}{l} 46.8 = AD:76.8 = AK, \\ 19.5 = DC:32 = KI, \end{array} \right\}$  whence (drawing  $IL$  parallel to  $KE$ )  $LB = BE - KI = 12.8$ , and  $KE = LI = \sqrt{IB^2 - BL^2} = 50.4$ ; and hence  $EA = AK - KE = 26.4$ , and  $AB = \sqrt{BE^2 + EA^2} = 52$ ; also  $8:5::52 = AB:32.5 = BC$ .

But  $\frac{1}{2} AE \times EB = 13.2 \times 44.8 = 591.36 =$  the area of the triangle  $ABC$ .

And

\* DEMONSTRATION.

Because, by similar  $\Delta$ s,  $IA:AG (AF)::IF:FH (FC)$ , we have, by division,  $AI:IF::FI:IC$ , or, joining  $I, B$ ,  $AI:IB::BI:IC$ ; but the  $\angle I$ , included by these proportional lines, is common to both the  $\Delta$ s  $ABI, IBC$ ,  $\therefore$  those  $\Delta$ s are similar, and consequently  $AB:BC::(AI:IB (IF)::AG:FH)::8:5$ .



And  $\overline{EB+CD} \times \frac{1}{2} DE = 64.3 \times 10.2 = 655.86 =$  the area of the trapezoid  $DEBC$ .

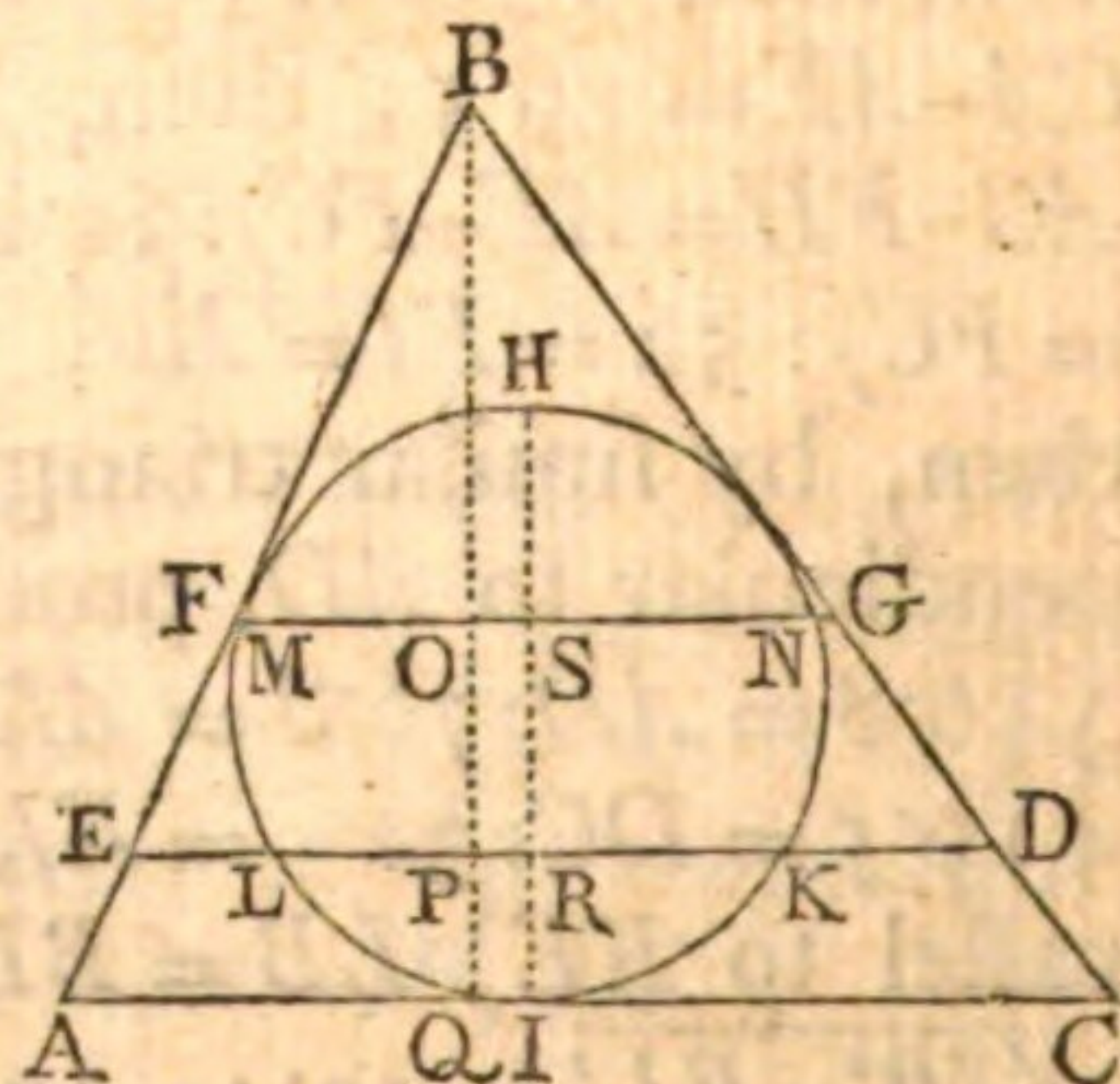
Therefore  $655.86 + 591.36 = 1247.22$  square chains  $= 124$  a. 2 r. 35.52 perches  $=$  the area of the quadrangle  $ABCD$  required.

### QUESTION XXVII.

Given the three sides  $AB = 13$ ,  $BC = 15$ ; and  $CA = 14$ , of a triangle  $ABC$ , divided into three equal parts by the two lines  $FG$ ,  $ED$  both parallel to  $AC$ ; it is required to find the areas of the two segments  $MHN$ ,  $KIL$ , and zone  $LMNK$  into which the inscribed circle is cut by those lines.

First,  $\sqrt{21 \times 8 \times 7 \times 6} =$   
 $\sqrt{7 \times 3 \times 2 \times 4 \times 7 \times 2 \times 3} = 7 \times 3 \times 4$   
 $= 84 =$  the area of the triangle  $ABC$ .

Then  $84 \div 7 (\frac{1}{2} AC) = 12 =$  the perpendicular  $BQ$ ; and  
 $84 \div 21 (\frac{AB+BC+CA}{2}) = 4 =$  the radius, or  $8 =$  the diameter  $HI$  of the inscribed circle.



Now, the triangles  $BFG$ ,  $BED$ ,  $BAC$ , are similar; but similar triangles are as the squares of their like dimensions; also those triangles are to one another as the numbers 1, 2, 3, respectively; wherefore  $\sqrt{3}:$

$$12 = BQ :: \begin{cases} \sqrt{1}:12\sqrt{\frac{1}{3}} = 4\sqrt{3} = BO, \\ \sqrt{2}:12\sqrt{\frac{2}{3}} = 4\sqrt{6} = BP; \end{cases}$$

Hence  $BQ - BP = 12 - 4\sqrt{6} = PQ =$  the versed sine  $IR$ ;  $BQ - BO = 12 - 4\sqrt{3} = OQ = IS$ ; and  $HI - IS = 4\sqrt{3} - 4 =$  the versed sine  $SH$ .

Then,



Then, dividing the versed fines *HS*, *RI*, by the diameter, we have  $\frac{4\sqrt{3}-4}{8} = \frac{\sqrt{3}-1}{2} = .366025404$ , and  $\frac{12-4\sqrt{6}}{8} = \frac{3-\sqrt{6}}{2} = .275255128$ ; the corresponding tabular versed fines; to which, in the table of circular segments, belong the areas .26034449, and .17577983 = the tabular area corresponding to the two segments; whose sum taken from .785398 &c. leaves .34927384 = that corresponding to the zone; each of which being multiplied into 64, the square of the diameter, gives their several areas, viz.

$$64 \times \begin{cases} .26034449 = 16.66204736 = \text{the segment } LIK, \\ .17577983 = 11.24990912 = \text{the segment } NHM, \\ .34927384 = 22.35352576 = \text{the zone } MLKN. \end{cases}$$



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## PART III.

### MENSURATION *of* SOLIDS.

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#### GENERAL DEFINITIONS.

**A** Solid or body is a figure extended in every direction. It is commonly said to consist of length, breadth, and thickness, which are three of its extentions; of which the direction of each is perpendicular to those of the other two.

The measure of a solid is called its solidity, capacity, or content.

By the mensuration of solids then are determined the spaces included by contiguous surfaces; and the sum of the measures of these including surfaces is the surface or superficies of the body.

Solids are measured by cubes, whose sides are inches, feet, yards, or any other assigned quantity; and hence the solidity of a body is said to be so many cubic inches, feet, yards, &c. as will fill its capacity or space, or another of an equal magnitude.

The least solid measure is the cubic inch, other cubes being taken from it according to the proportion in the following

TABLE



## TABLE of SOLID MEASURE.

| Cubic inches    | cubic feet         |                   |            |          |         |
|-----------------|--------------------|-------------------|------------|----------|---------|
| 1728            | 1                  | cubic yards       |            |          |         |
| 46656           | 27                 | 1                 | cub. poles |          |         |
| 7762392         | 4492 $\frac{1}{8}$ | 166 $\frac{1}{8}$ | 1          | c. furl. |         |
| 946793088000    | 287496000          | 10648000          | 64000      | 1        | c. mile |
| 254358061056000 | 147197952000       | 5451776000        | 32768000   | 512      | 1       |

## SECTION I.

*Of Prisms, Pyramids, and the Sphere, with the Parts into which some of them may be cut by planes.*

**A** Prism is a solid all whose plane sections, parallel to its end or bases, are equal and similar figures.—It is among solids what a parallelogram is among planes.

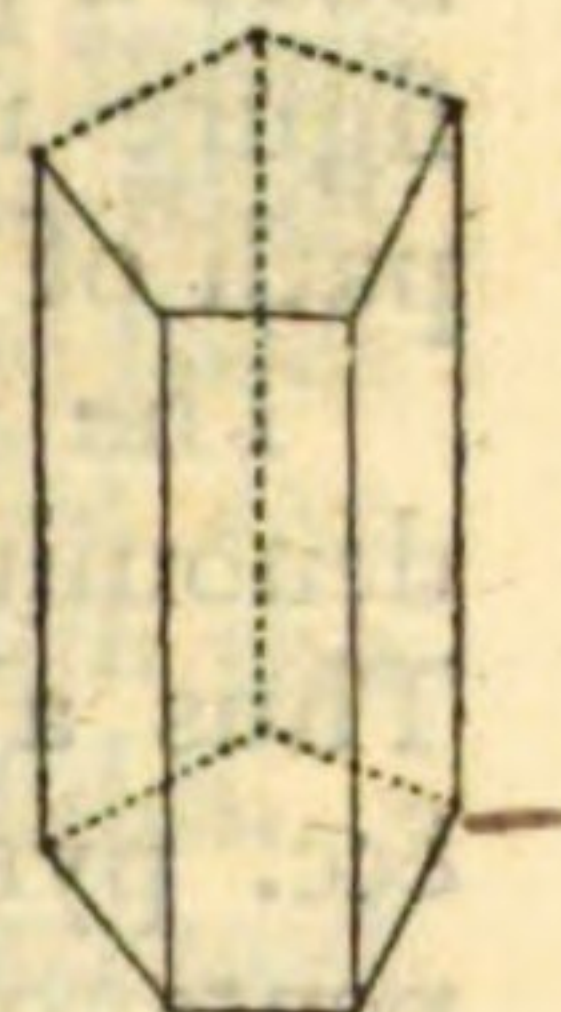
Or a prism is a solid whose sides are parallelograms, the ends of all which parallelograms form the bases or ends of the prism, which ends or bases are two equal, parallel, and similar plane figures.

Prisms receive several particular denominations according to the figure of their ends or bases.

If the base be a triangle, it is called a triangular prism. And is something like a hat box.

If its base be a square, it is a square prism; if a pentagon, a pentagonal prism; if a hexagon, a hexagonal prism; &c.

If the length or height of a square prism be equal to one side of its base, it will be of the form of a die, and is called a cube.



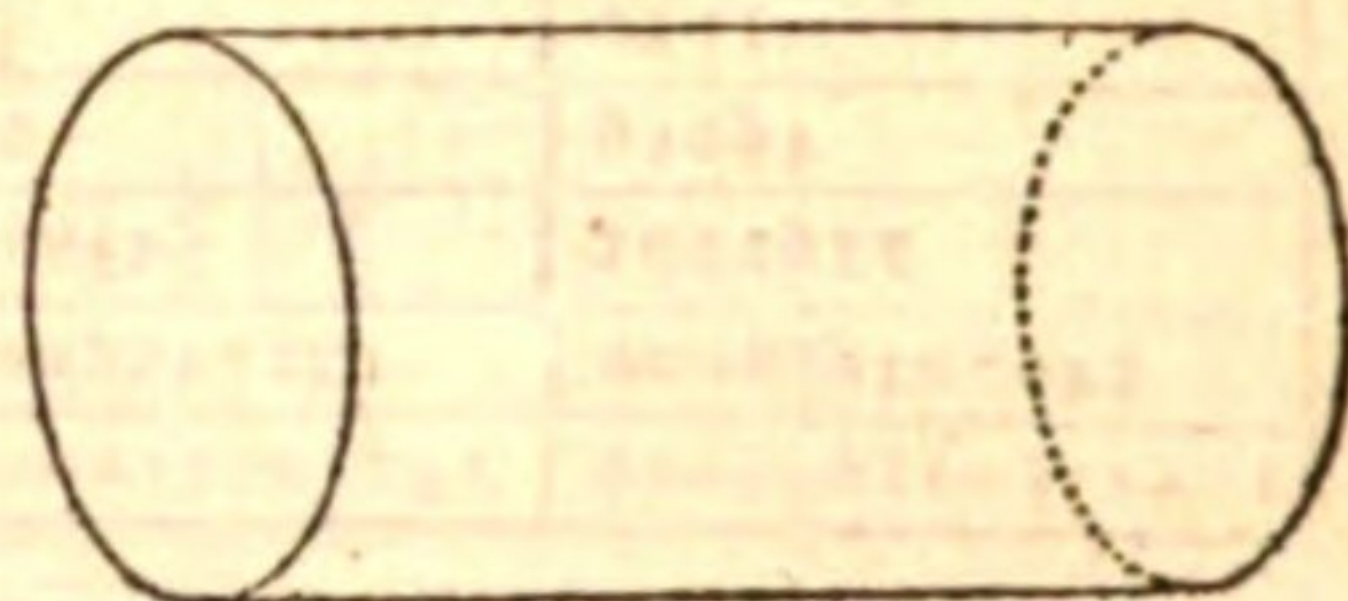
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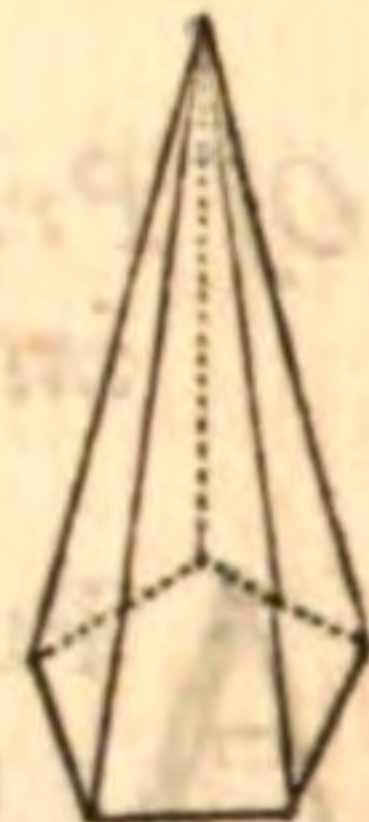
If the base be a rectangle, it is called a parallelopiped: like a box or chest whose length and breadth or height are unequal.

If the base of a prism be a circle, it is called a cylinder: like a rowling-stone.

In this prism the number of parallelograms forming its sides is infinite.



A pyramid is a solid whose sides are triangles, and whose vertexes meet in a point at the top called the vertex of the pyramid. Also the bases of the triangles form the base of the pyramid.



If a right line, moveable about a point fixed above a plane, be moved round so as to describe any figure upon the plane, the space inclosed by the path of the line will be in the form of a pyramid.

The pyramid, like the prism, receives particular denominations according to the figure of its base. Thus a triangular, square, pentagonal, hexagonal, &c. pyramid hath for its base a triangle, square, pentagon, hexagon, &c.

A cone is a round pyramid, or a pyramid having a circular base.

Here the number of triangles forming the upright surface of the pyramid is infinite.

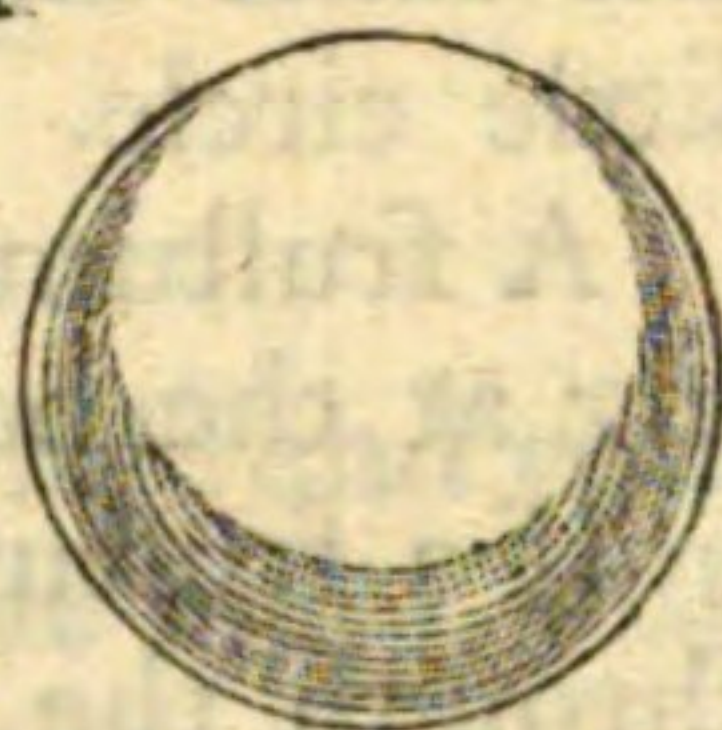


Hence, if a sector of a circle be conceived to be rowled up till its two bounding radiuses coincide, it will take the form of a cone. And, in general, a pyramid



pyramid may be formed by rolling up a part of a plane figure, cut off from it by two lines drawn to the perimeter from any point within it; having cut the figure through from the said point to every angular point of the perimeter, to make it bend or join so as that every part into which it is cut may form a right-lined angle with the next part.

A sphere is a solid bounded by a convex surface, every point of which is equally distant from a point within called the center.



If a semicircle be moved round its diameter fixed, the space through which it moves will be in form of a sphere or globe.

The axe or axis of a solid is a line connecting the middle of its opposite ends. So the axe of a prism is a line from the middle of the one end to the middle of the other; the axe of a pyramid is a line from the vertex to the middle of the base; and the axe or diameter of a sphere is a line drawn through the center and terminated at the surface on both sides.

A right prism, or pyramid is a prism or pyramid whose axe is perpendicular to its base. And

An oblique prism or pyramid is one whose axe is not perpendicular to its base.

Also, a prism, or a pyramid, is said to be regular or irregular according as the base is a regular or an irregular plane figure.

A segment of a pyramid, sphere, or any other tapering solid, is a part cut off the top by a plane parallel to the base.—If the plane pass through the

2 N

center



center of the sphere, it will cut it equally in two, and each half is called a hemisphere.

The section made by the plane is a plane similar to the base of the figure; and every section of a sphere is a circle. If the section be made through the center, it is a great circle of the sphere, having the same diameter with the sphere; if not, it is a little circle.

A frustum, truncus, or trunk, is the part remaining at the bottom after the segment is cut off.

If a frustum be cut by a plane diagonally passing through the extremity of one side at the less end, and through the extremity of the opposite side at the greater end, the two parts into which it is cut are called ungulas or hoofs; the greater hoof being that including the greater end, and the less that including the less.

A zone of a sphere is a part intercepted between two parallel planes.

The middle zone is that whose parallel ends are equally distant from the center.

A sector of a sphere is a solid composed of a segment less than a hemisphere, and of a cone having the same base with the segment, and whose vertex is the center of the sphere.

The height of a solid is the perpendicular drawn from its vertex or top to the plane of its base.

A solid is convex where the surface, being curved, rises outwards; and concave where it sinks inwards.

Similar pyramids are those which are contained under an equal number of similar, and similarly posited plane figures.

Similar



Similar solids are those which are made up of an equal number of similar, and similarly posited pyramids; or that are contained under an equal number of similar plane figures, alike placed.

## PROBLEM I.

*To find the surface of a prism.*

## General RULE.

It is evident, that, if the area of each side and end be calculated separately, the sum of those areas will be the whole surface of any prism, whether right or oblique, or, indeed, of any other body whatever.\*  
—But for a right prism observe the following

## Particular RULE.

Multiply the perimeter of the end into the height, and the product will be the sum of the sides, or upright surface. Or, if the ends of the prism be regular plane figures, multiply the perimeter of the end into the sum of the height of the prism and the radius of the circle inscribed in the end, and the product will be the whole surface.

EXAMPLE I. What is the upright surface of a triangular prism whose length is 20 feet, and the ends of its base each 18 inches?

Here  $18 \times 3 = 54$  inches  $= 4\frac{1}{2}$  feet equal to the perimeter of the base.

Therefore

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\* The surfaces of similar prisms, and indeed of any other similar bodies, are as the squares of their like lineal dimensions; this follows from their being composed of similar plane figures, alike placed.



Therefore  $4\frac{1}{2} \times 20 = 90$  square feet equal to the upright surface.

Again, by rule 2 problem 4 section 1, we have  $2 \times \frac{3}{2} \times \frac{3}{2} \times 433013 = 1.9485585 =$  the area of the two ends.  
Whence  $91.9485585$  is the whole surface.

EXAMPLE II. What is the surface of a cube the length of each of whose sides is 20 feet?

$20 \times 20 = 400$  equal to the area of one side,

And  $400 \times 6 = 2400 =$  the whole surface required.

EXAMPLE III. What must be paid for lining a rectangular cistern with lead at 2d. a lb. the lead being 7 feet to the lb. supposing (on the inside) the length to be 3 feet 2 inches, the breadth 2 feet 8 inches, and height 2 feet 6 inches?

First,  $38 + 32 \times 2 \times 30 = 70 \times 60 = 4200$  square inches equal to the two sides and two ends together.

Then,  $38 \times 32 = 1216 =$  to the area of the bottom.

Wherefore  $4200 + 1216 = 5416$  square inches =  $37\frac{1}{4}$  square feet = the whole area.

But  $37\frac{1}{4} \times 7 = 263\frac{1}{4}$  lb. = the whole weight.

Therefore 1 lb. : 2d. ::  $263\frac{1}{4}$  : 2l. 3s.  $10\frac{1}{2}$ d. the cost.

EXAMPLE IV. What is the convex surface of a round prism or cylinder whose length is 20 feet, and the diameter of whose base is 2 feet?

$3.1416 \times 2 = 6.2832 =$  the circumference.

Therefore  $6.2832 \times 20 = 125.664 =$  the convex surface.

EXAMPLE V. What is the whole surface of a cylinder whose length is 10 feet, and the circumference is 3 feet?

Here



Here  $10 + \frac{3}{2 \times 3 \cdot 1416} \times 3 = 30 + \frac{9}{6 \cdot 2832} = 31 \cdot 43239 =$  the whole surface required.

## PROBLEM II.

*To find the solidity of a prism.*

## RULE.

Multiply the area of the base into the height, and the product will be the solidity.\*

## EXAMPLE I.

How many solid feet are in a square prism whose length is  $5\frac{1}{2}$  feet, and each side of its base is  $1\frac{1}{3}$  feet?

$1\frac{1}{3} \times 1\frac{1}{3} = \frac{4}{3} \cdot \frac{4}{3} = \frac{16}{9} =$  the area of the end. And  $\frac{16}{9} \times 5\frac{1}{2} = \frac{16}{9} \times \frac{11}{2} = \frac{88}{9} = 9\frac{7}{9}$  solid feet equal to the content required.

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Ex-

\* For, if we conceive to be cut off from the prism, by a plane parallel to the ends, a part whose height is equal to the lineal measuring unit; and then imagine both its ends to be divided in a similar manner into so many squares as are expressed by the area of each, the side of each of those squares being equal to the lineal measuring unit; and, lastly, suppose planes to be drawn through the corresponding lines of division; it will be evident that the part cut off will be divided, by those planes, into so many cubes as there are squares in each end, and also having the same dimension with those squares, viz. the lineal measuring unit; and this number of cubes is the measure of the part.

But the magnitude of the whole prism, or, indeed, of any other of an equal base, is, evidently, to the magnitude of the part whose height is the lineal measuring unit, as the length of the whole is to that unit (1).

And, therefore, the measure of the whole is equal to that of the part so often repeated as there are lineal measuring units in the height; that is, equal to the base drawn into the height.

And that the rule is true for oblique prisms as well as right ones, will be evident by conceiving, according to the method of *Cavalieri*, a right and an oblique prism, of equal bases and heights, to be made up of an indefinite number of equal thin plates all parallel to the base; for the prisms being both of a height, the one of them will require as many of such plates to compose it as the other.



## EXAMPLE II.

How many ale gallons of water will the cistern hold whose length, breadth, and height are 3 feet 2 inches, 2 feet 8 inches, and 2 feet 6 inches?

$38 \times 32 \times 30 = 36480 =$  the solidity in inches. And  $36480 \div 282 = 6080 \div 47 = 129\frac{17}{47}$  ale gallons.

## EXAMPLE III.

What is the capacity of a cylinder whose height is 20 feet, and the circumference of its base 20 feet, also?

First,  $\frac{20}{3.1416} =$  the diameter, and  $10 \times \frac{10}{3.1416} = \frac{100}{3.1416} = \frac{25}{.7854} =$  the area of the end.

Then,  $\frac{25}{.7854} \times 20 = \frac{250000}{3927} = 636.61828 =$  the capacity required.

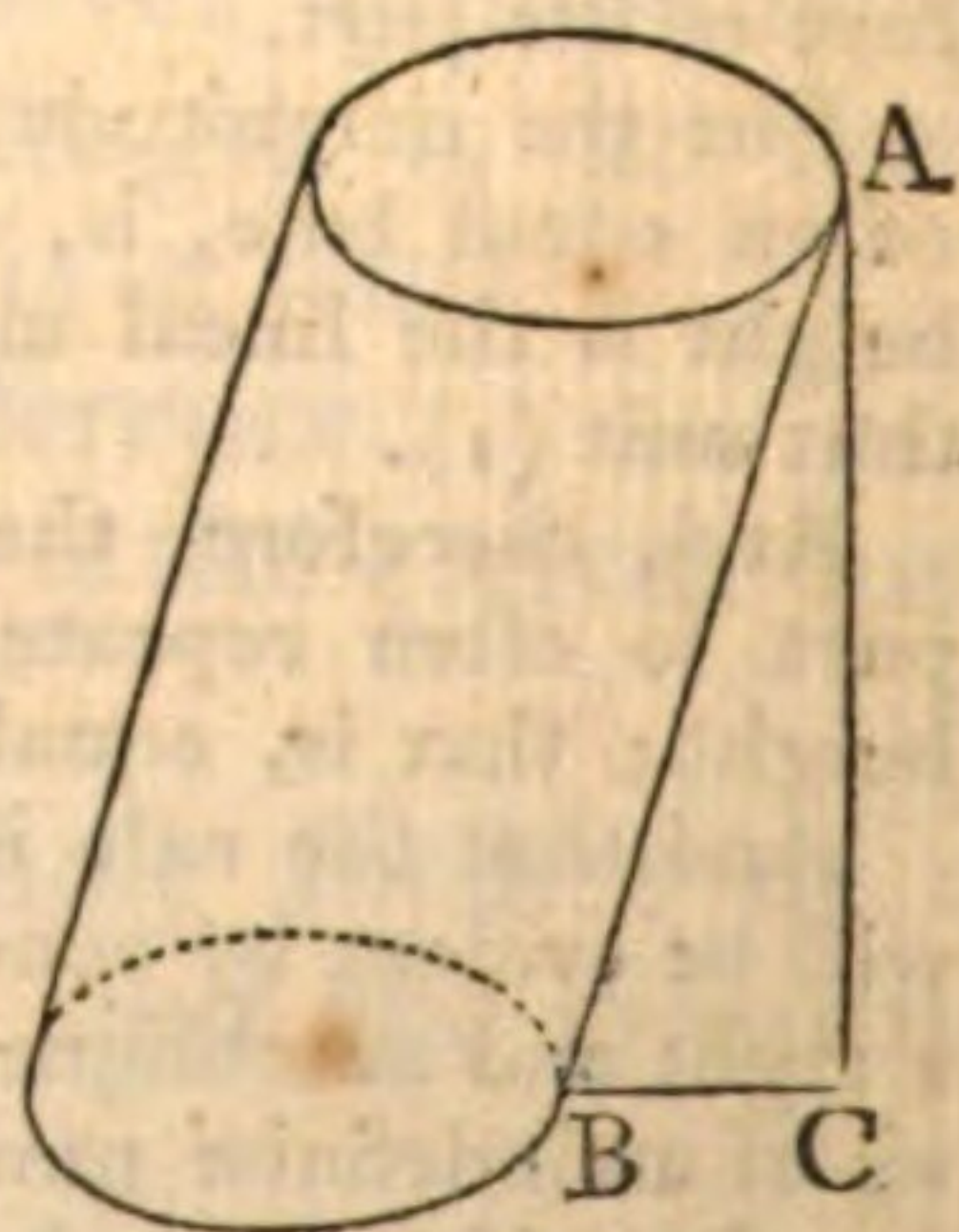
## EXAMPLE IV.

What is the capacity of the oblique cylinder whose axe is 20 feet, and the circumference of whose base is 20 feet also; the axe making an angle of  $75^\circ$  with the base?

As in the last example, the area of the base is  $\frac{25}{.7854}$  or  $\frac{125000}{3927}$ .

But as  $1(\text{rad.}):.9659258(\text{s. } 75^\circ = \angle B)::20(AB):.9659258 \times 20 = 19.318516 = AC =$  the height of the cylinder.

Therefore  $\frac{125000}{3927} \times 19.318516 = 125000 \times .0049194082 = 614.92602$  the capacity required.



Other-







Then  $\frac{b}{v} \times ds - ac = \text{the convex surface.}^*$

That is, From the product of the diameter and sine, subtract the product of the arc and cosine, and multiply the difference by the quotient arising from the division of the height by the versed sine.

*Note 1.* When  $F$  is the center of the base; then  $v = s = \frac{1}{2}d$ , and  $c = 0$ ; and then the theorem becomes  $dh$ , viz. the product of the diameter and height equal to the curve surface.

*Note 2.* When  $AF$  exceeds  $\frac{1}{2} AB$ , then  $ac$  must be added.

#### EXAMPLE I.

Given the diameter  $AB$  equal to 100, the height  $AD$  equal to 140, and the versed sine  $AF$  equal to 10: required the curve surface.

Here

\* For, having drawn  $HI$ ,  $IK$  parallel to  $FA$  and  $AD$  respectively, and joined the points  $H$ ,  $K$ , since it is evident that the surface is generated by the motion of  $IK$  along the arc  $AIG$ ,  $KI \times$  the fluxion of  $IA$  will be the flux. of the surface. Wherefore put  $z = AI$ ,  $x =$  its sine  $IL \parallel GE$ , and  $y =$  its cosine; then  $HI = LF = FA - AL = v - \frac{1}{2}d + y$ , and, by similar triangles,  $FA : AD :: HI : IK = \frac{b}{v} \times \overline{v - \frac{1}{2}d + y}$ ; and hence the fluxion of the surface, or  $\dot{z} \times IK$  is  $\frac{b}{v} \times \dot{v}z - \frac{1}{2}\dot{d}z + y\dot{z}$ . But  $y\dot{z} = \frac{1}{2}\dot{x}$ , and therefore  $\dot{z} \times IK = \frac{b}{v} \times \dot{v}z - \frac{1}{2}\dot{d}z + \frac{1}{2}\dot{x}$ ; the fluent of which is  $\frac{b}{v} \times \overline{vz - \frac{1}{2}dz + \frac{1}{2}dx}$   $= \frac{b}{v} \times \frac{1}{2}dx - cz =$  (when  $IA = AG$ )  $\frac{b}{v} \times \frac{1}{2}ds - \frac{1}{2}ac$ , the double of which is  $\frac{b}{v} \times ds - ac =$  the whole convex surface  $DEAGD$ .

#### COROLLARY I.

If  $F$  be the center; then  $v = s = \frac{1}{2}d$ , and  $c = 0$ ; and then the theorem becomes barely  $dh = 4$  times the triangle  $DAF$ .

#### COROLLARY II.

When  $AF$  exceeds  $\frac{1}{2}d$ ,  $c$  is negative, and then  $-ac$  becomes  $+ac$ .

#### COROLLARY III.

If  $F$  coincide with  $B$ ; then  $s = 0$ , and  $c = -\frac{1}{2}v$ ; and the theorem becomes  $\frac{1}{2}ab =$  half the surface of the cylinder.



Here  $d=100$ ,  $b=140$ , and  $v=10$ : wherefore  $\frac{1}{2}d-v=50-10=40=c$ . And  $\sqrt{\frac{1}{4}dd-cc}=\sqrt{2500-1600}=\sqrt{900}=30=s$ .

But  $\frac{s}{\frac{1}{2}d}=\frac{30}{50}=\frac{3}{5}=.6$ =the sine reduced to the radius 1, to which, in a table of sines, belong 36 degrees, 52.268 minutes = 36.87113 degrees. Then, by rule 1 for the length of a circular arc,  $.01745329 \times 36.87113 \times 100 = 64.352252$  = the arc  $a$ .

Whence  $\frac{b}{v} \times ds - ac = 14 \times 3000 - 2574.09008 = 14 \times 425.90992 = 5962.73888$  = the convex surface required.

## EXAMPLE II.

If the diameter and height be 100 and 140, as before, and the section be made through the center of the base, or  $v=\frac{1}{2}d=50$ ; what is the convex surface?

Here, by note 1,  $db=100 \times 140=14000$  = the convex surface required.

## EXAMPLE III.

Supposing  $d$  and  $b$  still the same, and  $v=90$ ; to find the convex surface.

Here  $\frac{1}{2}d-v=50-90=-40=c$ ,  $s=30$  the same as before, but is, here, the sine of the supplemental arc, which therefore is  $180-36.87113=143.12887$  degrees. Hence  $.01745329 \times 143.12887 \times 100 = 249.807013$  = the arc  $a$ . Or the arc may be sooner found by only subtracting the arc in the first example, viz. 64.352408, from 314.159 &c. the whole circumference.

Then, by note 2,  $\frac{b}{v} \times ds + ac = \frac{14}{90} \times 3000 + 9992.28052 = \frac{14}{90} \times 12992.28052 = 20210.21414$  = the convex surface required.



## PROBLEM IV.

*To find the solidity of the hoof of a cylinder.*

## RULE.

From  $\frac{2}{3}$  of the cube of the right sine subtract the product of the base and cosine of half the arc of the base, then multiply the difference by the quotient of the height divided by the versed sine, and the product will be the solidity required.

That is, putting  $b$  = the height  $AD$

$v$  = the versed sine  $AF$

$s$  = the right sine  $FG$

$c$  = the cosine  $= \frac{1}{2}AB - AF$

and  $b$  = the base, or area of the seg.  $GAEG$ ;

Then  $\frac{2}{3}s^3 - bc \times \frac{b}{v}$  = the solidity.\*

*Note*

\* For the fluxion of the solid is = the  $\triangle HIK$  drawn into the fluxion of  $LI$ , which fluxion will, therefore, be  $\dot{x} \times \frac{b}{2v} \times HI^2$  (using the same characters as in the demonstration of the last problem)  $= \frac{bx}{2v} \times y - c^2 = \frac{bx}{2v} \times yy - 2cy + cc = \frac{bx}{2v} \times \frac{1}{4}dd - xx - 2cy + cc = \frac{bx}{2v} \times \frac{1}{4}dd - xx - dc + 2c \times AL + cc = \frac{bx}{2v} \times ss - xx - 2c \times FL$ ; whose fluent  $\left( \frac{bx}{2v} \times ss - \frac{1}{3}xx - \frac{bc}{v} \times \text{area } FAIH \right)$  when  $I$  coincides with  $G$ , is  $\frac{b}{2v} \times \frac{2}{3}s^3 - bc$ , the double of which is  $\frac{b}{v} \times \frac{2}{3}s^3 - bc$  = the content of the solid  $DEAGD$  required.

## COROLLARY I.

If  $F$  fall in the center of the base; then  $c = 0$ , and  $s = v = \frac{1}{2}d$ ; and the rule will be  $\frac{1}{6}ddh$ .

## COROLLARY II.

If  $AF$  exceed  $FB$ ,  $c$  will be negative, and then  $-bc$  will become  $+bc$ .

## COROLLARY III.

If  $F$  fall in  $B$ ;  $s = 0$ , and  $c = -\frac{1}{2}v$ ; and then the theorem becomes  $\frac{1}{2}bb$  = half the cylinder.



*Note 1.* If  $F$  be the center, that is, if the base be equal to the semicircle, then  $v = s$ , and  $c = 0$ ; and therefore  $\frac{2}{3}bss = \frac{1}{6}ddh$  is the solidity in that case.

*Note 2.* If  $v$  exceed  $\frac{1}{2}d$ , that is, if the base exceed the semicircle, then  $c$  is negative, and  $bc$  must be added.

## EXAMPLE I.

If the diameter  $AB$  be 50, the height  $AD$  120, and the versed sine  $AF$  10; what is the solidity of the hoof.

Or supposing a cylindric vessel  $ABCD$ , containing a fluid, to be placed in such a position that the surface of the fluid, disposing itself parallel to the horizon, may cut the base in  $GE$  leaving 40 inches of the diameter dry, and the side of the cylinder in  $D$  120 inches distant from the base; to find how many ale gallons are in it; the diameter of the base being 50 inches.

Here  $h = 120$ ,  $d = 50$ , and  $v = 10$ . Then  $\frac{1}{2}d - v = 25 - 10 = 15 = c$ , and  $\sqrt{\frac{1}{4}dd - cc} = \sqrt{25^2 - 15^2} = \sqrt{5^2 \times 5^2 - 3^2} = \sqrt{5^2 \times 4^2} = 5 \times 4 = 20 = s$ .

And, to find the base by the table of segments,  $\frac{v}{d} = \frac{10}{50} = .2$ , this found in the column of versed sines, opposite to it I find the area .1118238: hence  $50 \times 50 \times .1118238 = 279.5595 = b$  = the segment or base.

Then  $\frac{h}{v} \times \frac{2}{3}s^3 - bc = 12 \times \frac{2}{3} \times 8000 - 15 \times 279.5595 = 12 \times 5333\frac{1}{3} - 4193.3925 = 12 \times 1139.9408 = 13679.2896 =$  the solidity in inches; which divided by 282, the inches in a gallon, give 48.50939 ale gallons for the content.

## EXAMPLE II.

Suppose the cylinder so placed, that the surface of the liquor may bisect the base, and rise up the side  
to



to the same distance of 120 inches from the base: to find the content.

Here, by note 1, we have  $\frac{1}{6}ddh = \frac{50 \times 50 \times 120}{6} = 50 \times 50 \times 20 = 50000$  solid inches = 177.3049645 gallons, for the content in this case.

### EXAMPLE III.

Suppose, now, the same vessel so placed as that the surface of the liquor may leave only 10 inches of the diameter dry, still rising to the same distance of 120 inches along the side; to find the content.

Here the part of the cylinder's base left dry is equal to the base in the first example, viz. 279.5595; which, therefore, taken from  $50 \times 50 \times 78539816 = 1963.4954$ , the whole circle, leaves  $1683.9359 = b$  = the base of the ungula in this example.

Now  $v = 40$ ,  $c = -15$ , and  $s = 20$ .

Whence  $\frac{b}{v} \times \frac{2}{3} s^3 - bc = \frac{1683.9359}{40} \times \frac{2}{3} \times 8000 + 25259.0385 = 3 \times 30592.3718 = 91777.1154$  solid inches = 325.45076 gallons = the content in this case.

### PROBLEM V.

*To find the surface of a right pyramid.*

#### RULE.

Multiply the perimeter of the base by the flant height or length of the side, and half the product will, evidently, be the surface, or the sum of the areas of all the triangles which form it.\*

Ex-

#### \* COROLLARY I.

Hence, because in a right cone the circumference of the base into half the side is equal to the curve surface, and into half the radius is equal



## EXAMPLE I.

What is the surface of a triangular pyramid, the flant height being 20 and each side of the base 3?

First,  $3 \times 3 = 9 =$  the perimeter of the base.

And  $\frac{9 \times 20}{2} = 9 \times 10 = 90 =$  the surface required.

## EXAMPLE II.

Required the surface of a square pyramid whose flant height is 20 and each side of the base 3.

$\frac{3 \times 4 \times 20}{2} = 12 \times 10 = 120 =$  the surface sought.

## EXAMPLE III.

Required the convex surface of a circular pyramid, or cone, whose flant side is 20 and the circumference of the base 9.

$\frac{20 \times 9}{2} = 10 \times 9 = 90 =$  the convex surface.

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PRO-

equal to the base, therefore as the radius of the base is to the side, so is the base to the curve surface.—And the same is true of any other pyramid whose base is a regular figure, viz. as the radius of the circle inscribed in the base is to the flant height, so is the base to the surface.

## COROLLARY II.

Let there be a cone and cylinder of the same base and altitude, the common altitude being equal to the radius of the base; then the base, the surface of the cone, and the surface of the cylinder, are to one another as the numbers 1,  $\sqrt{2}$ , and 2, and so are in continual proportion.

And the same is true of any other regular prism and pyramid of the same base and altitude, the altitude being equal to the radius of the circle inscribed in the base.



## PROBLEM VI.

*To find the surface of the frustum of a right pyramid.*

## RULE.

Multiply the sum of the perimeters of the ends by the slant height, and half the product will be the surface required.\*

## EXAMPLE I.

How many square feet are in the surface of the frustum of a square pyramid whose slant height is 10 feet, each side of the greater base being 3 feet 4 inches, and each side of the less 2 feet 2 inches?

$$4 \times 40 + 26 = 4 \times 66 = 264 = \text{the sum of the perimeters.}$$

$$\frac{264 \times 10 \times 12}{2} \text{ square inches} = \frac{264 \times 10 \times 12}{2 \times 144} \text{ square feet} =$$

$$\frac{22 \times 10}{2} = 11 \times 10 = 100 \text{ feet} = \text{the surface required.}$$

## EXAMPLE II.

If a segment of 6 feet slant height be cut off a cone whose slant height is 30 feet, and circumference of its base 10 feet, what will be the surface of the frustum?

As 30 = the height of the whole : 6 = the height of the upper part :: 10 = the bottom circumference :  $\frac{6}{30} \times 10 = 2$  = the circumference at the top of the frustum.

$$\text{Then } \frac{10+2}{2} \times 24 = 6 \times 24 = 144 = \text{the surface required.}$$

PRO-

## \* DEMONSTRATION.

For the surface is composed of a number of equal trapezoids whose common height is equal to the slant height of the frustum, and the sums of whose parallel sides make up the perimeters of the ends of the frustum.



## PROBLEM VII.

To find the solidity of a pyramid.

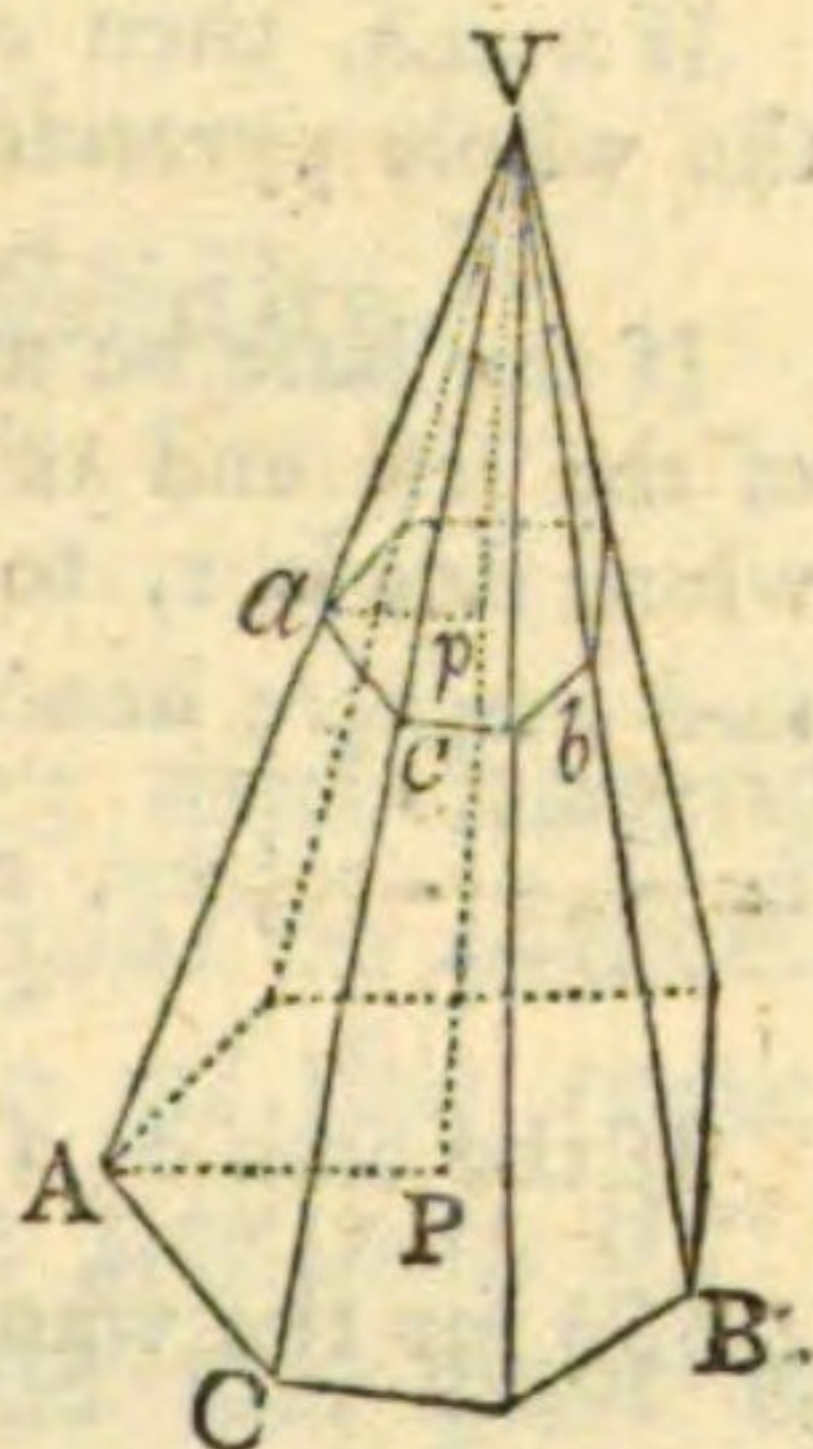
## RULE.

Multiply the base by the perpendicular height, and  $\frac{1}{3}$  of the product will be the content.\*

Ex-

\* Let  $VAB$  be a pyramid whose base is any plane figure, regular or irregular,  $VP$  a  $\perp$  to the base in  $P$  and meeting the section  $ab$ , parallel to the base, in  $p$ ; and put  $A$  for the area of the base  $AB$ ,  $a$  for that of the section  $ab$ ,  $h$  the height  $PV$  of the pyramid, and  $x$  the height  $Pp$  of the frustum  $Aabb$ .

First, the parallel sections  $AB$ ,  $ab$ , are to each other as the squares of their distances,  $PV$ ,  $Vp$ , from the vertex. For, because they are similar plane figures, which are as the squares of their like sides, we shall have  $A:a::AC^2:ac^2::$  (by sim.  $\Delta$ s)  $AK^2:Va^2::$  (by sim.  $\Delta$ s again)  $PV^2:(h-x)^2::$



Hence we find  $a = \frac{A}{h^2} \times (h-x)^2$ , and  $ax = \frac{Ax}{h^2} \times (h-x)^2$ , the fluent of

which is  $Ax \times 1 - \frac{x^2}{h} + \frac{xx^2}{3hb} =$  the content of the frustum  $Aabb$ .

## COROLLARY I.

Since  $\sqrt{A}:\sqrt{a}::h:h-x$ , it will be  $\sqrt{A}-\sqrt{a}:\sqrt{A}::x:(h-h-x):h = \frac{x\sqrt{A}}{\sqrt{A}-\sqrt{a}}$ , which being substituted for  $b$  in the Quantity  $Ax \times 1 - \frac{x^2}{h} + \frac{xx^2}{3hb}$ , expressing the content of the frustum, that content will become  $\frac{1}{3}x \times \frac{A+a+\sqrt{Aa}}{\sqrt{A}-\sqrt{a}}$ .

## COROLLARY II.

Or, if  $S$  be a side or any other line in the greater end, and  $s$  a similar side or line in the less, since  $S-s:S::x:h$ , hence  $\frac{x}{h} = \frac{S-s}{S}$ , which being substituted instead of it, gives  $\frac{1}{3}Ax \times \frac{SS+Ss+ss}{SS}$  for the value of the said frustum.

Co-



## EXAMPLE I.

Required the solidity of a triangular pyramid whose height is 30 and each side of the base 3.

First,  $3 \times 3 \times \cdot 433013 = 3 \cdot 897117 =$  the area of the base.

Then  $\frac{3 \cdot 897117 \times 30}{3} = 38 \cdot 97117 =$  the solidity.

E x-

## COROLLARY III.

If  $x = b$ , then  $a = 0$ , and the general expression becomes  $\frac{1}{3} Ab =$  the whole pyramid; which is our rule.

## COROLLARY IV.

If the base be a regular figure; let  $S$  be one of its sides,  $s =$  a side of the less end of the frustum, and  $n =$  the area of a similar figure whose side is 1, to be found in the table in page 81; then  $A = nSS$ , and  $a = nss$ ; hence the solidity of the frustum will be  $\frac{SS + Ss + ss}{3} \times \frac{1}{3} nx = \frac{S^3 - s^3}{S - s} \times \frac{1}{3} nx$ , and that of the pyramid  $= \frac{1}{3} nSSb$ .

## COROLLARY V.

If the base be a square; then  $n = 1$ , and the last expressions become  $\frac{1}{3} SSb$  for the whole pyramid, and  $\frac{1}{3} x \times \frac{SS + Ss + ss}{S - s}$  or  $\frac{S^3 - s^3}{S - s} \times \frac{1}{3} x$  for the frustum.

## COROLLARY VI.

If the base be a circle; then  $n = \cdot 7854$ ,  $\cdot 2618SSb =$  the cone, and  $\frac{SS + Ss + ss}{3} \times \cdot 2618x$  or  $\frac{S^3 - s^3}{S - s} \times \cdot 2618x =$  the frustum; where  $S$  is the diameter of the base, and  $s$  that of the top.

## COROLLARY VII.

If  $a = A$ ; then  $\frac{A + a + \sqrt{Aa}}{3} \times \frac{1}{3} x$  becomes  $3 A \times \frac{1}{3} x = Ax =$  the prism.

## COROLLARY VIII.

Hence a prism is to a pyramid of the same base and height as 3 to 1, and to the frustum of the same base and height as 1 to  $1 - \frac{x}{b} + \frac{xx}{3bb}$ , or as 3 to  $1 + \frac{a}{A} + \sqrt{\frac{a}{A}}$ , or as 3 to  $\frac{SS + Ss + ss}{SS}$  or  $1 + \frac{s}{S} + \frac{ss}{SS}$ .

## COROLLARY IX.

Similar pyramids are as the cubes of their like sides. For the pyramid is as  $Ab$ , or (because  $A$  is as  $bb$ ) as  $b^3$ , or (by similar  $\Delta$ s) as  $V A^3$ .

## COROLLARY X.

All similar solids are as the cubes of their like sides. This follows from their being composed of similar pyramids.



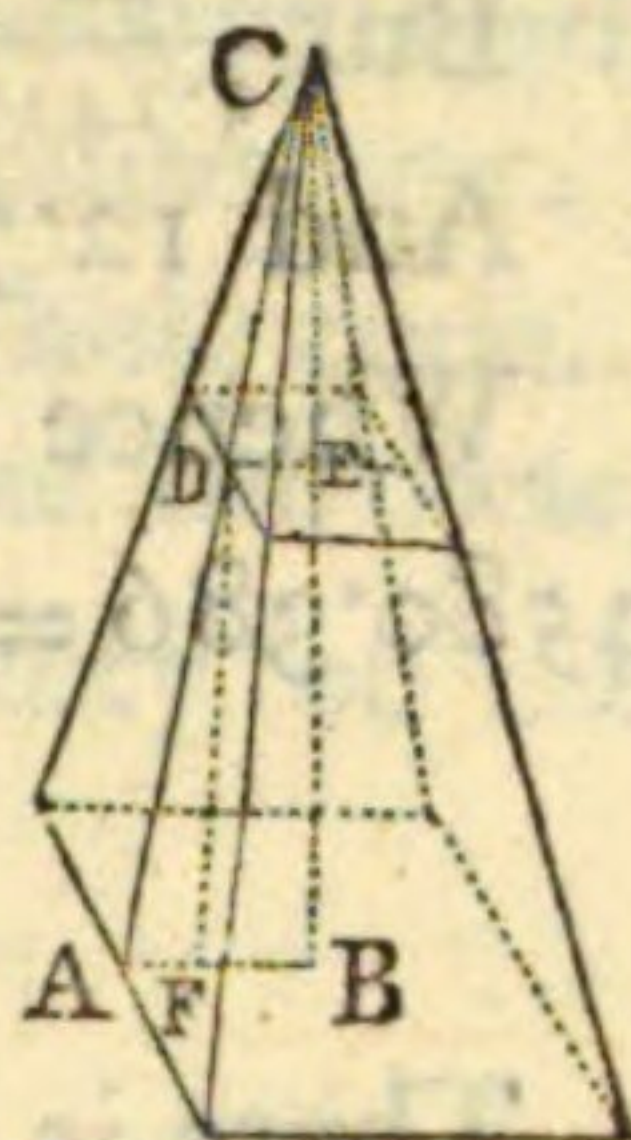
## EXAMPLE II.

Required the solidity of the square pyramid each side of whose base is 30 and the slant height 25.

First,  $30 \times 30 = 900 =$  the base.

But  $BC = \sqrt{CA^2 - AB^2} = \sqrt{25^2 - 15^2}$   
 $= \sqrt{5^2 \times 5^2 - 3^2 \times 5^2} = \sqrt{16 \times 5^2} = 4 \times 5 =$   
 $20 =$  the perpendicular height.

Then  $\frac{900 \times 20}{3} = 300 \times 20 = 6000 =$  the solidity.



## EXAMPLE III.

Being to measure an oblique cone; and having taken the circumference of the base equal to 40 feet, and the outward angle which the side of the cone, where it was shortest, made with the horizon  $85^\circ$ ; going thence in a direct line towards the part to which the vertex inclined, to the distance of 100 feet, I found there the angle of elevation of the vertex of the cone to be  $50^\circ$ : required the solidity.

The  $\angle B + \angle E = 85^\circ + 50^\circ = 135^\circ$ .

And  $180^\circ - 135^\circ = 45^\circ = \angle BCE$ .

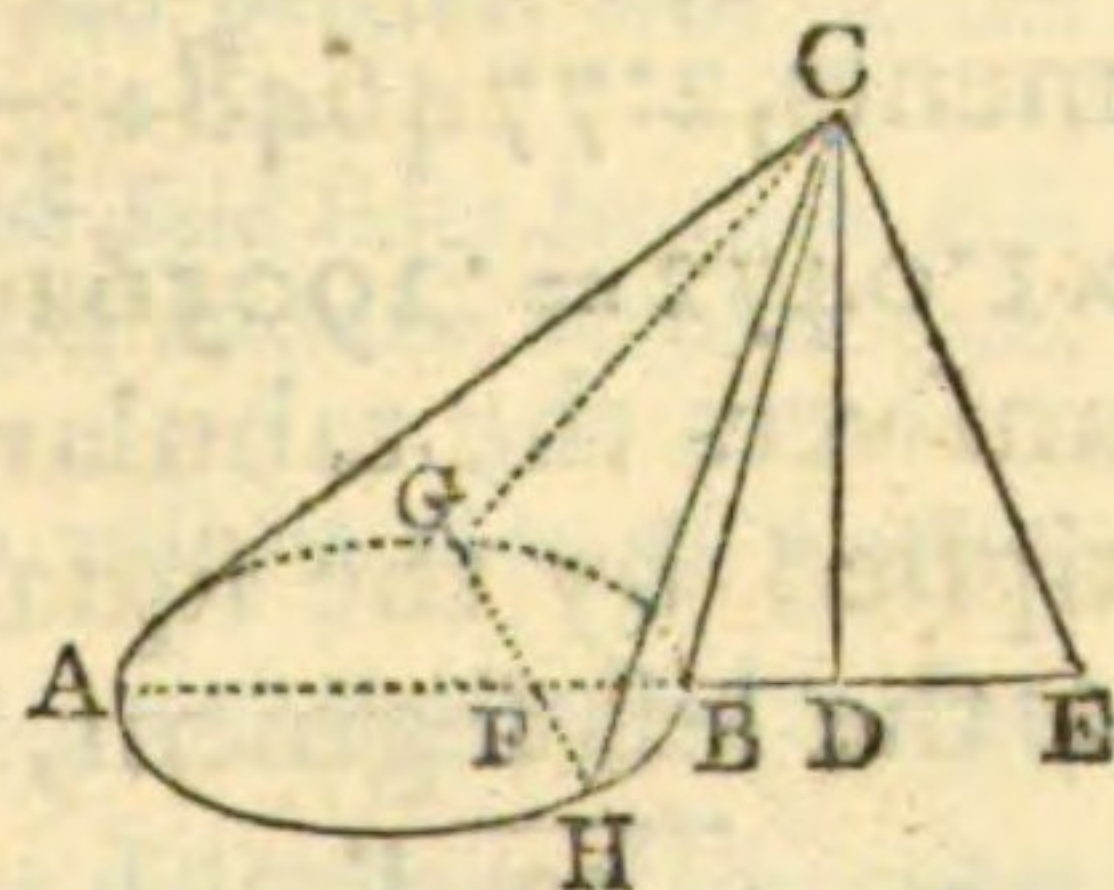
Then

s.  $\angle BCE = 45^\circ$  — 9.8494850

$BE = 100$  — — 2.0000000

s.  $\angle E = 50^\circ$  — 9.8842540

$BC$  — — 2.0347690



And s.  $\angle D = 90^\circ$  — — 10.0000000

$BC$  — — — 2.0347690

s.  $\angle B = 85^\circ$  — — 9.9983442

$DC = 107.9228$  — 2.0331132

2 R

But



But  $\frac{40}{3 \cdot 1416} = \frac{10}{\cdot 7854} = 12 \cdot 732396 = \text{the diameter.}$

And  $12 \cdot 732396 \times 10 = 127 \cdot 32396 = \text{the base.}$

Whence  $\frac{DC \times 127 \cdot 32396}{3} = 42 \cdot 44132 \times 107 \cdot 9228 = 4580 \cdot 386 = \text{the content of the cone.}$

#### EXAMPLE IV.

There is a cone whose perpendicular altitude is 40 feet, and the circumference of its base 30 feet, which is cut by a plane drawn from the vertex, and passing through the base at the distance of 2 feet from the center; what is the solidity of the whole cone and of the two pyramids, *GHBC*, *GHAC*, into which it is cut? [The last fig.]

Now  $\frac{30}{3 \cdot 1416} = 9 \cdot 54929 = \text{the diameter of the base.}$

And  $\frac{9 \cdot 54929}{2} = 4 \cdot 774648 = \text{the radius.}$

Therefore  $4 \cdot 774648 - 2 = 2 \cdot 774648 = \text{the versed sine of the less segment of the circle.}$

Then, to find its area by the table of circular segments,  $2 \cdot 774648 \div \frac{30}{3 \cdot 1416} = 2 \cdot 774648 \times \frac{3 \cdot 1416}{30} = 2 \cdot 774648 \times 1 \cdot 0472 = \cdot 2905611$  the tabular versed sine, to which answers the tabular segment  $\cdot 18955709$ , which multiplied by the square of the diameter gives  $17 \cdot 2856 = \text{the less segment, or base of the less pyramid.}$

But  $\left(\frac{30}{3 \cdot 1416}\right)^2 \times \cdot 7854 = \frac{900}{4 \times 3 \cdot 1416} = 71 \cdot 6197 = \text{the whole base.}$

And hence  $71 \cdot 6197 - 17 \cdot 2856 = 54 \cdot 3341 = \text{the greater segment, or base of the greater pyramid.}$

Then



Then  $\frac{4}{3} \times \begin{cases} 17.2856 = 230.474\frac{2}{3} = \text{the less pyramid,} \\ 54.3341 = 724.454\frac{2}{3} = \text{the greater pyr.} \end{cases}$   
 their sum is  $954.929\frac{1}{3} = \text{the whole cone.}$

Or, since  $71.6197$  is the area of the base of the cone,  
 therefore  $\frac{71.6197 \times 40}{3} = 954.929\frac{1}{3} = \text{the solidity of the cone.}$

## P R O B L E M VIII.

*To find the solidity of the frustum of a pyramid.*

## R U L E.

Add into one sum the areas of the two ends and the mean proportional between them, multiply the sum by the perpendicular height, and  $\frac{1}{3}$  of the product will be the solidity.

That is, If  $A$  be the area of the greater end,  
 $a$  that of the less, and  $b$  the height,

Then  $A + a + \sqrt{Aa} \times \frac{1}{3} b$  will be the solidity.

*Note 1.* If the ends be regular polygons, the particular rule for them will be easier, thus: Add together the square of a side of each end of the frustum, and the product of those sides, multiply the sum into the height and the product into the tabular area answering to the particular figure of the ends, and  $\frac{1}{3}$  of the last product will be the content.

Or, Divide the difference of the cubes of the said sides by their difference, and multiply the quotient by the height, and the tabular area, and take  $\frac{1}{3}$  of the product.

*Note 2.* If the ends be circles, the frustum will be that of a cone, and then multiply  $.2618 \left( \frac{.7854}{3} \right)$  by the height, and the product either by the quotient arising from the division of the difference of the cubes of the diameters by the difference of the diameters, or by the sum arising from the addition of the square of each diameter and the product of the diameters.\*

E x-

\* All these rules come from the demonstration of prob. 7 and its corollaries.



## EXAMPLE I.

How many solid feet are there in a tree whose bases are squares, each side of the one being 15 inches, and each side of the other 6, and the length measures along the side 24 feet? [Fig. to ex. 2 prob. 7.]

Here  $5 \times 15 = 225 =$  the greater base,

$6 \times 6 = 36 =$  the less,

and  $15 \times 6 = 90 =$  their mean.

their sum is 351

But,  $AB - DE = 7\frac{1}{2} - 3 = 4\frac{1}{2} = AF$ , and  $\sqrt{DA^2 - AF^2} = \sqrt{24 \times 12^2 - 4\frac{1}{2}^2} = 287.9649$  inches  $= FD =$  the perpendicular height.

Therefore  $\frac{351 \times 287.9649}{3} = 117 \times 287.9649 = 33691.8933$  inches  $= 19.49762$  feet  $=$  the solidity.

## EXAMPLE II.

If a cask which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, its head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold?

Here  $20^2 \times .7854 = 314.16 =$  area at the end,

$28^2 \times .7854 = 615.7536 =$  area of the bung circle,

and  $\sqrt{20^2 \times .7854 \times 28^2 \times .7854} = 20 \times 28 \times .7854 = 439.824 =$  their mean proportional; their sum is 1369.7376, which multiplied by  $\frac{40}{3}$  ( $\frac{1}{3}$  of the length of both frustums together) produces 18263.168 solid inches; which divided by 231, the inches in a wine gallon,

thus,  $231 = 3 \times 7 \times 11$   $\left\{ \begin{array}{l} 3) 18263.168 \\ 7) 6087.7226 \\ 11) 869.6746 \end{array} \right.$

gives 79.0613 wine gallons.

Or,



Or,  $20^2 + 28^2 + 20 \times 28 \times 40 \times .2618 = 1744 \times 10.472$   
 $= 18263.168 =$  the solidity the same as before.

## PROBLEM IX.

*To find the solidity of a cuneus or wedge.*

A wedge is a solid having a rectangular base *AC*, and two of the opposite sides ending in an acies or edge *EF*.

## RULE.

To twice the length of the base add the length of the edge, multiply the sum by the product of the height of the wedge and breadth of the base, and  $\frac{2}{3}$  of the last product will be the solidity.

That is, using the letters in the demonstration below,  $bb \times \frac{2L+l}{6} =$  the content.\*

2 S

Note.

## \* DEMONSTRATION.

Put  $L = BC =$  the length of the base,  
 $l = EF =$  the length of the edge,  
 $b = BA =$  the breadth of the base,  
 $h = EP =$  the height of the wedge.

Then, since it is evident that, according as the edge is shorter or longer than the base, the wedge is greater or less than half a prism of the same height and breadth with the wedge and length equal to that of the edge, by a pyramid of the same height and breadth at the base also, and the length of whose base is equal to the difference of the lengths of the edge and base of the wedge; we shall have  $\frac{1}{2} b h \pm b h \times \frac{\pm L \mp l}{3} = \frac{1}{2} b h + b h \times \frac{L-l}{3} = b h \times \frac{3l+2L-2l}{6} = b h \times \frac{2L+l}{6}$ . Q.E.D.

## COROLLARY.

If  $l = L$ , the rule will become  $b h \times \frac{3L}{6} = \frac{1}{2} b h L = \frac{1}{2}$  a prism of the same base and height, as it ought.

## SCHOLIUM.

It is evident that whether the two ends or the two sides of the wedge be equally or unequally inclined to the base, it will make no difference in the rule.







Hence, as  $s. LE = 60^\circ : s. LQ = 55^\circ :: QR = 20 : RE = 18.9175$ ; and as  $s. LP = 90^\circ : s. LR = 65^\circ :: RE = 18.9175 : EP = 17.14508$  = the height of the wedge.

Wherefore  $bb \times \frac{2L+l}{6} = 15 \times 17.14508 \times \frac{70+55}{6} = 8.57254 \times 5 \times 125 = 5357.8375$  solid inches =  $3.1006$  solid feet = the content required.

## PROBLEM X.

*To find the solidity of a prismoid.*

## Definition.

A prismoid is a solid having for its two ends any dissimilar parallel plane figures of the same number of sides, and all the sides of the solid plane figures also.

It is evident that the sides of this solid are all trapezoids.

If the ends of the prismoid be bounded by curves, as ellipses, &c. the number of its sides or trapezoids will be infinite, and it is then called, sometimes, a cylindroid.

## GENERAL RULE.

To the sum of the areas of the two ends add four times the area of a section parallel to and equally distant from both ends, multiply the last sum by the height and  $\frac{1}{6}$  of the product will be the solidity.\*

PAR-

\* It is evident that the rectangular prismoid is composed of two wedges whose bases are the two ends of the prismoid, and whose heights are each equal to that of the prismoid; wherefore, by the last problem, its solidity will be  $= \frac{2L+l}{6} \times B + \frac{2l+L}{6} \times b \times \frac{1}{6} h$ . Which is the particular rule.

Co-

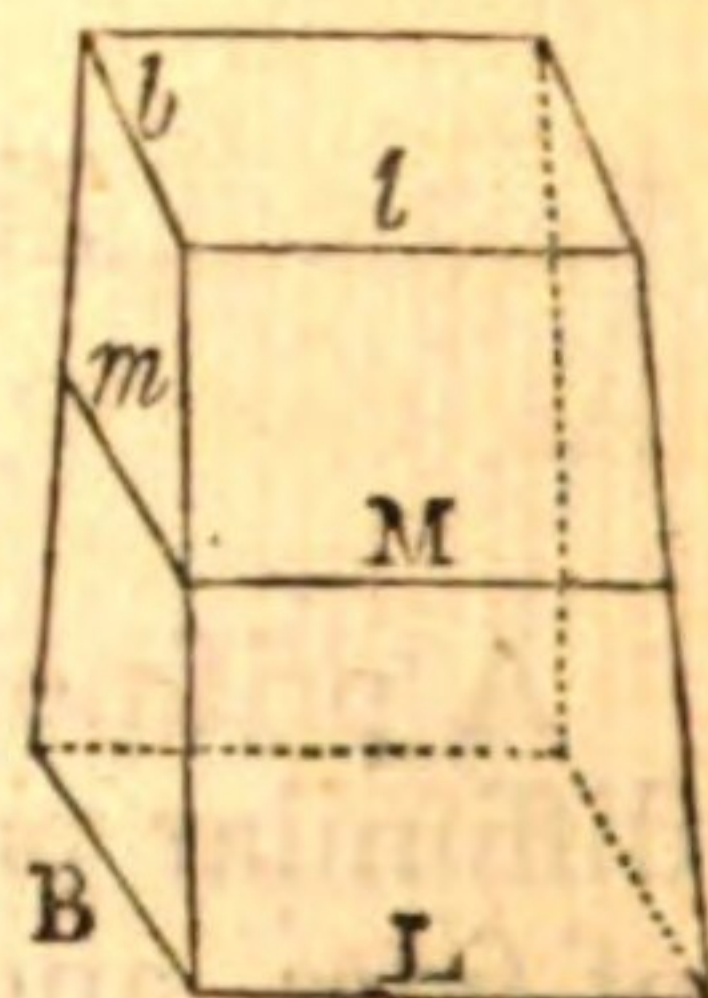


## PARTICULAR RULE.

Or, If the bases be dissimilar rectangles, take two corresponding dimensions, and multiply each by the sum of double the other dimension of the same end and the dimension of the other end corresponding to this other dimension; then multiply the sum of the products by the height, and  $\frac{1}{6}$  of the last product will be the solidity.

That is, if  $L, l$  be one dimension of each corresponding to each other, and  $B, b$  the other corresponding dimensions, and  $h$  the height.

Then  $2L+l \times B + 2l+L \times b \times \frac{1}{6} h =$  the solidity.



*Note.* Corresponding dimensions are those connected by a side of the solid, as is evident from the figure.

## EXAMPLE I.

How many solid feet of timber are in a tree whose ends are rectangles, the length and breadth of the one

## COROLLARY I.

Since  $\frac{L+l}{2} = M$ , and  $\frac{B+b}{2} = m$ , are the length and breadth of a section parallel to and equally distant from each end, the above rule  $2L+l \times B + 2l+L \times b \times \frac{1}{6} h$  or  $2BL+Bl+2bl+bL \times \frac{1}{6} h$ , will become  $BL+bl+4Mm \times \frac{1}{6} h$ . That is, the sum of the areas of the two ends four times the section in the middle multiplied into  $\frac{1}{6} h$ .

## COROLLARY II.

This last rule will serve for any prismoid or cylindroid of whatever figures the ends may be, inasmuch as they may be conceived to be composed of an infinite number of rectangular prismoids. Which is the general rule.



one being 14 and 12 inches, and their corresponding sides of the other 6 and 4 inches; the perpendicular length being  $30\frac{1}{2}$  feet?

First, by the general rule,

$$\begin{array}{rcl} 14 \times 12 & = & 168 \\ 6 \times 4 & = & 24 \end{array} \left. \vphantom{\begin{array}{rcl} 14 \times 12 & = & 168 \\ 6 \times 4 & = & 24 \end{array}} \right\} \text{areas of the ends}$$

$$\begin{array}{rcl} \frac{14+6}{2} & = & \frac{20}{2} = 10 \\ \frac{12+4}{2} & = & \frac{16}{2} = 8 \end{array} \left. \vphantom{\begin{array}{rcl} \frac{14+6}{2} & = & \frac{20}{2} = 10 \\ \frac{12+4}{2} & = & \frac{16}{2} = 8 \end{array}} \right\} \text{the dimensions in the middle}$$

$10 \times 8 \times 4 = 320$  = four times the middle area,  
their sum is 512 square inches =  $\frac{512}{144}$  square feet =  $3\frac{2}{9}$  square feet.

Then  $\frac{32}{9} \times \frac{30\frac{1}{2}}{6} = \frac{16 \times 30\frac{1}{2}}{9 \times 3} = \frac{488}{3 \times 9} = \frac{162\frac{2}{3}}{9} = 18\frac{2}{7}$  feet = the solidity.

Secondly, by the particular rule,

Here  $L=14$ ,  $B=12$ ,  $l=6$ ,  $b=4$  inches, and  $h=30\frac{1}{2}$  feet; therefore  $2L+l \times B+2l+L \times b \times \frac{1}{6}h = 28+6 \times 12+12+14 \times 4 \times \frac{30\frac{1}{2}}{6 \times 144} = 17 \times 3 + 13 \times \frac{4 \times 30\frac{1}{2}}{3 \times 144} = 64 \times \frac{30\frac{1}{2}}{3 \times 36} = \frac{16 \times 30\frac{1}{2}}{3 \times 9} = \frac{488}{3 \times 9} = \frac{162\frac{2}{3}}{9} = 18\frac{2}{7}$  feet, the same as before.

#### EXAMPLE II.

What is the capacity of a coal waggon whose inside dimensions are thus: at the top the length and breadth  $81\frac{1}{2}$ , and 55 inches; at the bottom, the length 41 and breadth  $29\frac{1}{2}$ ; and the perpendicular depth  $47\frac{1}{4}$ .

$$\text{Here } 2L+l \times B+2l+L \times b \times \frac{1}{6}h =$$

$$163+41 \times 55+82+81.5 \times 29.5 \times \frac{47.25}{6} =$$

$$102 \times 55+81.75 \times 29.5 \times \frac{47.25}{3} = 8047.75 \times 15.75 =$$

126752.0625 cubic inches = 449.4754 ale gallons = the content required, which is near one Newcastle chaldron of coals.

2 T

PRO-

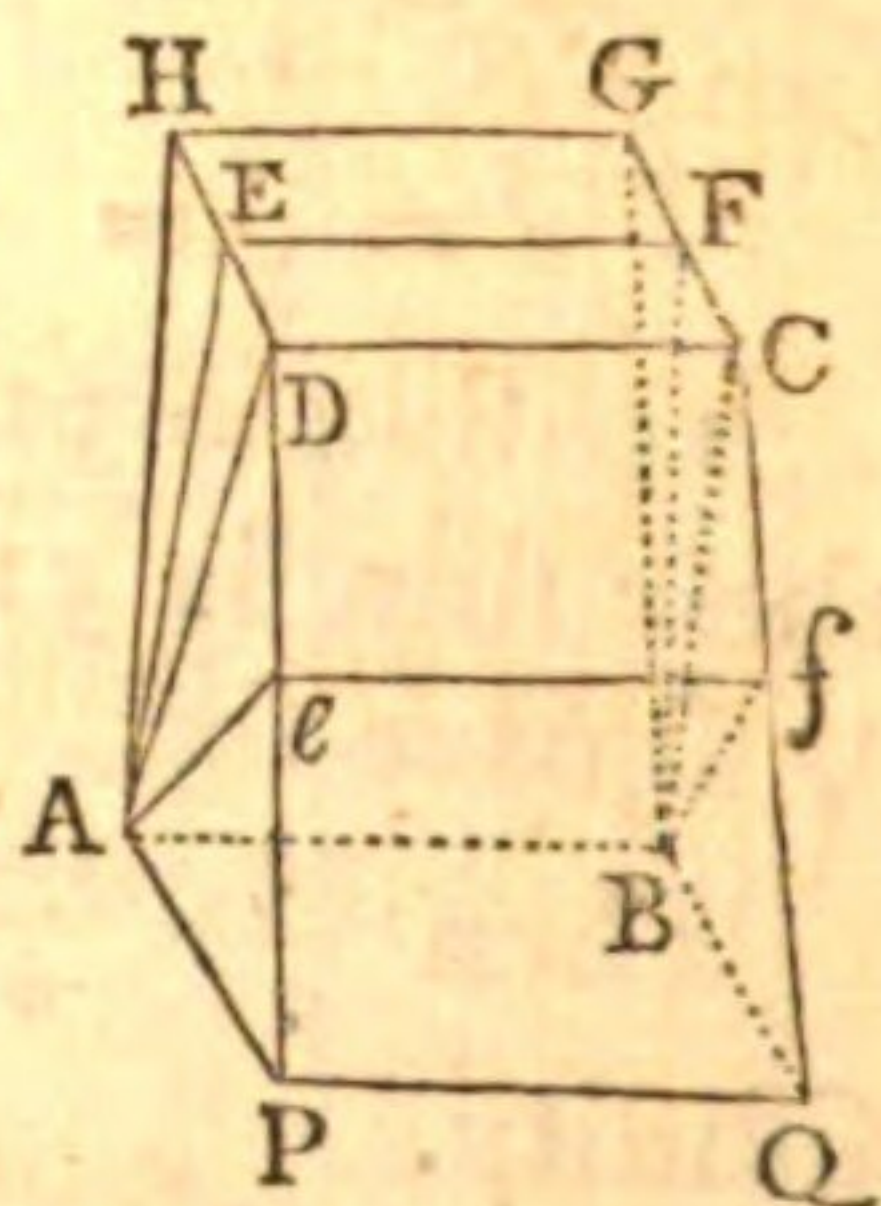


## P R O B L E M X I.

To find the solidity of the two Parts (commonly called ungulas or hoofs) into which the frustum of a rectangular or a square pyramid, or a rectangular prismoid, is cut by a plane inclined to its base.

## C A S E I.

If the plane, passing through *A* and *B*, cut the end in *EF*, between *GH* and *DC*; it will cut off the wedge *AGEFHB*, whose base is *GEFH*, edge *AB*, and height the same with that of the frustum, or prismoid, and the remaining part *AEDCFBQP* will be a prismoid.



Then, by problem 9, find the content of the wedge *ABHFEG*, and that of the prismoid *ABQPDCFE* by problem 10.

## E X A M P L E.

If the frustum of a square pyramid be cut by a plane passing through one side of the less end and through the middle of the greater end; what are the contents of the two parts, supposing each side of the greater end to be 15 inches, each side of the less 6, and the slant height 24 feet?

Since by example 1 problem 8, the perpendicular height is 287.9649 inches, therefore,

First,  $\frac{2 \times 15 + 6}{6} \times 7\frac{1}{2} \times 287.9649 = 45 \times 287.9649 =$   
 $12958.4205$  inches  $= 7.499086$  feet  $=$  the content of the wedge.

And



And  $\overline{30+6 \times 7\frac{1}{2}+12+15 \times 6 \times \frac{287.9649}{6}} = \overline{18 \times 15 + 27 \times 6 \times \frac{287.9649}{6}} = 72 \times 287.9649 = 20733.4728$  inches =  
 11.998536 feet = the prismoid.

## C A S E II.

If the plane pass through *D* and *C* as well as *A* and *B*, the solid is thereby divided into two wedges or hoofs *AGDCHB*, *DPABQC*, whose two bases are the ends or bases of the solid.

And then the contents of the two parts will be found by problem 9.

## E X A M P L E.

Let there be taken here the same figure as in the last example, to find the content of the two wedges into which it is cut.

First,  $\frac{30+6}{6} \times 15 \times 287.9649 = 90 \times 287.9649 = 25916.841$  inches = 14.99817 feet = the greater wedge.

And  $\frac{12+15}{6} \times 6 \times 287.9649 = 27 \times 287.9649 = 7775.0523$  inches = 4.49945 feet = the less wedge.

## C A S E III.

If the plane cut the side of the solid in *ef*; the part cut off *ePABQf* will be a wedge whose base is *AQ*, and which being taken from the whole figure will leave the content of the part *ABfeDCHG*.

## E X A M P L E.

Let there be taken here the same figure, supposing the plane to cut the side *PC* in *ef* at the distance of  
 10 feet



10 feet from  $PQ$ ; to find the solidity of the two parts.

First, as  $PD = 24 : Pe = 10 :: 287.9649$  inches = the height of the whole solid:  $119.98536$  inches = the height of the wedge  $AQe$ . And as  $PD = 24 : PE = 10 :: DC - PQ = 9 : ef - PQ = \frac{1}{4} = 3.75$ ; hence  $ef = 3.75 + 6 = 9.75$  = the length of the edge of the cuneus. Wherefore  $\frac{12 + 9.75}{6} \times 6 \times 119.98536 = 21\frac{3}{4} \times 119.98536 = 87 \times 29.99634 = 2609.68158$  inches =  $1.51023$  feet = the solidity of the wedge  $AQe$ .

Which taken from  $19.49762$  = the content of the whole frustum found by problem 9, leaves  $17.98739$  for the content of the part  $ABfeDGHc$ .

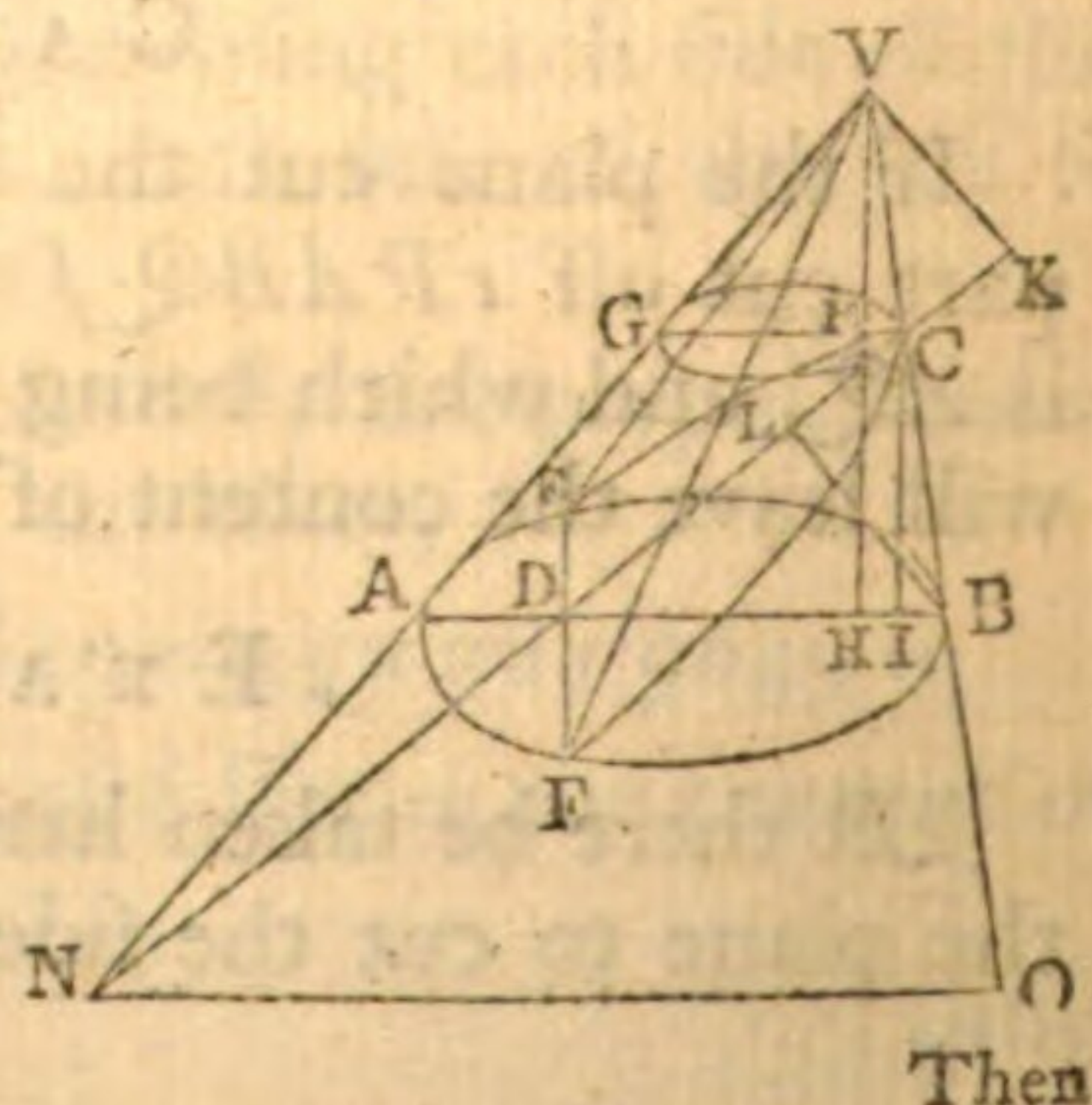
#### PROBLEM XII.

*To find the solidity of the elliptic hoofs of the frustum of a cone, made by a plane cutting diagonally the opposite extremities of the ends.*

#### \* RULE.

From the square of the diameter of the base of the hoof, subtract the product of the diameter of the other

\* Let  $AEBF$  be the base of a cone, or of any other pyramid, right, or oblique;  $AVB$  a section through the vertex by a plane perpendicular to the base; and  $VFE$ ,  $ECF$ , two other sections perpendicular to  $AVB$ , the former through the vertex and the latter through the side, at  $C$ , between  $V$  and  $B$ . On  $AB$  let fall the perpendiculars  $VH$ ,  $CI$ ; and on  $DC$  the perpendiculars  $VK$ ,  $BL$ ; and draw  $CG$  parallel to  $AB$ , and meeting  $AV$  and  $VH$  in  $G$  and  $M$ .



Then



other end of the frustum and a mean proportional between the diameters; divide the difference by the difference of the diameters; multiply the quotient by the height of the hoof and the product by the diameter of its base; so shall the last product multiplied by  $\cdot 2618 \left( \frac{.7854}{3} \right)$  produce the content of the hoof.

That is, Putting  $D=AB$  the diameter of the greater end,

$d=CG$  the less diameter,

and  $h=CI$  the perpendicular height, or distance of the base and end.

Then  $\frac{D^2-d\sqrt{Dd}}{D-d} \times 0.2618 dh =$  the greater hoof  $ABC$ .

And  $\frac{D\sqrt{Dd}-d^2}{D-d} \times 0.2618 dh =$  the less or complementary hoof  $ACG$ . A plane being conceived to be drawn through  $C$  and  $A$ .—The proof in corollary 2.

2 U

Ex-

Then it will be evident that  $EFBV$  is a pyramid whose base is  $EFB$  ( $A$ ), altitude  $VH$  ( $a$ ); and therefore its content is equal to  $\frac{1}{3}Aa$ ; and that  $EFCV$  is a pyramid whose base is  $EFC$  ( $B$ ), height  $VK$  ( $b$ ), and therefore its content equal to  $\frac{1}{3}Bb$ ; and moreover that the difference of these pyramids, or  $\frac{1}{3}Aa - \frac{1}{3}Bb$  is the content of the hoof  $EFBC$ .

But, by the similar triangles  $ABV$ ,  $VCG$ , it will be  $AB-GC:CI$  ( $HV-VM$ ) ::  $AB:HV$  ( $a$ ) =  $\frac{AB \times CI}{AB-GC}$ ; also  $AB-GC:CI$  (::  $AB:HV$ ) ::  $GC:VM$  =  $\frac{GC \times CI}{AB-GC}$ ; and  $CD:DB$  :: (by the similar triangles  $ICD$ ,  $DBL$ )  $CI:BL$  :: (because of the similar triangles  $BCI$  and  $CMV$ ,  $VKC$  and  $CLB$ )  $MV$  ( $= \frac{GC \times CI}{AB-GC}$ ) :  $VK$  ( $b$ ) =  $\frac{GC \times CI \times BD}{DC \times AB-GC}$ .

Wherefore the hoof  $EFBC$  will be =  $\frac{\frac{1}{3}CI}{AB-GC} \times A \times AB - B \times \frac{GC \times BD}{DC}$ , which is a general theorem for any hoof of any pyramid.

Co-



## EXAMPLE I.

If a conical vessel whose bottom diameter is 30 inches, be inclined to the horizon till the liquor in it just cover the bottom, and its surface, disposing itself into the position  $AC$ , cut the side  $BC$  of the vessel in  $C$  at the distance of 18 inches from the bottom  $AB$ ; how many wine gallons of liquor are in it, supposing the diameter  $GC$  of the vessel at  $C$  to be 19.2 inches?

Here  $D=30$ ,  $d=19.2$ , and  $h=18$ .

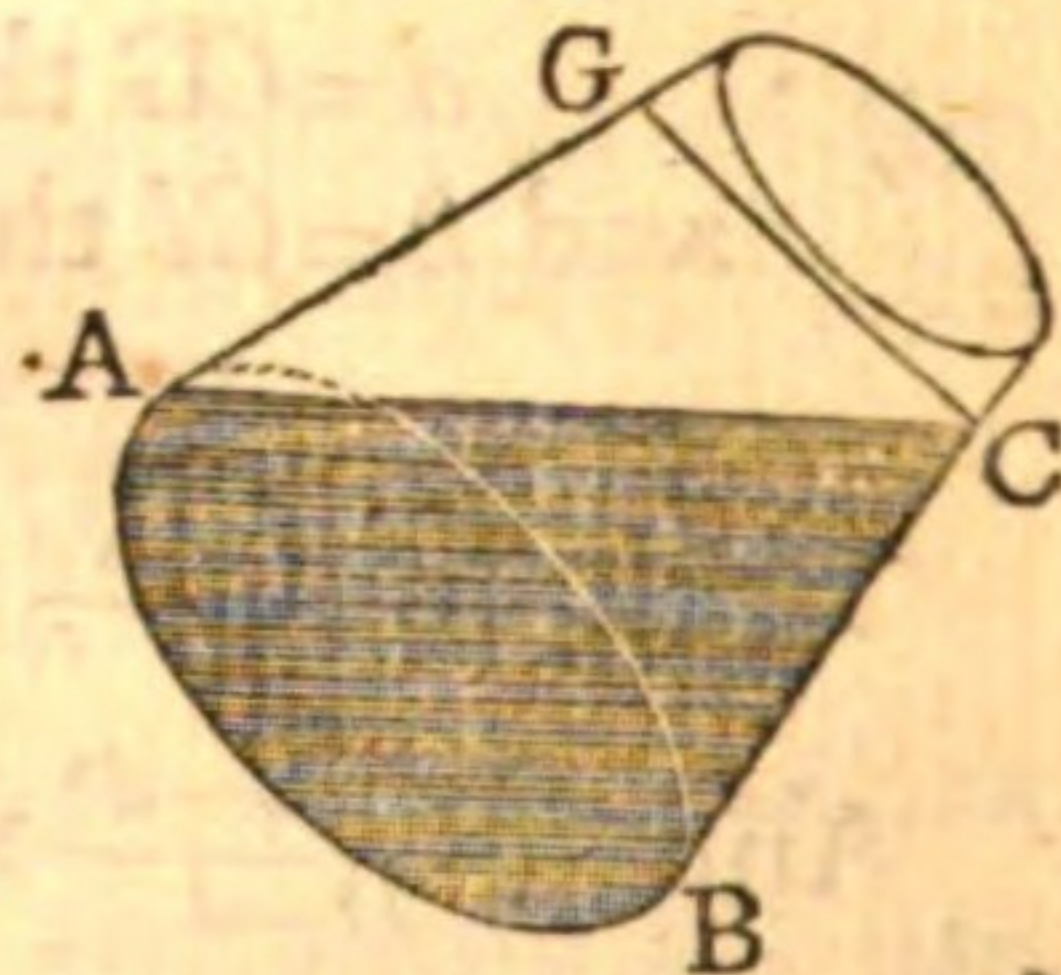
Whence  $\frac{D^2-d\sqrt{Dd}}{D-d} \times .2618 D h =$

$$\frac{30^2 - 19.2\sqrt{30 \times 19.2}}{30 - 19.2} \times 30 \times 18 \times .2618 =$$

$$\frac{439\frac{1}{2} \times 30 \times 18 \times .2618}{10\frac{4}{5}} = \frac{2196 \times 30 \times 18 \times .2618}{54}$$

$$= 2196 \times .2618 \times 10 = 5749.128 \text{ inches. Which being}$$

divided by 231



thus,  $231 = 3 \times 7 \times 11$   $\left\{ \begin{array}{l} 3 \overline{) 5749.128} \\ 7 \overline{) 1916.376} \\ 11 \overline{) 273.768} \end{array} \right.$

gives 24.888 wine gallons  
for the content required.

Ex-

## COROLLARY I.

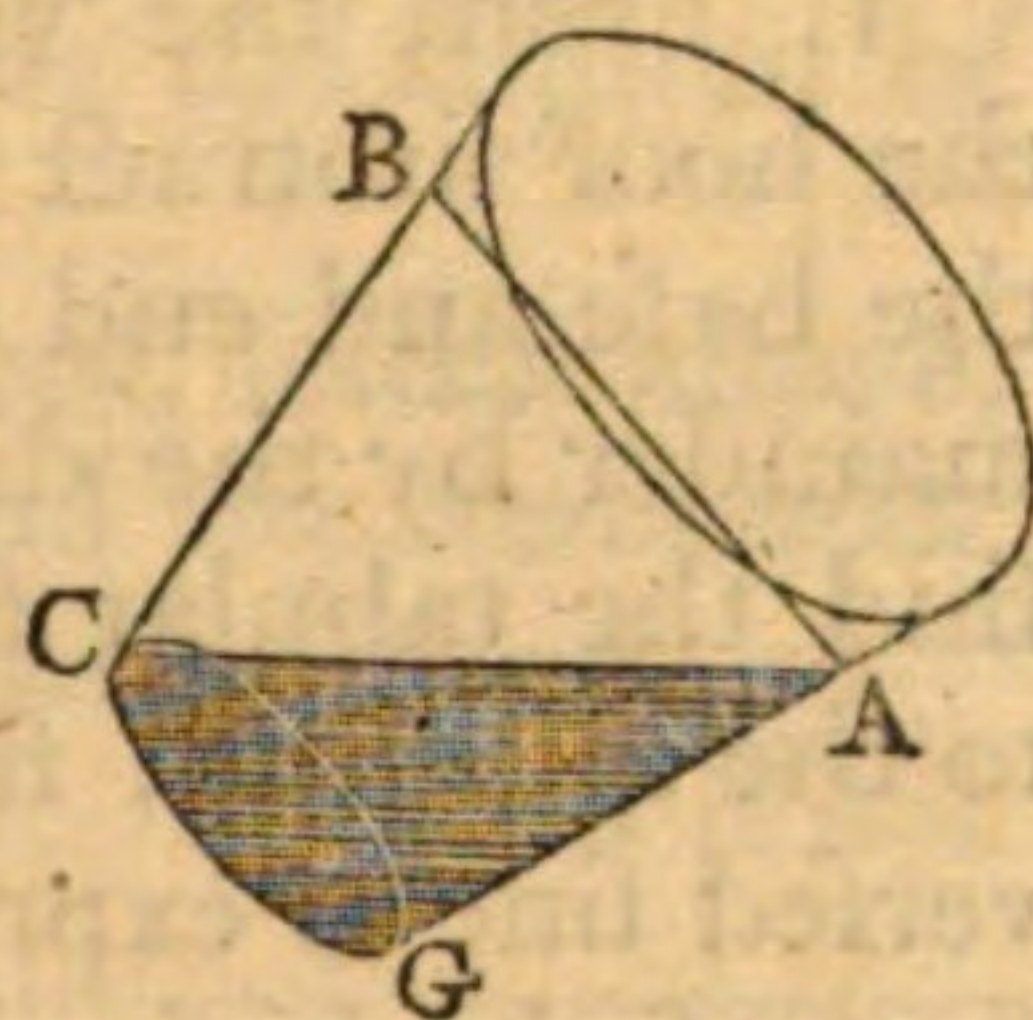
If the base be circular, or the pyramid a cone, and the angle  $CDB$  be less than the angle  $VAD$ ; or, which is the same, if  $CD$  and  $VA$ , produced, intersect in  $N$ ; the section  $ECF$  will be a segment of an ellipse whose transverse axis is  $CN$ , and conjugate  $\sqrt{NO \times CG}$ ,  $NO$  being drawn parallel to  $AB$  and meeting  $VB$ , produced, in  $O$ . And then the above general theorem will become  $\frac{\frac{1}{2}CI}{AB-CG} \times AB \times \text{circular seg-}$   
ment



EXAMPLE II.

Let there be taken here the same dimensions as in the last example, supposing the vessel to be narrowest at the bottom, to find how many ale gallons are in it.

Here  $\frac{D\sqrt{Dd-dd}}{D-d} \times .2618 db =$   
 $\frac{30\sqrt{30 \times 19\frac{1}{2}} - 19 \cdot 2^2}{30 - 19\frac{1}{2}} \times .2618 \times 19\frac{1}{2} \times 18 =$   
 $\frac{351\frac{2}{3} \times 19\frac{1}{2} \times 18 \times .2618}{10\frac{4}{5}} = \frac{351\frac{2}{3} \times 96 \times 18 \times .2618}{54}$   
 $= 117\frac{3}{5} \times 96 \times .2618 = 117 \cdot 12 \times 96 \times$   
 $.2618 = 2943 \cdot 55 \text{ inches. Which}$   
 being divided by 282, give 10.43 for the number of  
 ale gallons required



PRO-

ment  $EBF = \frac{CG \times BD}{DC} \times \text{elliptic segment } ECF = (\text{by prob. 6 sect. 3 part 3})$   
 $\frac{\frac{1}{2}CI}{AB - CG} \times : AB \times \text{cir. seg. } EBF = \frac{CG \times BD \times \sqrt{NO \times CG}}{DC \times CN} \times \text{cir. seg. whose}$   
 diameter is  $CN$  and height  $CD =$  (since similar segments are as the  
 squares of their diameters)  $\frac{\frac{1}{2}CI}{AB - CG} \times : AB \times \text{circular segment } EBF =$   
 $\frac{CG \times BD \times CN \times \sqrt{NO \times CG}}{DC \times AB^2} \times \text{seg. of the cir. } AEBF \text{ whose height is } \frac{AB \times DC}{CN}$   
 $= \text{the elliptic hoof } EFCB.$

But, by sim.  $\Delta$ s,  $CG - AD : DC :: CG : CN = \frac{CG \times CD}{CG - AD}$ , and  $CG -$   
 $AD : DB :: CG : NO = \frac{CG \times DB}{CG - AD}$ ; which values of  $NO$  and  $CN$  being  
 substituted instead of them in the above expression of the elliptic hoof,  
 will give  $\frac{\frac{1}{2}CI}{AB - CG} \times : AB \times \text{cir. seg. } EBF = \frac{CG^3}{AB^2} \times \frac{BD}{CG - AD} \Big|^{3/2} \times \text{seg. of}$   
 the cir.  $AEBF$  whose height is  $\frac{AB \times CG - AD}{CG} = \frac{\frac{1}{2}b}{D - d} \times D \times \text{circular seg.}$   
EBF



## PROBLEM XIII.

*To find the solidity of the elliptic hoofs of the frustum of a cone made by a plane cutting off a part of the base.*

## RULE.

1. From the versed sine or height of the base of the hoof subtract the difference of the diameters of the base and end of the frustum, and divide the remainder by the diameter of the end of the frustum; find the tabular segment whose versed sine is equal to the quotient; find also the tabular segment whose versed sine is expressed by the quotient of the versed sine of the base of the hoof divided by the diameter of the base of the frustum; multiply the former segment by the cube of the diameter of the end, by the quotient of the versed sine of the base of the hoof divided

---

$EBF - \frac{d^3}{D^2} \times \frac{BD}{BD - D - d} \Big| \frac{3}{2} \times \text{segment of the circle } AB \text{ whose height is}$   
 $\frac{D \times BD - D - d}{d} = \text{the elliptic hoof } EFCB; \text{ putting } h \text{ for the height of}$   
the hoof, and  $D$  and  $d$  for the diameters of the base and end or top of the frustum respectively.

Or, if we would reduce the circular segments in the last expression to those of a circle whose diameter is 1, to be found in the table of circular segments at the end of the book, that expression will become  $\frac{\frac{1}{2}h}{D-d} \times D^3 \times \text{tab. seg. whose height is } \frac{BD}{D} - d^3 \times \frac{BD}{BD - D - d} \Big| \frac{3}{2} \times$   
 $\text{tab. seg. whose height is } \frac{BD - D - d}{d} = \text{the elliptic hoof } EFCB. \text{ And}$   
this is the rule in prob. 13.

And if this value of the hoof be taken from  $\left( \frac{1}{3} h n \times \frac{D^3 - d^3}{D - d} = \right)$   
that of the whole conic frustum, the remainder  $\frac{\frac{1}{2}h}{D-d} \times D^3 - d^3 \times n -$   
 $D^3$



divided by the difference between the said versed sine and the difference of the diameters, and by the root of the said quotient; and multiply the latter segment by the cube of the greater diameter; multiply the difference of these two products by  $\frac{1}{3}$  of the height of the hoof, and the product divided by the difference of the diameters will give the content of the hoof required.

2. Subtract the measure of the hoof, found above, from that of the whole frustum, and the remainder will be the measure of the complementary hoof.

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That

$D^3 \times \text{tab. seg. whose height is } \frac{BD}{D} + d^3 \times \frac{BD}{BD-D-d} \Big| \frac{3}{2} \times \text{tabular seg.}$

whose height is  $\frac{BD-D-d}{d}$  or  $\frac{\frac{1}{3}h}{D-d} \times: -nd^3 + D^3 \times \text{tab. seg. whose}$

height is  $\frac{AD}{D} + d^3 \times \frac{BD}{BD-D-d} \Big| \frac{3}{2} \times \text{tab. seg. whose height is } \frac{BD-D-d}{d}$

will express the measure of the complementary elliptic hoof  $EFCGA$ .

## COROLLARY II.

If the points  $D$  and  $A$  coincide, the section  $EFC$  will be a whole ellipse, and the rules in corollary I will become  $\frac{1}{3}Dhn \times \frac{DD-d\sqrt{Dd}}{D-d}$

= the elliptic hoof  $ACB$ , and  $\frac{1}{3}dhn \times \frac{D\sqrt{Dd-dd}}{D-d}$  = the complementary elliptic hoof  $ACG$ . And these are the rules in prob. 12.

## COROLLARY III.

Since  $\frac{h}{D-d} = \frac{H}{D}$  ( $H$  denoting the height of the whole cone), the last theorems will become  $\frac{1}{3}nH \times \frac{D^2-d\sqrt{Dd}}{D} = ACB$ , and  $\frac{1}{3}nH \times \frac{D\sqrt{Dd-dd}}{D} = ACG$ .

## COROLLARY IV.

But  $\frac{1}{3}nHD^2$  = the whole cone  $VAB$ , and therefore  $\frac{1}{3}nHD^2 - \frac{1}{3}nH \times \frac{D^2-d\sqrt{Dd}}{D} = \frac{1}{3}nHd\sqrt{Dd}$  = the top part  $VAC$ . Which top part, conse-



That is, if  $D$  and  $d$  be the extreme diameters of the frustum, and  $b$  its height; also  $P$  = the tabular segment whose versed sine is  $\frac{BD}{D}$ ,  $\mathcal{Q}$  = that whose versed sine is  $\frac{BD-D-d}{d}$ , and  $n = .785398$  &c.

Then  $P \times D^3 - \mathcal{Q} \times d^3 \times \frac{BD}{BD-D-d} \sqrt{\frac{BD}{BD-D-d}} \times \frac{\frac{1}{3}b}{D-d} =$   
the content of the elliptic hoof  $EFCB$ .

And  $\overline{D^3 - d^3} \times n - P \times D^3 + \mathcal{Q} \times d^3 \times \frac{BD}{BD-D-d} \sqrt{\frac{BD}{BD-D-d}} \times \frac{\frac{1}{3}b}{D-d} =$  to that of the complementary hoof  $EFCGA$ .  
—These are proved in corollary 2 to prob. 12.—See the fig. in page 168.

Ex-

consequently, is to the whole cone  $VAB$  as  $(\frac{1}{3}nHd\sqrt{Dd}$  to  $\frac{1}{3}nHD^2$  as)  $d^{\frac{3}{2}}$  to  $D^{\frac{3}{2}}$ ; or the square of the whole cone  $VAB$  is to the square of the top part  $VAC$ , as  $D^3$  to  $d^3$ .

#### COROLLARY V.

If the angle  $CDB$  be equal to the angle  $VAB$ , or if the section  $ECF$  be parallel to the side  $AG$  of the cone, it will be a parabola whose axis is  $CD$ , base  $EF = 2\sqrt{BD \times DA} = 2\sqrt{D-d} \times d$ , and its area (by prob. 5 sect. 4 part 3)  $= \frac{2}{3}CD \times EF = \frac{4}{3}CD \sqrt{D-d} \times d$ ; and therefore the gen. theor. will become  $\frac{\frac{1}{3}b}{D-d} \times D \times \text{cir. seg. } EBF - \frac{d \times D-d}{CD} \times \frac{4}{3}CD \sqrt{D-d} \times d$   
 $= \frac{1}{3}b \times \frac{D^3}{D-d} \times \text{tab. seg. whose height is } \frac{D-d}{d} - \frac{4}{3}d \sqrt{D-d} \times d =$  the parabolic hoof  $EFBC$ .

Which being taken from  $\frac{1}{3}bn \times \frac{D^3-d^3}{D-d} =$  the measure of the whole frustum  $ABCG$ , the remainder  $\frac{1}{3}b \times : \frac{4}{3}d \sqrt{D-d} \times d - \frac{nd^3}{D-d} + \frac{D^3}{D-d} \times \text{tab. seg. whose height is } \frac{d}{D}$  or  $\frac{\frac{1}{3}b}{D-d} \times : \overline{D-d} \times \frac{4}{3}d \sqrt{d \times D-d} - nd^3 + D^3 \times \text{tab. seg. whose height is } \frac{d}{D}$  will express the measure of the complementary parabolic hoof  $EFCGA$ . C o.



## EXAMPLE I.

If a vessel in the form of the frustum of a cone, whose bottom diameter is 30 inches, be inclined to the horizon till the surface of the liquor in it cut the bottom, leaving 10 inches of its diameter dry, and meeting with the side in  $C$  at the distance  $CI$  of 18 inches from the base: How many Winchester gallons are in it, supposing the diameter  $CG$  of the vessel at the top of the liquor to be  $19\frac{1}{2}$  inches?

Here  $D=30$ ,  $d=19\frac{1}{2}$ ,  $b=18$ , and  $BD=30-10=20$ .

Whence  $\frac{BD}{D} = \frac{20}{30} = \frac{2}{3} = .6666\frac{2}{3}$ , the tabular area corresponding to which is .55622573; and  $\frac{BD-\overline{D}-d}{d} = \frac{20-30-19\frac{1}{2}}{19\frac{1}{2}} = \frac{20-10\frac{1}{2}}{19\frac{1}{2}} = \frac{9\frac{1}{2}}{19\frac{1}{2}} = \frac{46}{96} = \frac{23}{48} = \frac{2.875}{6} = .479\frac{5}{6}$ , the tabular area answering to which is .37187178.

Again,

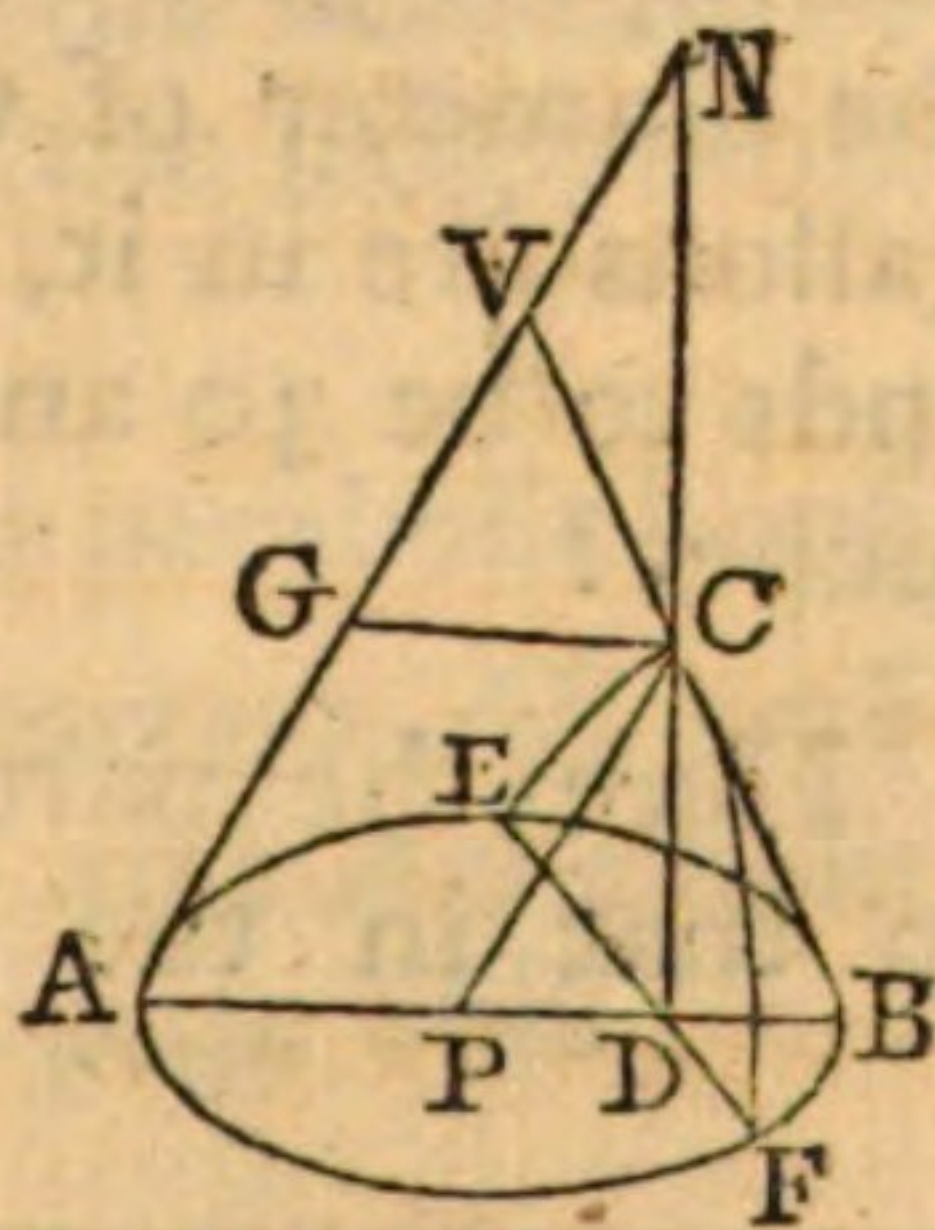
## COROLLARY VI.

If the  $\angle CDB$  exceed the  $\angle VAB$ , or, which is the same thing, if  $CD$  and  $VA$ , produced, intersect above  $V$ , in  $N$ , the section  $ECF$  will be an hyperbola, whose transverse axe is  $CN$ . But in the hyperbola, as well as in the other conic sections, the transverse axe  $CN = \frac{GC \times CD}{DP}$

$CP$  being drawn parallel to  $VA$ ,  $= \frac{d \times CD}{D-d-BD}$ ,

and the conjugate of each also  $= GC \sqrt{\frac{BD}{DP}}$

$= d \sqrt{\frac{BD}{D-d-BD}}$ ; and the area of the hyperbolic section being found by prob. 6 sect. 5 part 3, and substituted in the general theorem, will give the solidity of the hyperbolic ungula.



Co-



Again,  $D' = 27000$ , which multiplied into  $\cdot 55622573$ , the former area, produces  $15018\cdot 09471$ , and  $d^3 \times \frac{BD}{BD-D-d} \times \sqrt{\frac{BD}{BD-D-d}} = 19\frac{1}{2} \times \frac{20}{9\frac{1}{2}} \times \sqrt{\frac{20}{9\frac{1}{2}}} = 19\frac{1}{2} \times \frac{19\frac{1}{2}}{9\frac{1}{2}} \times 20 \sqrt{184} = \frac{19\frac{1}{2} \times 48^2 \times 40}{23 \times 23} \sqrt{46} = 22686\cdot 470698$ , which multiplied by  $\cdot 37187178$ , the latter area, gives  $8436\cdot 45824$ .

Then  $6581\cdot 63647$ , the difference of these two products, being multiplied by  $\frac{6}{10\frac{1}{2}} = \frac{30}{54} = \frac{5}{9} = \frac{\frac{1}{2}b}{D-d} =$  the quotient of  $\frac{1}{2}$  of the height divided by the difference of the diameters, will produce  $3656\cdot 4647$  for the solidity in inches; which being divided by  $268\frac{4}{5}$ , the inches in a corn gallon, give  $13\cdot 6029$  corn or winchester gallons for the quantity of liquor in the vessel.

## EXAMPLE II.

If a vessel, in form of the frustum of a cone, close at both ends, be placed in such position that the liquor just covers the less end and 10 inches of the diameter of the greater; what number of wine gallons are in it, supposing the diameters of the two ends to be 30 and  $19\frac{1}{2}$  inches, and their distance 18 inches?

Here the part filled is the complementary hoof to that in the last example. Wherefore if from  $D^2 +$

## COROLLARY VII.

If  $D$  coincide with  $I$ , or  $CDB$  be a right angle, the transverse and conjugate axes of the hyperbolic section will become  $\frac{2db}{D-d}$  and  $d$  respectively.



$D^2 + d \times \overline{D+d} \times \frac{1}{3}nb = 30^2 + 19\frac{1}{2} \times 49\frac{1}{2} \times 6 \times 785398 \text{ \&c.} = 8692.661$ , the content of the whole frustum, be taken  $3656.4647$ , that of the hoof in the last example, the remainder  $5036.1963$  will be the content, in inches, of the complementary hoof; which being divided by  $231$ , will give  $21.8018$  wine gallons for the content required.

## PROBLEM XIV.

To find the parabolic hoofs of the frustum of a cone; that is, of the hoofs made by a plane passing parallel to the side of the cone.

## RULE.

Multiply the base of the hoof by the greater diameter of the frustum, and divide the product by the difference of the diameters; from the quotient subtract  $\frac{1}{3}$  of the product arising from the multiplication of the less diameter by the square root of the product of the less diameter and difference of the diameters; then the remainder multiplied by  $\frac{1}{3}$  of the height will be the content of the hoof required.

That is,  $\frac{A \times D}{D-d} - \frac{1}{3}d\sqrt{D-d} \times \frac{1}{3}b = \text{the parabolic hoof } EFCB$ ; where  $D$  and  $d$  are the diameters  $AB$  and  $CG$ , of the frustum,  $b$  the height  $CI$ , and  $A$  the base  $EBF$  of the hoof.—See the figure in page 168.

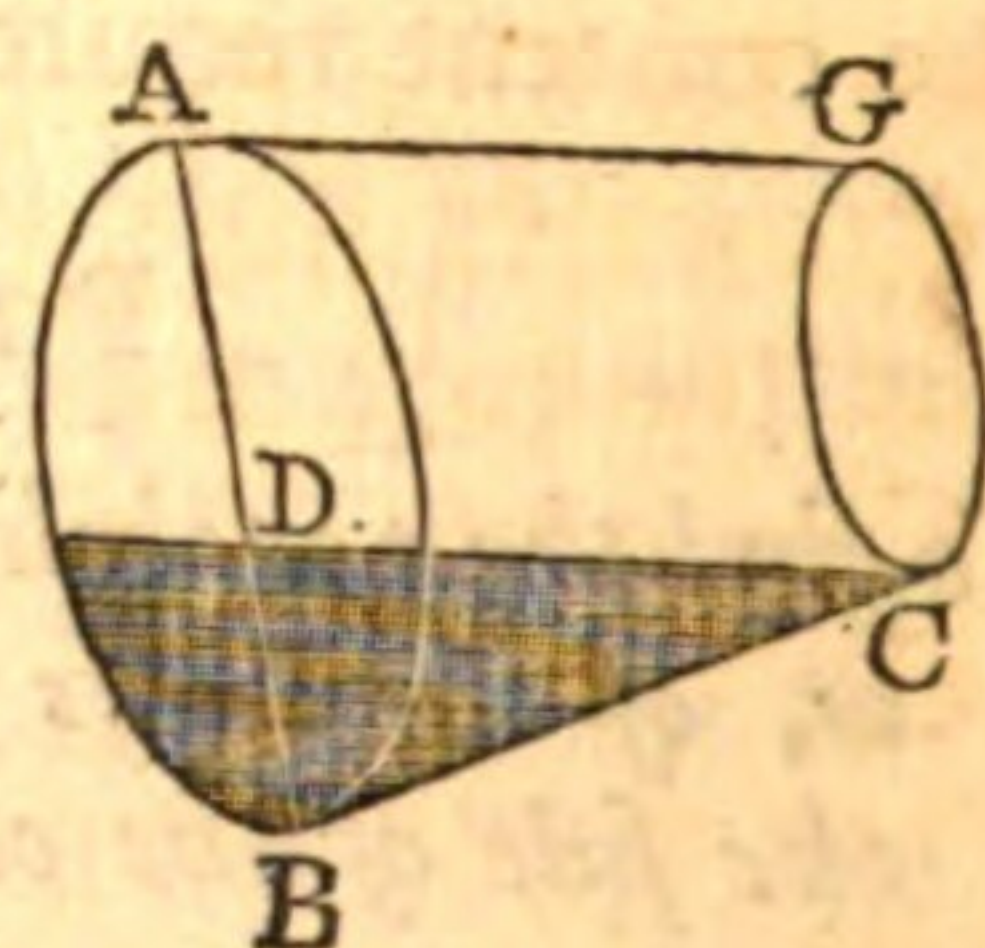
## EXAMPLE.

If a vessel in the form of the frustum of a cone be laid with its upper side parallel to the horizon,  
 2 Y and



and if the greatest distance of the surface of the liquor from the bottom, whose diameter is 30 inches, be 18 inches; how many ale gallons of liquor are in it, supposing the diameter of the vessel at the greatest distance of the surface of the liquor from the bottom to be  $19\frac{1}{2}$  inches.

Here the dimensions are the same as in the last problems; and  $BD = D - d = 30 - 19\frac{1}{2} = 10\frac{1}{2} = 10.8$  the versed sine or height of the base; which divided by the diameter 30, gives  $.36 =$  the tabular versed sine, the area answering to which is  $.25455056$ ; which multiplied by 900, the square of the diameter, gives  $229.095504$  for the circular segment or base of the hoof. Then  $229.0955 \times \frac{30}{10.8} = \frac{2290.955}{3.6} = \frac{3818.25834}{6} = 636.37639$ .



Again  $\frac{4}{3}d\sqrt{D-d} \times d = \frac{4}{3} \times 19\frac{1}{2} \sqrt{10\frac{1}{2} \times 19\frac{1}{2}} = \frac{4}{3} \times 19.2 \times \sqrt{\frac{4}{3} \times \frac{9}{2}} = \frac{4}{3} \times 19.2 \times 14.4 = 4 \times 6.4 \times 14.4 = 368.64$ .

Then  $636.37639 - 368.64 \times \frac{1}{3} = 267.73639 \times 6 = 1606.41934$  inches, equal to the solidity; which being divided by 282, give  $5.696526$  ale gallons for the quantity of liquor required.

*Note 1.* If from  $8692.661$ , the content of the whole frustum, found in the second example of the last problem, be taken  $1606.419$ , the content of the hoof, above found, the remainder,  $7086.242$ , will be that of the complementary parabolic hoof  $DAGC$ .

*Note 2.* If the section  $ECF$  be an hyperbola, the hoofs may then be found after the manner described in the sixth corollary to prob. 13.

PRO-



## PROBLEM XV.

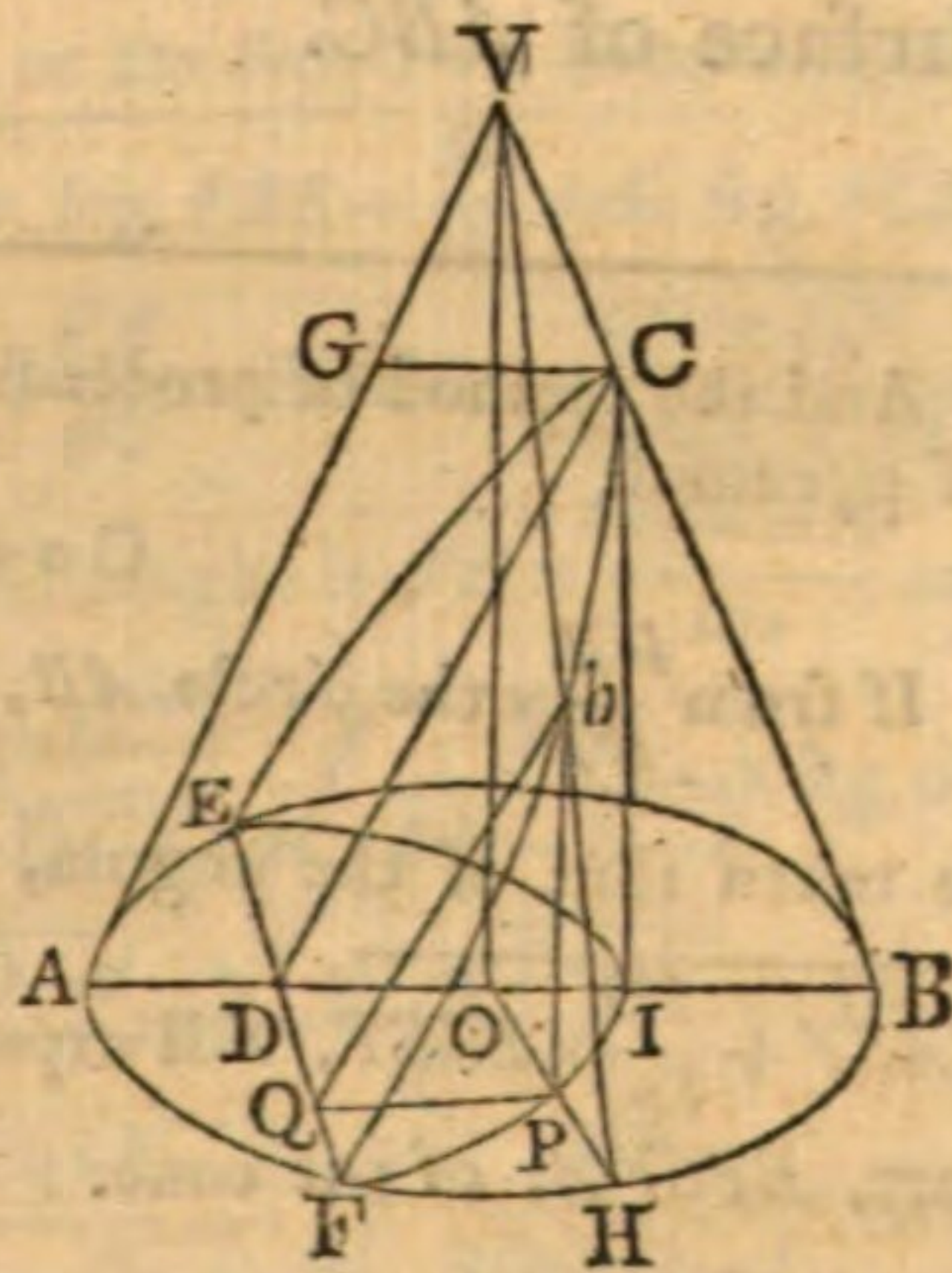
\* To find the convex surface of the elliptic hoofs of the frustum of a cone, made by a plane cutting the opposite extremities of the ends.

## RULE.

To four times the square of the height of the frustum add the square of the difference of the diameters of the base and end, divide the square root of the sum by the difference of the said diameters, and call the quotient  $\mathcal{Q}$ .

Multiply

\* Let the right-angled triangle  $VOH$  be conceived to revolve about the axis  $VO$ , the part  $Hb$  of the hypotenuse describing the surface of the ungula, whilst the part  $HP$  of the radius of the base of the cone, cut off by the perpendicular  $bP$ , describes the space  $FPIEBF$ .—Then since, by the similar  $\Delta s$   $VOH$ ,  $HbP$ , it is every-where as  $PH : Hb :: OB (OH) : BV (HV)$ , the spaces generated by these lines will be, likewise, in the same ratio, viz.  $OB : BV ::$  the area  $FIEBF$  : the surface  $FCEB (S) = \frac{VB}{BO} \times FIEBF$ .—And in the same manner,



by reason of the similar  $\Delta s$   $DIC$ ,  $\mathcal{Q}Pb$ , ( $\mathcal{Q}P$  being drawn parallel to  $BD$ ) we shall have  $CD : DI :: FCEF (= B = \text{the space described by } b\mathcal{Q}) : FIEF$  (the space described in the same time by  $\mathcal{Q}P) = \frac{ID}{DC} \times FCEF = \frac{ID}{DC} \times B$ .—But the space

$FIEBF = \text{the circular seg. } FBEF (A) - FIEF = A - \frac{ID}{DC} \times B$ . And

consequently  $S = \frac{VB}{BO} \times FIEBF = \frac{VB}{BO} \times A - \frac{ID}{DC} \times B = \frac{CB}{BI} \times A - \frac{ID}{DC} \times B =$   
the convex surface of the ungula  $EFBC$ . And



Multiply half the sum of the said diameters by the square root of their product, and call the product  $P$ . Then

1.  $\mathcal{Q}$  multiplied by  $P$  and by  $\cdot 785398$  &c. will produce the convex surface of the oblique cone  $ACV$ .

That is,  $\frac{\sqrt{4b^2 + D-d}^2}{D-d} \times \frac{D-d}{2} \sqrt{Dd} =$  the curve surface of  $ACV$ .

2.  $\mathcal{Q}$  multiplied by the difference between  $P$  and the square of the diameter of the base, and by  $\cdot 785398$  &c. will produce that of the hoof,  $ABC$ , including the base.

That is,  $\frac{\sqrt{4b^2 + D-d}^2}{D-d} \times DD - \frac{D+d}{2} \sqrt{Dd} =$  the curve surface of  $ABC$ .

3.  $\mathcal{Q}$

And the method of proceeding will be the same for any other kind of pyramid.

#### COROLLARY I.

If from  $\frac{VB}{BO} \times$  the circle  $AB$ , the convex surface of the whole cone, be taken that of the ungula, above found, the remainder  $\frac{VB}{BO} \times FAEF + \frac{ID}{DC} \times FCEF$ , will express the convex surface of the remaining part,  $EFCVA$ , of the cone.

#### COROLLARY II.

And if from the value of the part last found be taken  $\frac{VB}{BO} \times$  circle  $CG (= \frac{VC}{OI} \times$  circle  $CG)$  the convex surface of the top cone  $CVG$ , the remainder,  $\frac{VB}{BO} \times FAEF + \frac{ID}{DC} \times FCEF -$  circle  $CG$ , will express that of the complementary ungula  $EFAGG$ .

C o -



3.  $Q$  multiplied by the difference between  $P$  and the square of the diameter of the top, and by .785398 &c. will produce that of the hoof,  $ACG$ , including the top.

That is,  $\frac{\sqrt{4b^2 + D - d^2}}{D - d} \times \frac{D + d}{2} \sqrt{Dd - dd} =$  the curve surface of  $ACG$ .

Where the letters denote the same Quantities as in the foregoing problems.—All these are proved in corollary 4.

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## COROLLARY III.

If the  $\angle CDB$  be less than the  $\angle GAB$ , or the sect.  $FCE$  be an ellipse, the surface of the ungula will be  $\frac{VB}{BO} \times \text{cir. seg. } FBE - \frac{ID}{DC} \times \text{ell. seg. } FCE$

= (by proceeding as in cor. 1 prob. 12)  $\frac{VB}{BO} \times EBF - \frac{CG^2 \times ID}{AB^2 \times CG - AD}$

$\sqrt{\frac{DB}{CG - AD}} \times \text{seg. of the circle } AB \text{ whose height is } AB \times \frac{CG - AD}{CG} =$

$\sqrt{\frac{4b^2 + D - d^2}{D - d}} \times EBF - \frac{dd}{DD} \times \frac{BD - \frac{1}{2} \times D - d}{BD - D - d} \sqrt{\frac{BD}{BD - D - d}} \times \text{seg. of}$

the circle  $AB$  whose height is  $D \times \frac{BD - D - d}{d} = \frac{\sqrt{4b^2 + D - d^2}}{D - d} \times D^2 \times$

tab. seg. whose height is  $\frac{BD}{D} - d^2 \times \frac{BD - \frac{1}{2} \times D - d}{BD - D - d} \sqrt{\frac{BD}{BD - D - d}} \times \text{tab.}$

seg. whose height is  $\frac{BD - D - d}{d}$ . Where  $b$  is the height, and  $D$  and

$d$  the diameters of the base and top of the frustum.

And the value of the surface of the remaining part  $EFCVA$ , in

corollary 1, will become  $\frac{VB}{BO} \times \text{cir. seg. } FAE + \frac{ID}{DC} \times \text{ellip. seg. } FCE$

$= \frac{VB}{BO} \times FAE + \frac{CG^2 \times ID}{AB^2 \times CG - AD} \sqrt{\frac{DB}{CG - AD}} \times \text{seg. of the circle } AB$

whose



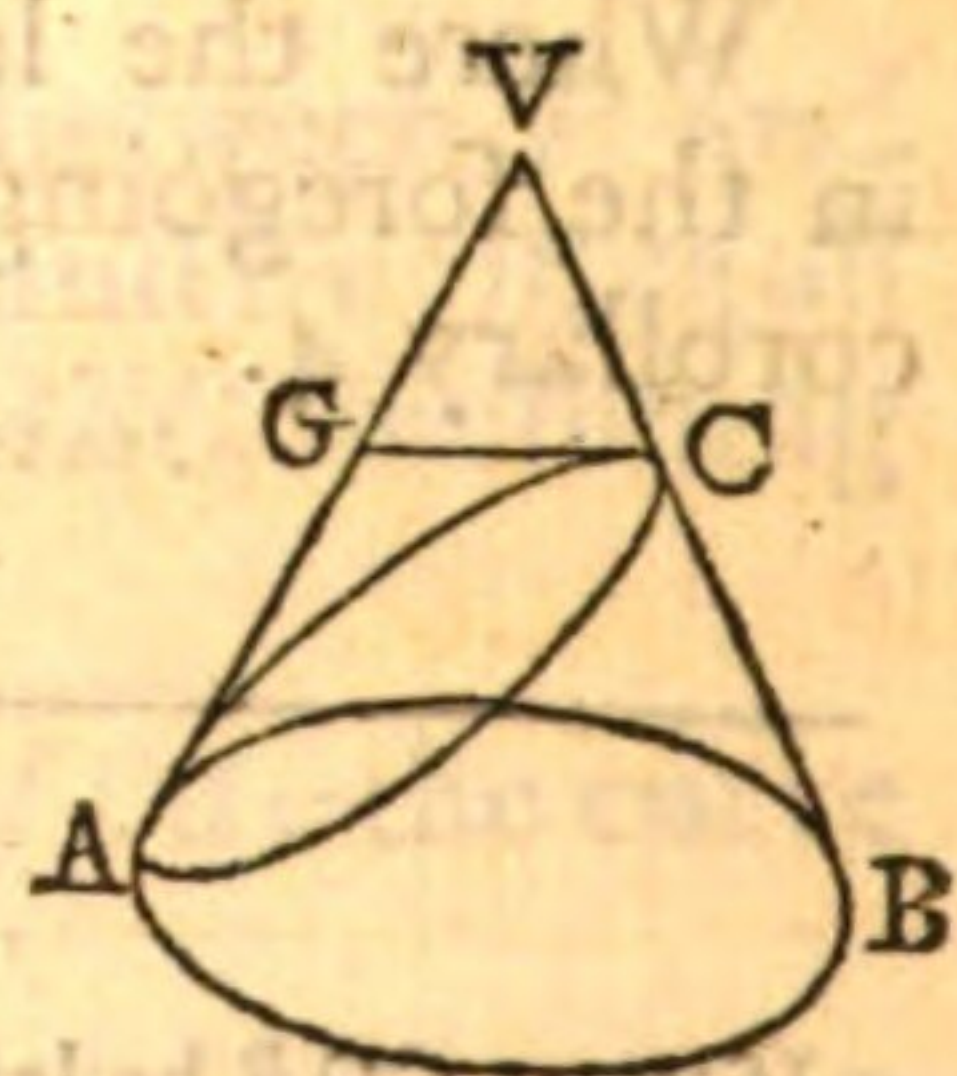
## EXAMPLE.

Required the convex surfaces of the two hoofs of a conic frustum whose base and top diameters are 30 and  $19\frac{1}{2}$ , and height 18 inches, the section being made through the contrary extremities of the diameters; together with that of the oblique cone  $ACV$ .

Here  $D=30$ ,  $d=19\frac{1}{2}$ , and  $h=18$ .

$$\text{Whence } Q = \frac{\sqrt{4b^2 + D - d}}{D - d} = \frac{\sqrt{4 \times 18^2 + 10 \cdot 8^2}}{10 \cdot 8} = \frac{\sqrt{4^2 + 1 \cdot 2^2}}{1 \cdot 2} = \frac{1}{2} \sqrt{10^2 + 3^2} \\ = \frac{1}{2} \sqrt{109} = 3 \cdot 48010217.$$

$$P = \frac{D+d}{2} \sqrt{Dd} = 24\frac{1}{2} \sqrt{30 \times 19\frac{1}{2}} = 24\frac{1}{2} \times 24 = 590\frac{1}{2}.$$



And

$$\text{whose height is } AB \times \frac{CG - AD}{CG} = \frac{\sqrt{4b^2 + D - d}}{D - d} \times : FAE + \frac{dd}{DD} \times \\ \frac{BD - \frac{1}{2} \times D - d}{BD - D - d} \sqrt{\frac{BD}{BD - D - d}} \times \text{feg. of the cir. } AB \text{ whose height is } D \times \\ \frac{BD - D - d}{d} = \frac{\sqrt{4b^2 + D - d}}{D - d} \times : D^2 \times \text{tab. feg. whose height is } \frac{AD}{D} + \\ d^2 \times \frac{BD - \frac{1}{2} \times D - d}{BD - D - d} \sqrt{\frac{BD}{BD - D - d}} \times \text{tab. feg. whose height is } \frac{BD - D - d}{d}. \\ \text{Or it is } = \frac{\sqrt{4b^2 + D - d}}{D - d} \times : FAE + \frac{dd}{DD} \times \frac{\frac{1}{2} \times D + d - AD}{d - AD} \sqrt{\frac{D - AD}{d - AD}} \\ \times \text{feg. of the cir. } AB \text{ whose height is } D \times \frac{d - AD}{d} = \frac{\sqrt{4b^2 + D - d}}{D - d} \\ \times : D^2 \times \text{tab. feg. whose height is } \frac{AD}{D} + d^2 \times \frac{\frac{1}{2} \times D + d - AD}{d - AD} \sqrt{\frac{D - AD}{d - AD}} \\ \times \text{tab. feg. whose height is } \frac{d - AD}{d}$$

Also



And then

1.  $Q \times P \times .785398 = 2.73326528 \times 590.4 =$   
 $1613.7198213$  inches  $= 11.2063876$  feet = the convex  
 surface of the oblique cone  $ACV$ .

2.  $Q \times .785398 \times \overline{D^2 - P} = .785398 \times 3.48010217 \times$   
 $900 - 590.4 = 2.73326528 \times 309.6 = 846.2189307$  inches  
 $= 5.87652$  feet = that of the hoof  $ABC$ .

3.  $Q \times .785398 \times \overline{P - d^2} = .785398 \times 3.48010217 \times$   
 $509.4 - 368.64 = 2.73326528 \times 221.76 = 606.1289085$   
 inches  $= 4.2092285$  feet = that of the complemental  
 hoof  $ACG$ .

PRO-

Also that of the complemental hoof in cor. 2, will become  $\frac{VB}{BO} \times$  cir.  
 seg.  $FAE + \frac{ID}{DC} \times$  ellip. seg.  $FCE -$  the cir.  $CG = \frac{VB}{BO} \times : - n \times CG^2 +$   
 $FAE + \frac{CG^2 \times ID}{AB^2 \times CG - AD} \sqrt{\frac{DB}{CG - AD}} \times$  seg. of the cir.  $AB$  whose height is  
 $AB \times \frac{CG - AD}{CG} = \frac{\sqrt{4b^2 + D - d^2}}{D - d} \times : - n d d + FAE + \frac{d d}{D D} \times \frac{\frac{1}{2} \times D + d - AD}{d - AD}$   
 $\sqrt{\frac{D - AD}{d - AD}} \times$  segment of the circle  $AB$  whose height is  $D \times \frac{d - AD}{d} =$   
 $\frac{\sqrt{4b^2 + D - d^2}}{D - d} \times : - n d^2 + D^2 \times$  tab. seg. whose height is  $\frac{AD}{D} + d^2 \times$   
 $\frac{\frac{1}{2} \times D + d - AD}{d - AD} \sqrt{\frac{D - AD}{d - AD}} \times$  tab. seg. whose height is  $\frac{d - AD}{d}$ .

#### COROLLARY IV.

If  $D$  coincide with  $A$ , the rules in the last cor. will become  $\frac{VB}{BO} \times$ :

$AB^2 \times n - n \times IA \sqrt{AB \times CG} = \frac{n \sqrt{4b^2 + D - d^2}}{D - d} \times D^2 - \frac{D + d}{2} \sqrt{D d}$  for  
 the convex surface of the ungula  $ABC$ .

And



## PROBLEM XVI.

To find the convex surfaces of the elliptic ungulas of a cone, made by a plane cutting off a part of the base.

## RULE.

Multiply continually together the four following quantities, viz. the quotient arising from the division of

$$\text{And } \frac{VB}{BO} \times IA \sqrt{AB \times CG} \times n = 2n \times VB \times IA \sqrt{\frac{CG}{AB}} = \frac{n \sqrt{4b^2 + D - d}}{D - d} \\ \times \frac{D + d}{2} \sqrt{Dd} \text{ for that of the oblique cone } ACV.$$

$$\text{Also } \frac{VB \times n}{BO} \times IA \sqrt{AB \times CG - CG^2} = \frac{n \sqrt{4b^2 + D - d}}{D - d} \times \frac{D + d}{2} \sqrt{Dd - d^2} \\ \text{for that of the complemental elliptic hoof } ACG.$$

## COROLLARY V.

If  $CD$  be parallel to  $VA$ , or the section a parabola; since its area ( $B$ ) is  $\frac{4}{3} CD \times DF = \frac{4}{3} CD \sqrt{AD \times DB}$ , the general theorem for the ungula will become  $\frac{VB}{BO} \times \text{feg. } FBE - \frac{4}{3} ID \sqrt{AD \times DB} = \frac{\sqrt{4b^2 + D - d}}{D - d} \\ \times \text{feg. } FBE \text{ (its height } BD = D - d) - \frac{2}{3} \times D - d \sqrt{d \times D - d}$  for the convex surface of the parabolic ungula  $FEBC$ .

And the rule in cor. 1 will be  $\frac{VB}{BO} \times \text{feg. } FAE + \frac{4}{3} ID \sqrt{AD \times DB} = \frac{\sqrt{4b^2 + D - d}}{D - d} \times \text{feg. } FAE \text{ (its height } AE = d) + \frac{2}{3} \times D - d \sqrt{d \times D - d} \\ \text{for that of the part } AEF CG.$

Also that in cor. 2 will be  $\frac{VB}{BO} \times \text{feg. } FAE + \frac{4}{3} ID \sqrt{AD \times DB} - AD^2 \\ \times n = \frac{\sqrt{4b^2 + D - d}}{D - d} \times \text{segment } FAE \text{ (its height } AD = d) + \frac{2}{3} \times D - d \sqrt{d \times D - d} - n d d, \text{ for that of the complemental parabolic ungula } FAE CG.$



of the square of the less diameter by that of the greater, the quotient arising from the division of the difference between the part of the diameter of the base not cut off and half the sum of the diameters by the difference between the said part of the diameter of the base and the less diameter, the square root of the quotient arising from the division of the part cut off the diameter of the base by the difference between the part not cut off and the less diameter, and such a segment of the base of the frustum whose height is equal to the product arising from the multiplication

3 A

tion

COROLLARY VI.

If the  $LCDB$  be greater than the  $LVAB$ , or the section be an hyperbola, its area being found and substituted for  $B$  in the general rules, will give the surfaces of the hyperbolic unguas.

COROLLARY VII.

If the hyperbolic section be perpendicular to the base,  $DI$  will vanish, and the general theorems will become  $\frac{VB}{BO} \times \text{seg. } FBE = \frac{CB}{BI} \times \text{seg.}$

$FBE$  (its height being  $BI$ ) =  $\frac{\sqrt{4b^2 + D - d^2}}{D - d} \times \text{seg. of the circle } AB$

whose height is  $\frac{D - d}{2}$  for the curve surface of the perpendicular ungula  $CIB$ .

Also  $\frac{VB}{BO} \times \text{seg. } FAE = \frac{CB}{BI} \times \text{seg. whose height is } AI = \frac{\sqrt{4b^2 + D - d^2}}{D - d} \times \text{seg. of the cir. } AB$  whose height is  $\frac{D + d}{2}$  for that of the remaining part  $AICV$ .

And  $\frac{VB}{BO} \times \text{seg. } FAE - \text{cir. } CG = \frac{\sqrt{4b^2 + D - d^2}}{D - d} \times \text{seg. of the circle}$

$AB$  whose height is  $\frac{D + d}{2} - nd^2$  for that of the complementary perpendicular ungula  $AICG$ .



tion of the quotient of the greater diameter divided by the less into the difference between the less diameter and the part of the base diameter not cut off; and call the last product  $R$ .

Then

1. The difference between  $R$  and the base of the hoof multiplied by the quantity  $\mathcal{Q}$ , in the last problem, will produce the convex surface of the ungula required.

That is,  $\frac{\sqrt{4b^2 + D-d|^2}}{D-d} \times \text{seg. } FBE - \frac{dd}{DD} \times \frac{\frac{1}{2} \times \overline{D+d} - AD}{d-AD}$   
 $\sqrt{\frac{BD}{d-AD}} \times \text{seg. of the circle } AB \text{ whose height is } D \times \frac{d-AD}{d} = \text{the convex surface of the hoof } FEBC.$  Using still the same letters as in the former problems. (The fig. in page 179.

2. The sum of  $R$  and the remaining segment of the base, not included by the hoof above, multiplied by  $\mathcal{Q}$ , will give the surface of the remaining part of the whole cone.

That is,  $\frac{\sqrt{4b^2 + D-d|^2}}{D-d} \times \text{seg. } FAE + \frac{dd}{DD} \times \frac{\frac{1}{2} \times \overline{D+d} - AD}{d-AD}$   
 $\sqrt{\frac{BD}{d-AD}} \times \text{seg. of the circle } AB \text{ whose height is } D \times \frac{d-AD}{d} = \text{the convex surface of the part } FEAVC.$

3. From the surface of the part in the last article subtract that of the little cone at the top, viz. the product of  $\mathcal{Q}$  by the area of the top of the frustum, and the remainder will be the surface of the complementary ungula.

That is,  $\frac{\sqrt{4b^2 + D-d|^2}}{D-d} \times : -n dd + \text{seg. } FAE + \frac{dd}{DD} \times \frac{\frac{1}{2} \times \overline{D+d} - AD}{d-AD} \sqrt{\frac{BD}{d-AD}} \times \text{seg. of the cir. } AB \text{ whose height is}$   
is



is  $D \times \frac{d-AD}{d}$  = the convex surface of the complemental hoof  $FEAGC$ .—All these are proved in cor. 3 prob. 15.

## EXAMPLE.

Required the convex surfaces of the parts into which a cone is cut by a plane which enters the side at the distance of 18 inches from the base, and cuts off 20 inches of the diameter of the base, that diameter being 30 and the diameter at the top of the section  $19\frac{1}{2}$  inches.

Here  $D=30$ ,  $d=19\frac{1}{2}$ ,  $h=18$ ,  $BD=20$ , and  $AD=10$ .  
Whence

$$Q = \frac{\sqrt{4b^2 + D-d^2}}{D-d} = \frac{1}{3} \sqrt{109} = 3.48010217.$$

The seg.  $FBE = 30^2 \times .55622573 = 500.603157$ .

The seg.  $FAE = 30^2 \times .22917243 = 206.255187$ .

$$R = \frac{19\frac{1}{2}}{30} \times \frac{14\frac{3}{4}}{9\frac{1}{2}} \sqrt{\frac{20}{9\frac{1}{2}}} \times \text{seg. whose diameter is 30 and height } \frac{30 \times 9\frac{1}{2}}{19\frac{1}{2}} = \frac{8^2 \times 146 \sqrt{46}}{5^3 \times 23^2} \times 30^2 \times \text{tab. seg. whose height is } \frac{23}{48} = \frac{6^2 \times 8^2 \times 146 \sqrt{46}}{5 \times 23^2} \times 37187178 = 320.761173.$$

$$\text{And } ndd = .785398 \times 19\frac{1}{2}^2 = 289.529179.$$

Wherefore

$$1. \quad Q \times FBE - R = 3.48010217 \times 179.841984 = 625.868479 = \text{the convex surface of the hoof } FEBC.$$

$$2. \quad Q \times FAE + R = 3.48010217 \times 527.01636 = 1834.070779 = \text{that of the part } FAECV.$$

$$3. \quad Q \times FAE + R - ndd = 3.48010217 \times 237.487181 = 826.479654 = \text{that of the complemental hoof } FAECG.$$

PRO-



## PROBLEM XVII.

*To find the convex surfaces of the parabolic hoofs made by a plane cutting a cone parallel to one side.*

## R U L E.

Multiply  $\frac{2}{3}$  of the difference of the diameters of the base and top of the frustum by the square root of the product of the said difference and the less diameter, and call the product  $S$ .

Then

1. The difference between  $S$  and the base of the hoof multiplied into  $\mathcal{Q}$ , in the 15th problem, will give the surface of the hoof.

That is,  $\frac{\sqrt{4b^2 + D-d}^2}{D-d} \times : \text{segment } FBE - \frac{2}{3} \times \overline{D-d}$   
 $\sqrt{d \times D-d} =$  the convex surface of the ungula  $FEBC$ .  
 Using still the same letters.

2. The Sum of  $S$  and the complementary base multiplied by  $\mathcal{Q}$ , will produce the convex surface of the remaining part of the cone.

That is,  $\frac{\sqrt{4b^2 + D-d}^2}{D-d} \times : \text{segment } FAE + \frac{2}{3} \times \overline{D-d}$   
 $\sqrt{d \times D-d} =$  the surface of the part  $FAECV$ .

3. The difference between the circle  $CG$  and the sum of  $S$  and the complementary base multiplied by  $\mathcal{Q}$ , will give the surface of the complementary hoof.

That is,  $\frac{\sqrt{4b^2 + D-d}^2}{D-d} \times : \text{segment } FAE + \cdot \times \overline{D-d}$   
 $\sqrt{d \times D-d} - n d d =$  the convex surface of the complementary hoof  $FAECG$ .

Ex-



## EXAMPLE.

Required the convex surfaces of the parabolic  
hoofs of the frustum of a cone whose height is 18  
and its diameters 30 and  $19\frac{1}{2}$  inches.

Here again  $\mathcal{Q} = 3.48010217$ .

$$S = \frac{2}{3} \times D - d \sqrt{d \times D - d} = \frac{2}{3} \times 10\frac{4}{5} \sqrt{19\frac{1}{2} \times 10\frac{4}{5}} = \frac{2 \times 54 \times 6 \times 4 \times 3}{3 \times 5 \times 5} \\ = \frac{2592}{25} = 103.68.$$

The seg.  $FBE = 30^2 \times \text{tab. seg. whose versed sine is}$   
 $\left(\frac{10\frac{4}{5}}{30} = \frac{3.6}{10} =\right) \cdot 36 = 900 \times .25455055 = 229.095495.$

The seg.  $FAE = 30^2 \times \text{tab. seg. whose height is}$   
 $\left(\frac{19\frac{1}{2}}{30} = \frac{6.4}{10} =\right) \cdot 64 = 900 \times .785398 - .25455055 = 900 \times$   
 $.53084761 = 477.762849.$

And  $ndd$ , as before,  $= 289.529179$ .

Whence

1.  $\mathcal{Q} \times FBE - S = 3.48010217 \times 125.415495 =$   
 $436.458737 = \text{the convex surface of the hoof } FEBC.$

2.  $\mathcal{Q} \times FAE + S = 3.48010217 \times 581.442849 =$   
 $2023.480521 = \text{that of the part } FAECV.$

3.  $\mathcal{Q} \times FAE + S - ndd = 3.48010217 \times 291.91367 =$   
 $1015.8889396 = \text{that of the complemental hoof } FAECG.$

## PROBLEM XVIII.

To find the curve surfaces of the hyperbolic hoofs of the  
frustum of a cone, made by a plane cutting it per-  
pendicular to the base.

## RULE.

1. Multiply the quantity  $\mathcal{Q}$ , found as in the pre-  
ceding problems, by the base of the hoof, and the  
product will be its surface.

3 B

That



That is,  $\frac{VB}{BO} \times \text{seg. } FBE = \frac{\sqrt{4b^2 + D-d^2}}{D-d} \times \text{seg. of the circle } AB \text{ whose height is } \frac{D-d}{2} = \text{the curve surface of the perpendicular hoof } FEBC. \text{ (See the figure at page 175.)}$

2. Multiply  $\mathcal{Q}$  by the complemental base, and the product will be that of the complemental part of the whole cone.

That is,  $\frac{VB}{BO} \times \text{seg. } FAE = \frac{\sqrt{4b^2 + D-d^2}}{D-d} \times \text{seg. of the circle } AB \text{ whose height is } \frac{D+d}{2} (=AD=AI) = \text{the curve surface of the part } FAECV.$

3. Multiply  $\mathcal{Q}$  by the difference between the complemental segment and the circle  $CG$ , and the product will be that of the complemental hoof.

That is,  $\frac{VB}{BO} \times \overline{\text{seg. } FAE - \text{cir. } CG} = \frac{\sqrt{4b^2 + D-d^2}}{D-d} \times : -ndd + \text{seg. of the circle } AB \text{ whose height is } \frac{D+d}{2} = \text{the curve surface of the complemental hoof } FAECG. \text{ Using still the same letters as in the former problems.}$

*Note.* The two first articles are the same rule, and serve for the whole cone as well as any perpendicular part.

#### EXAMPLE.

Required the curve surfaces of the perpendicular hoofs of a cone whose base diameter is 30 inches; the height of the section being 18 and the diameter of the cone at the top of the section  $19\frac{1}{2}$  inches.

Here again  $\begin{cases} \mathcal{Q} = 3.48010217, \\ ndd = 289.529179. \end{cases}$

The



The segment  $FBE = 30^\circ \times \text{tab. seg.}$  whose height is  $\left(\frac{30 - 19\frac{1}{2}}{2 \times 30}\right) \cdot 18 = 900 \times .09613453 = 86.521077$ .

The segment  $FAE = 30^\circ \times \text{tab. seg.}$  whose height is  $\left(\frac{30 + 19\frac{1}{2}}{2 \times 30}\right) \cdot 82 = 900 \times .68926363 = 620.337267$ .

Whence

1.  $Q \times FBE = 3.48010217 \times 86.521077 = 301.102188$   
= the curve surface of the hoof  $FEBC$ .

2.  $Q \times FAE = 3.48010217 \times 620.337267 = 2158.837068$  = that of the part  $FEACV$ .

3.  $Q \times FAE - n d d = 3.48010217 \times 330.808088 = 1151.245945$  = that of the complemental hoof  $FEACG$ .

### PROBLEM XIX.

To find the curve surface of a sphere, or of any segment or zone of it.

### RULE.

Multiply the circumference of the sphere into the height of the part required, and the product will be the curve surface, whether it be segment, zone, hemisphere, or the whole sphere.\*

Note. The height of the whole sphere is its diameter.

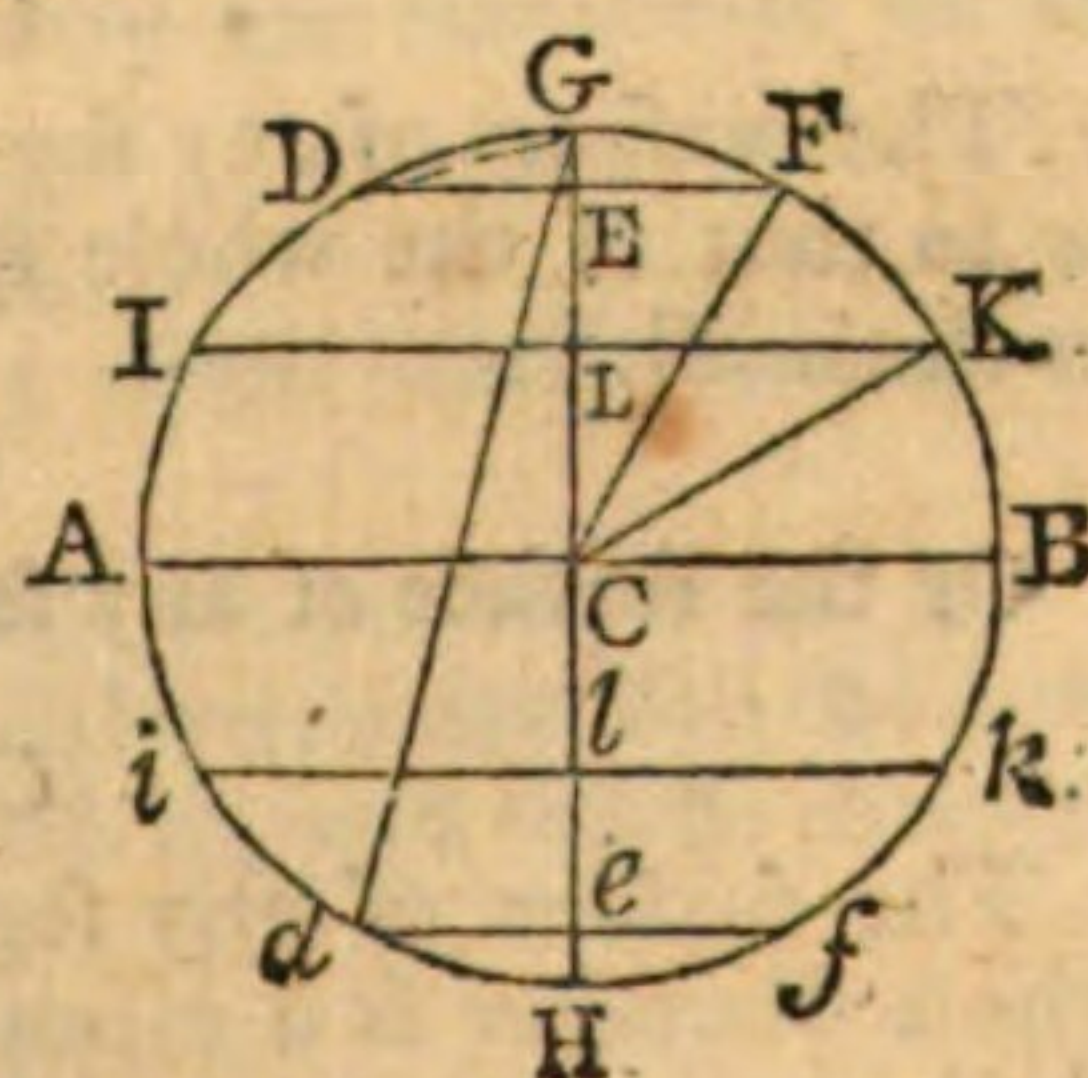
Ex-

### \* DEMONSTRATION.

Put  $d$  = the diameter  $AB$  or  $2CG$ ,  $x = CE$  the height of the zone  $ADFB$ ,  $y = ED$ ,  $z = DA$ ,  $p = 3.14159$  &c. and  $s$  = the surface required.

Then  $s = 2pyz$ ; but as  $y : \frac{1}{2}d :: x : z =$

$\frac{dx}{2y}$ ; therefore  $s = pdx$ , and consequently  $s = pdx$ ; viz. the product of the circumference of the sphere and height of the zone, for  $pd$  is = the circumference of the circle whose diameter is  $d$ .



Co-



## EXAMPLE I.

If the diameter or axe of the earth be  $7957\frac{1}{4}$  miles, what is the whole surface, supposing it a perfect sphere?

$7957\frac{1}{4} \times 3.141592 = 25000$  miles, (extremely near) = the circumference.

Then  $7957\frac{1}{4} \times 25000 = 198943750$  square miles = the whole surface required. And the half is  $99471875$  = the surface of the hemisphere.

Ex-

## COROLLARY I.

When  $x$  becomes  $= CG = \frac{1}{2}d$ ,  $s$  will be  $= \frac{1}{2}pdd$  = the surface of the hemisphere. And consequently that of the whole sphere is  $pdd$ , the product of the circumference and height or diameter.

## COROLLARY II.

To or from  $\frac{1}{2}pdd$ , the surface of the hemisphere, add or subtract  $pdx$ , that of the zone  $ADFB$ , and the remainder  $pd \times \frac{1}{2}d \pm x = pd \times GE$  will be the surface of the segment  $DGF$ , viz. the product of the circumference and height. So that the rule is general. Q.E.D.

## COROLLARY III.

The surfaces of spheres, and also of their similar parts, are to each other as the squares of their diameters. For, by exterminating the common given quantity  $p$ , they are as  $d$ , the diameter, into the height of the part; but the heights of similar parts are as the diameters; wherefore &c.

## COROLLARY IV.

The surfaces of any segments or zones of a sphere are to each other, or to that of the whole sphere, as their heights. For  $pd$  is common to them all.

## COROLLARY V.

Or the surface of any segment or zone of a sphere is as its height.

## COROLLARY VI.

The surface of any segment or zone is equal to 4 times a circle whose diameter is a mean proportional between its height and the diameter of the sphere. For this circle is  $= \frac{1}{4}p \times \sqrt{dx}^2 = \frac{1}{4}pdx$ .

Co-



## EXAMPLE II.

To find the surface of the two frigid zones of the earth.

*Note.* The frigid zones are the two opposite segments  $DGF$ ,  $dHf$ , in which each of the arcs  $DG$ ,  $dH$ , or half the breadth of the zone, is  $23\frac{1}{2}$  degrees.

Draw the radius  $CF$ ; then the angle  $FCE = 23\frac{1}{2}$  degrees, and the angle  $EFC = 90 - 23\frac{1}{2} = 66\frac{1}{2}$  degrees; also the angle  $E = 90$  degrees.

Then, ass.  $LE:s.LF::FC=3978\frac{7}{8}:CE=3648.8675054$ .

And  $EG = GC - CE = 3978.875 - 3648.8675054 = 330.0074946$  = the height.

Hence  $25000$  (the circumference)  $\times 330.0074946 = 8250187.365$  = the surface of the segment  $DGF$  or frigid zone.

3 C

Ex-

## COROLLARY VII.

Hence the surface of a segment  $DGF$  is equal to 4 times the circle whose diameter is the chord  $GD$  drawn from the vertex to the extremity of the base; or equal to a circle whose radius is that chord. For  $GD = \sqrt{HG \times GE} = \sqrt{dx}$ .

## COROLLARY VIII.

And hence the surface of a sphere is equal to 4 times the area of a great circle of it; that is, 4 times the area of a circle of the same diameter with the sphere. And consequently the surface of an hemisphere is double the area of its base.

## COROLLARY IX.

Or the surface of a sphere is equal to a circle whose diameter is double to that of the sphere.

## COROLLARY X.

The surface of any segment or zone of a sphere is equal to the curve surface of a cylinder of the same height with it, and whose diameter is equal to that of the sphere.

## COROLLARY XI.

Hence the surface of a whole sphere, or of a hemisphere, is equal to the curve surface of the circumscribed cylinder.



## EXAMPLE III.

To find the surface of the torrid zone  $IKki$ , which extends to the distance of ( $IA$  or  $Ai$ )  $23\frac{1}{2}$  degrees on each side from  $AB$  the diameter.

Here the  $\angle LKC = 23\frac{1}{2}$  degrees.

Then, s.  $\angle L$  : s.  $\angle LKC$  ::  $KC$  :  $CL = 1586.572$ .

And  $Ll = 2CL = 3173.144$  = the height.

Wherefore  $25000 \times 3173.144 = 79328600$  square miles = the surface of the torrid zone.

## EXAMPLE IV.

Required the convex surface of each of the temperate zones  $DFKI$ ,  $dfki$ , which are included between the frigid and torrid zones.

By the 1st ex.  $CE = 3648.8675$   
and by the 2d ex.  $CL = 1586.572$

their difference is  $LE = 2062.2955$  = the height.

Wherefore  $25000 \times 2062.2955 = 51557387\frac{1}{2}$  square miles = each temperate zone.

|                            |   |           |
|----------------------------|---|-----------|
| Hence the two frigid zones | = | 16500375  |
| the two temperate zones    | = | 103114775 |
| the torrid zone            | = | 79328600  |
| their sum is               |   | 198943750 |

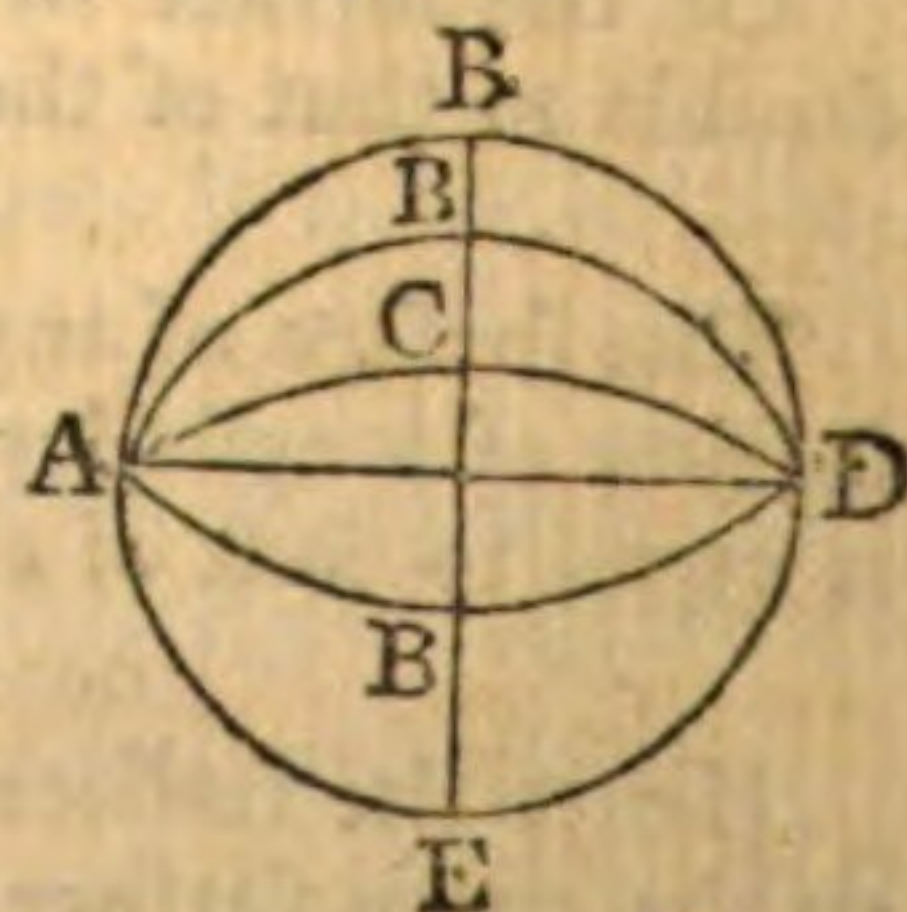
= the surface of the sphere.

## PROBLEM XX.

To find the lunar surface  $ABDCA$  included between two great circles  $ABD$ ,  $DCA$  of a sphere.

## RULE I.

Multiply the diameter  $AD$  into the breadth  $BC$  of the surface in the middle, that is, into the arc that measures the angle  $BAC$  of inclination of the two circles, and the product will be the surface  $ABDCA$ .



That



That is, if  $d = AD$  the diameter,  
 $a$  = the length of the arc  $BC$  over the  
 middle of the surface.

Then  $ad$  is the surface.

\* RULE II.

As four right angles are to the surface of the sphere;  
 Or, As one right angle is to a great circle of the sphere;  
 So is the angle made by the two great circles,  
 To the surface included by them.

E X A M P L E.

Required the surface included by two great circles forming an angle of 25 degrees, the diameter of the sphere being 10 feet.

1. By rule 1.

First,  $3.1415926 \times 10 = 31.415926 =$  the circumference.

And

---

\* DEMONSTRATION.

If the great circle  $BCE$ , whose poles are  $A$  and  $D$ , be conceived to be divided into an indefinite number of equal parts, and great circles be conceived to be drawn through the points of division and through  $A$  and  $D$ ; it is evident that the whole surface will be divided, by the circles, into the same number of parts similar and equal to one another, and that, therefore, the surface included between any two of those circles will be as the number of parts, or as the arc of the circle  $BE$  included by them; wherefore as the whole circumference  $BE$  is to the arc  $BC$ , or as four right angles to the angle  $BAC$ , so is the surface of the sphere to the lunar surface  $ABDCA$ ; or also as one right angle is to the angle  $BAC$ , so is the area of a great circle of the sphere ( $\frac{1}{4}$  of the surface) to the lunar surface. Which is rule 2.

COROLLARY.

If  $d$  be the diameter and  $c$  the circumference of the sphere, and  $a$  the arc  $BC$ ; then, from the process above,  $c : a :: cd$  (the surface of the sphere) :  $ad =$  the lunar surface. Which is rule 1.







That is,  $pdd \times \frac{s-180}{720}$  = the area of the triangle, putting  $p = 3.14159$  &c.  $d$  = the diameter of the sphere, and  $s$  = the sum of the 3 angles of the triangle.

## EXAMPLE.

If the angles be 55, 60, and 85 degrees, what is the trilineal surface; supposing the diameter to be 10?

Here  $.7853982 \times 10 \times 10 = 78.53982 = \frac{1}{4}$  of the surface of the sphere.

Then  $180 : 20 (= 55 + 60 + 85 - 180) :: 78.53982 : \frac{78.53982}{9} = 8.72664$  = the area of the triangle required.

## PROBLEM XXII.

To find the area of a spheric polygon, or to find the spherical surface included by any number of intersecting great circles.

## RULE.

|        |                                                                                                |      |
|--------|------------------------------------------------------------------------------------------------|------|
| As     | 8 right angles or $720^\circ$ ,                                                                |      |
| To     | the surface of the sphere;                                                                     |      |
| Or, As | 2 right angles or $180^\circ$ ,                                                                |      |
| To     | a great circle of the sphere;                                                                  |      |
| So     | is the excess of all the angles above the product of 180 and 2 less than the number of angles, |      |
| To     | the area of the spheric polygon.                                                               | That |
|        | 3 D                                                                                            |      |

## COROLLARY.

When  $P = 0$ , then  $(A+B+C)$  or  $s = 180$ ; but when  $P = \frac{1}{2}pdd$ , half the surface of the sphere, then  $s - 180 = 360$ , or  $s = 540$ ; consequently  $A+B+C$  is always between 180 and 540, that is, greater than 2 and less than 6 right angles.



That is, putting  $n$  = the number of angles,  
 $s$  = the sum of all the angles,  
 $d$  = the diameter of the sphere,  
 and  $p = 3.14159$  &c.

Then  $pd^2 \times \frac{s-180 \times n-2}{720}$  = the area of the spheric polygon.\*

#### EXAMPLE.

What surface is included by the intercepted arcs of five intersecting great circles of a sphere of 10 feet diameter; supposing the sum of the angles formed by those arcs to be 640 degrees.

Here the surface of the sphere is  $3.1415926 \times 10 \times 10 = 314.15926$ .

And

#### \* DEMONSTRATION.

For if the polygon be supposed to be divided in as many triangles as it has sides by great circles drawn to all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by the last problem, it will be in any one triangle,

As  $720 : S ::$  sum of its angles  $- 180 : \text{its area},$

Wherefore by composition, As  $720 : S :: s - 180n : \text{sum of all the triangles or area of the polygon}.$

But all the vertical angles  $= 360$  or  $180 \times 2$ .

Wherefore, As  $720 : S :: s - 180 \times n - 2 : S \times \frac{s-180 \times n-2}{720}$  = the area of the polygon. Q.E.D.

#### COROLLARY.

When the polygon is  $= 0$ , then  $s$  is  $= 180 \times n - 2$ ; and when the polygon is  $=$  the semi-spheric surface, then  $s = 180 \times n - 2 + 360 = 180 \times n$ ; consequently  $s$  the sum of all the angles of any polygon is always between  $180 \times n - 2$  and  $180 \times n$ , that is, less than  $n$  times 2 right angles, but greater than  $n - 2$  times 2 right angles,  $n$  being the whole number of angles.



And  $180 \times n - 2 = 180 \times 3 = 540$ , which taken from 640 leaves 100.

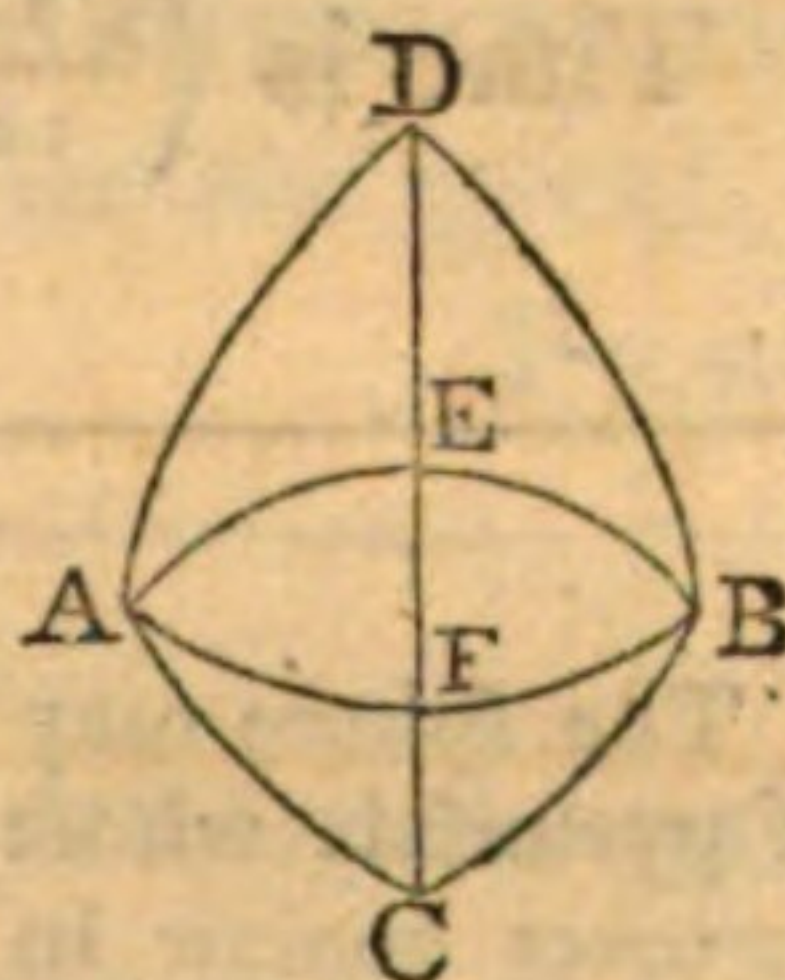
Whence, as  $720 : 100 :: 314 \cdot 15926 : \frac{392 \cdot 69908}{9} = 43 \cdot 63323$  = the area of the polygon required.

PROBLEM XXIII.

To find the surface AEBF included between two intersecting little circles of a sphere.

RULE.

Through the poles  $D, C$ , and intersections  $A, B$ , of the two little circles  $AFB, AEB$ , draw the great circles  $AD, DB, BC, CA$ ; and also the great circle  $DEFC$ .



By prob. 21 find the area of the triangle  $BCD$ , having first found the angles at  $B, C$ , and  $D$ , from the given sides, by the principles of spherical trigonometry.

By prob. 19 find the surface of the segment cut off by the circle of which  $AFB$  is a part, thus, viz. As the diameter is to the versed sine of the arc  $DF$  or  $DB$ , so is the surface of the sphere to that of the segment.

Then as 4 right angles, to the  $\angle BDC$ , so is that surface, to the part of it  $DBF$ .

In the same manner find the surface  $CBE$ .

Then from the sum of  $DBF$  and  $CBE$  take the triangle  $DBC$ , and the remainder will be the part  $BEF$ ; the double of which will be the whole  $AEBFA$ .

PRO-



## PROBLEM XXIV.

*To find the solidity of a sphere or globe.*

## \* RULE I.

Multiply the surface by  $\frac{1}{3}$  of the radius or by  $\frac{1}{6}$  of the diameter, and the product will be the solidity.

## RULE II.

Multiply the cube of the diameter by  $\cdot 5236$ , and the product will be the solidity.

That is  $\left(\frac{3 \cdot 1416}{6}\right) \cdot 5236 d^3 = \text{the solidity.}$

Ex-

## \* DEMONSTRATION.

The sphere may be considered as constituted of an infinite number of pyramids whose bases compose the spheric surface, and all whose vertexes concur in the center, their common height being equal to the radius of the sphere. And consequently the sphere, or any spheric pyramid (being a part contained within right lines drawn from the surface to the center) is equal to a pyramid whose base is equal to the spheric surface and height equal to the radius. And therefore the surface of the whole, or of any such part, being drawn into  $\frac{1}{3}$  of the radius will give the solidity, as in rule 1.

## COROLLARY I.

Since the surface of the sphere is  $= pdd$ , we shall have  $pdd \times (\frac{1}{3} \text{ of the radius or } \frac{1}{6}d = \frac{1}{6}pd^3 = \cdot 523598775 \text{ \&c.} \times d^3 \text{ (which is rule 2.)} = \frac{2}{3} \text{ of a cylinder of the same diameter and height.}$

## COROLLARY II.

If  $h$  be the height or versed sine of any segment; then, since  $pdb =$  its surface,  $\frac{1}{6}pd^2h = \cdot 523598775 \text{ \&c.} \times d^2h$  will be the solidity of the spheric pyramid or cone whose base is the surface of the segment.

## COROLLARY III.

Hence spheres and their similar pyramids, and also any other similar parts of them, are as the cubes of the diameters.

Co-



EXAMPLE.

Supposing the earth to be spherical, and its diameter  $7957\frac{1}{4}$  miles, what is its solidity.

1. By rule 1.

By exam. 1 of prob. 19 the surface is 198943750, then  $198943750 \times \frac{7957\frac{1}{4}}{6} = 263857437760$  miles = the solidity.

2. By the 2d rule.

Here  $\cdot 5236d^3 = \cdot 5236 \times 7957\frac{1}{4}^3 = 263857624944$  miles = the solidity by this rule: the difference arising by taking the number  $\cdot 5236$  a little too great.

3 E

PRO-

COROLLARY IV.

If to or from  $(\frac{1}{6}pd^2b)$  the spheric cone be added or subtracted  $(\frac{1}{4}p \times \frac{1}{3}d \pm b)$  the cone whose base is the same with the base of the segment, and whose vertex is in the center, the sum or difference  $\frac{1}{6}pd^2b \pm \frac{1}{4}pd^2b \mp \frac{1}{12}ph^3$  will be the spheric segment whose height is  $b$ , either greater or less than the hemisphere, or of whatever magnitude  $bi$  s, not exceeding  $d$ . Or if  $r$  be the radius of the segment's base, since  $d = \frac{r^2 + b^2}{b}$ , the segment will be  $\frac{1}{2}pr^2b + \frac{1}{6}ph^3 = \frac{1}{6}ph \times 3r^2 + b^2$ .

COROLLARY V.

Hence the difference between two segments whose heights are  $H$ ,  $h$ , and the radiuses of their bases  $R$ ,  $r$ , will give for the frustum or zone  $\frac{1}{6}p \times 3R^2H + H^3 - 3r^2h - h^3$ , which, putting  $a$  for the altitude of the frustum, and exterminating  $H$  and  $h$  by means of the two equations  $\frac{R^2 + H^2}{H} = \frac{r^2 + h^2}{h}$ , and  $a = H - h$ , will become  $\frac{1}{2}ap \times R^2 + r^2 + \frac{1}{3}a^2$ .

COROLLARY VI.

If one end of the frustum pass through the center, then  $R^2 = \frac{1}{4}d^2 = r^2 + a^2$ , and the last theorem will become  $ap \times r^2 + \frac{1}{3}a^2 = ap \times \frac{1}{4}d^2 - \frac{1}{3}a^2$ .

Co-



## P R O B L E M XXV.

*To find the solidity of the segment of a sphere.*

## R U L E I.

To three times the square of the radius of its base add the square of its height, multiply the sum by the height and the product by  $\cdot 5236$ ; and the last product will be the solidity.

That is, if  $r = DE$  the radius of its base,

$h = EG$  the height;

Then  $\cdot 5236h \times 3rr + hh =$  the solidity of the segment  $DGF$ . By cor. 4 to the last problem. [See the figure in page 191.]

## R U L E II.

From three times the diameter of the sphere subtract twice the height of the frustum, multiply the difference into the square of the height and into  $\cdot 5236$ , and the last product will be the solidity.

That is, if  $d = GH$  the diameter of the sphere,

$h = EG$  the height of the frustum;

Then  $\cdot 5236h^2 \times 3d - 2h =$  the solidity of  $DGF$ . By cor. 4 to the last problem.

## E X A M P L E.

What is the solidity of each of the frigid zones of the earth, the axe being  $7957\frac{1}{4}$  miles, and half the

## C O R O L L A R Y VII.

Hence the middle zone, or the double of the last expression will be  $2ap \times r^2 + \frac{2}{3}a^2 = 2ap \times \frac{1}{4}d^2 - \frac{1}{3}a^2$ , where  $a$  is  $\frac{1}{2}$  its altitude and  $r = \frac{1}{2}$  the diameter of each end. But if  $A$  be its whole altitude, and  $D$  the diameter of each end, those theorems will become  $\frac{1}{4}Ap \times D^2 + \frac{2}{3}A^2 = \frac{1}{4}Ap \times d^2 - \frac{1}{3}A^2$ .



the breadth, or arc  $VA$ , of the zone being  $23\frac{1}{2}$  degrees?

By rule 2.

As  $1 = \text{tabular radius} : 3978\frac{7}{8} = \text{radius of the earth} ::$   
 $\cdot 0829399 = \text{tab. versed sine of } 23\frac{1}{2} \text{ degrees} : \cdot 0829399$   
 $\times 3978\frac{7}{8} = 330\cdot 0074946 = \text{the versed sine or height of}$   
 the segment.

Then  $\cdot 5236b^2 \times 3d - 2b = \cdot 5236 \times 330\cdot 0074946 \times$   
 $330\cdot 0074946 \times 23213\cdot 2350108 = 1323679710 = \text{the}$   
 content of the segment.

By rule 1.

As  $1 : 3978\frac{7}{8} :: 3987491 = \text{tabular sine of } 23\frac{1}{2} \text{ de-}$   
 grees :  $\cdot 3987491 \times 3978\frac{7}{8} = 1586\cdot 57282526 = \text{the ra-}$   
 dius of the segment's base.

Then  $\cdot 5236b \times 3r^2 + b^2 = \cdot 5236 \times 330\cdot 0074946 \times$   
 $7660544\cdot 936 = 1323680299\cdot 69 = \text{the solidity of the}$   
 segment.

#### PROBLEM XXVI.

*To find the solidity of a frustum or zone of a sphere.*

#### RULE.

Add together the squares of the radiuses of the ends and  $\frac{1}{3}$  of the square of their distance or of the height, multiply the sum into the said height, and the product multiplied into  $1\cdot 5708$  will produce the content required.

That is,  $R^2 + r^2 + \frac{1}{3}b^2 \times \frac{1}{2}pb = \text{the solidity of the fru-}$   
 stum whose height is  $b$ , and the radiuses of its ends  
 $R$  and  $r$ ,  $p$  being equal  $3\cdot 1416$ . By cor. 5 to prob.



## EXAMPLE I.

What is the solidity of the frustum of a sphere, the diameter of whose greater end is 4 feet, the diameter of the less 3 feet, and the height  $2\frac{1}{2}$  feet.

Here  $R^2 + r^2 + \frac{1}{3}h^2 \times 1.5708h = 2^2 + 1.5^2 + \frac{1}{3} \times 2.5^2 \times 1.5708 \times 2\frac{1}{2} = 8\frac{1}{3} \times 3.927 = 32.725 =$  the solidity of the frustum required.

## EXAMPLE II.

What is the solidity of each temperate zone of the earth, they extending from  $23\frac{1}{2}$  degrees to  $66\frac{1}{2}$  degrees of latitude, and the diameter of the earth being  $7957\frac{3}{4}$  miles.

By the example to the last problem the radius of the top is 1586.57282526.

And as  $1 : 3978\frac{7}{8} :: .9170601 =$  tabular sine of  $66\frac{1}{2}$  degrees :  $.3978\frac{7}{8} \times .9170601 = 3648.86750538 =$  the radius of the base.

Also by example 4 prob. 19, the height is 2062.2955.

Then  $R^2 + r^2 + \frac{1}{3}h^2 \times 1.5708h = 17249136 \times 2062.2955 \times 1.5708 = 55877778668 =$  the solidity of each temperate zone.

*Otherwise.*

Since the radiuses of the ends of this zone are the sines of  $23\frac{1}{2}$  and  $66\frac{1}{2}$  degrees, which are complements the one of the other, the sum of the squares of those radiuses will be equal to the square of the radius of the sphere; and therefore  $\frac{1}{4}dd + \frac{1}{3}hb \times 1.5708h = 17249136 \times 2062.2955 \times 1.5708 = 55877778668 =$  the content the same as before.

P R O-



## PROBLEM XXVII.

*To find the solidity of the middle zone of a sphere.*

## RULE.

Multiply either the sum of the square of the diameter of the end and  $\frac{2}{3}$  of the square of the height, or the difference between the square of the diameter of the sphere and  $\frac{1}{3}$  of the square of the height, into the height, and the product by .7854 for the content.

That is,  $\overline{dd + \frac{2}{3}hb} \times .7854b$  or  $\overline{DD - \frac{1}{3}hb} \times .7854b =$  the content of the middle zone whose height is  $b$ , the diameter of each end  $d$ , and the diameter of the sphere  $D$ . By cor. 6. to prob. 24.

## EXAMPLE I.

Required the solidity of the middle zone of a sphere whose top and bottom diameters are each 3 feet, and height 4 feet.

Here  $\overline{dd + \frac{2}{3}hb} \times .7854b = 3^2 + \frac{2}{3} \times 4^2 \times 4 \times .7854 = 59 \times 4 \times .7854 = 186.18 = 61.7848 =$  the content required.

## EXAMPLE II.

What is the solidity of the torrid zone of the earth, which extends to  $23\frac{1}{2}$  degrees on each side of the equator; the diameter of the earth being  $7957\frac{3}{4}$  miles?

By the example to prob. 25, the sine of  $23\frac{1}{2}$  degrees, or  $\frac{1}{2}$  the height of the zone, is  $1586.57282526$ , and the whole height is  $3173.14565052 = b$ .

Then  $\overline{DD - \frac{1}{3}hb} \times .7854b =$   
 $7957.75^2 - \frac{1}{3} \times 3173.14565052^2 \times 3173.14565052 \times$   
 $.7854 = 149455081137 =$  the content required.



## S C H O L I U M.

From the last three problems we find that

The two frigid zones = 2647359420

The two temperate zones = 111755557336

The torrid zone = 149455081137

whose sum 263857997893 is  
the whole sphere, nearly the same as found in prob.  
24.

## P R O B L E M XXVIII.

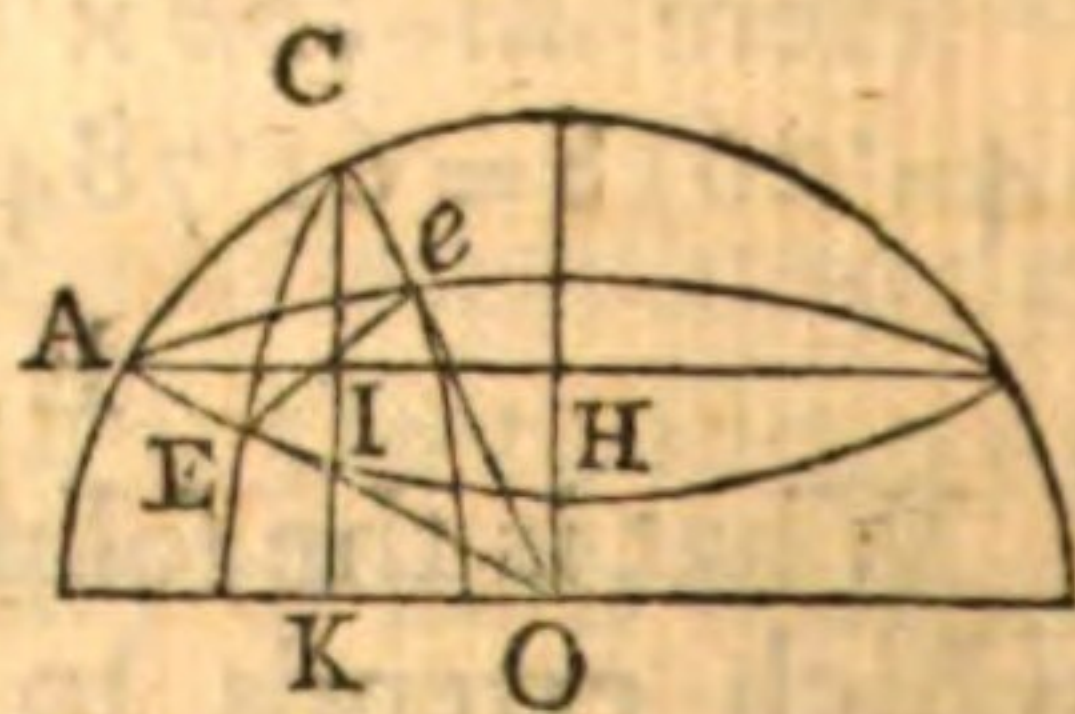
*To find the solidity of the second segment of a sphere.*

## D E F I N I T I O N.

A second segment is a part cut off a segment by a plane perpendicular to the base.

## R U L E.

By prob. 23 find the curve surface of the second segment  $AECe$ , which being drawn into  $\frac{4}{3}$  of the radius of the sphere, will produce the content of the spheric sector  $OEAc$ ; from which if there be taken the pyramid  $OEAc$  whose base is the segment  $AEE$  and height  $OH$ , and from the remainder be taken the pyramid  $OECe$  whose base is the segment  $CEe$  and height  $KO$  or  $HI$ , it is evident that the last remainder will be the second segment  $AEEc$ : for these two pyramids and the second segment compose the spheric pyramid.



PRO-



## PROBLEM XXIX.

*To find the surface of a circular spindle, or of any segment or frustum of it.*

## DEFINITION.

A circular spindle is a solid  $ACBDA$  conceived to be generated by the rotation of a segment  $ACB$  of a circle about its chord  $AB$  as an axe.—The distance  $FE$  of the middle of the chord, or center of the spindle, from the center of the circle, or of the revolving arc, is called the central distance.

## RULE.

From the product of the height of the solid and radius of the revolving arc subtract the product of the said arc and central distance, multiply the remainder by 3.1416, and double the product will be the surface described by that arc, whether it be the whole or any part of the spindle.\*

That

## \* DEMONSTRATION.

Put  $z$  = the arc  $CP$ ,  $x$  = its sine  $RE$ ,  $r$  and  $c$  for the radius and central distance, and  $p = 3.14159$  &c. Then the fluxion of the surface  $\dot{s}$  is  $= 2pz \times RP = 2p \times z \sqrt{rr - xx - cz}$ ; but, by the property of the circle,  $z \sqrt{rr - xx}$  is  $= rx$ ; therefore  $\dot{s} = 2p \times rx - c\dot{z}$ , and  $s = 2p \times rx - cz = 2p \times r \times ER - c \times CP$ .

## COROLLARY I.

When  $ER = EA$ , the rule becomes  $2p \times r \times EA - c \times CA$  for the surface of half the spindle, or  $2p \times r \times AB - c \times ACB$  for that of the whole.

## COROLLARY II.

If from the surface of the semi-spindle be taken that of the frustum, there will remain  $2p \times r \times RA - c \times AP$  for that of the segment  $PAQ$ : so that the rule is general.

Co-







Also, as  $20\frac{1}{2}:20::1:24\div25=96\div100=.96=$  the tabular sine of the arc  $AC$ , to which belong  $73.7398$  degrees, the double of which  $147.4796$  are the number of degrees in the whole arc  $ACB$ . And therefore by rule 1 prob. 6 sect. 1 part 2, we have  $.01745329 \times 147.4796 \times 20\frac{1}{2} = 53.62508 =$  the length of the revolving arc.

Wherefore, by the rule,  $\overline{br-ac} \times 2p =$   
 $20\frac{1}{2} \times 40 - 53.62508 \times 5\frac{1}{2} \times 6.2832 = 25 \times 40 - 53.62508 \times 7$   
 $\times 5\frac{1}{2} \times 6.2832 = 624.62444 \times 5.236 = 3270.5335 =$  the surface required.

## PROBLEM XXX.

*To find the solidity of a circular spindle.*

## RULE.

From  $\frac{1}{3}$  of the cube of half the length of the spindle subtract the product of the central distance and half the generating circular segment, multiply the remainder by four times  $3.14159$  &c. and the product will be the content of the spindle.\*

3 G

That

## \* GENERAL INVESTIGATION.

Putting  $RE = x$  (see the fig. to the last prob.)  $EF = c$ ,  $3.14159$  &c.  $= p$ ; the fluxion of the solid  $S$  will be  $= p \dot{x} \times PR^2 = p \dot{x} \times \overline{HP-c}^2 =$   
 $p \dot{x} \times \overline{HP^2 - c \times 2HP - c^2} = p \dot{x} \times \overline{HP^2 - c \times 2RP + c^2} = p \dot{x} \times \overline{r^2 - c^2 - x^2 - 2c \times RP}$ ;  
 and  $S = p \times \overline{r^2 - c^2 - \frac{1}{3}x^2 - \frac{2c}{x} \times RPCE} = p \times \overline{AE^2 - \frac{1}{3}x^2 \times x - 2c \times RPCE}$   
 $=$  the frustum generated by  $RPCE$ .

## COROLLARY I.

When  $x$  becomes  $= AE$ , the theorem above becomes  $\frac{1}{3}AE^3 - c \times ACE$   
 $\times 2p = \frac{1}{2}$  the spindle, and  $\frac{1}{3}AE^3 - c \times ACE \times 4p =$  the whole spindle.

Co-



That is, if  $l = AE = \frac{1}{2}$  the length,  
 $c = EF =$  the central distance,  
 $a =$  the area  $ACE$ ,  
 and  $p = 3.14159$  &c.

Then  $\frac{1}{3}l^3 - ac \times 4p =$  the whole spindle  $ACBDA$ .

#### EXAMPLE.

If the length of a circular spindle be 40, and its greatest diameter 30; what is its solidity.

By the last problem the radius is  $20\frac{1}{2}$ , the central distance  $5\frac{1}{2}$ , and the length of the whole arc  $53.62508$ ; hence,  $53.62508 \times 20\frac{1}{2} =$  the double of the sector  $FAB$ , (lines being supposed to be drawn from the center to  $A$  and  $B$ ) and  $40 \times 5\frac{1}{2} =$  the double of the triangle  $FAB$ , and consequently  $53.62508 \times 20\frac{1}{2} - 40 \times 5\frac{1}{2} = 53.62508 \times 25 - 40 \times 7 \times \frac{1}{2} = 883.8558 =$  the double of the segment  $ACB$ ,  $\frac{1}{4}$  of which, or  $220.9639$  is the half of that segment.

Then  $\frac{1}{3}l^3 - ac \times 4p = \frac{1}{3} \times 20^3 - 220.9639 \times 5\frac{1}{2} \times 4 \times 3.1416 = \frac{4}{3} \times 20^3 - 220.9639 \times 1\frac{1}{2} \times 20 \times 3.1416 = 275.542 \times 62.832 = 17312.858 =$  the solidity required.

PRO-

#### COROLLARY II.

If from the  $\frac{1}{2}$  spindle be taken the frustum, there will remain

$$p \times AE^2 \times AR - \frac{1}{3} \times AE^3 - x^3 - 2c \times APR = p \times \frac{1}{3} AR^2 \times 3AE - AR - 2c \times APR \text{ for the segment } PAQ \text{ of the spindle,}$$

#### COROLLARY III.

If  $E$  coincide with  $F$ , the spindle will become a sphere,  $c$  will vanish, and the theorems above become  $\frac{4}{3}p \times GF^3 = \frac{1}{6}p \times GI^3$  for the whole sphere;  $p \times FH \times GF^2 - \frac{1}{3}FH^3$  for the spheric frustum; and  $\frac{1}{3}p \times GH^2 \times 3GF - GH$  for the spheric segment; and are the same with the theorems in problems 24, 25, and 27.



## PROBLEM XXXI.

*To find the content of the middle frustum or zone of a circular spindle.*

## RULE.

From the square of half the length of the spindle take  $\frac{1}{3}$  of the square of half the length of the middle zone, and multiply the remainder into the said half length of the zone, from the product subtract the product of the generating circular area and central distance; then the remainder drawn into 2 times 3.14159 &c. will be the content of the middle zone.

That is,

putting  $l = ER =$  half the length of the zone,  
 $L = EA =$  half the length of the spindle,  
 $c = FE$  the central distance,

and  $a =$  the generating area  $RPYW$ .

Then  $LL - \frac{1}{3}ll \times l - ac \times 2p =$  the zone  $PYZQ$ .

## EXAMPLE.

If a cask, in form of the middle frustum of a circular spindle, have its head diameter 24, bung diameter 32, and length 40 inches; how many ale gallons will it hold.

Here  $CE - PR = \frac{32^2 - 24^2}{2} = 4 = TC$ ,  $\frac{PT^2}{TC} = \frac{20^2}{4} = \frac{400}{4} = 100 = TK$ ; hence  $100 + 4 = 104 =$  the diameter, and  $52 =$  the radius of the generating circle.

$FC - CE = 52 - 16 = 36 = EF$  the central distance, and  
 $\sqrt{AF^2 - FE^2} = \sqrt{52^2 - 36^2} = 4\sqrt{13^2 - 9^2} = 4\sqrt{88} = 8\sqrt{22} = EA =$  half the length of the whole spindle.

$PR \times RW = 12 \times 40 = 480 =$  the area  $RPTYW$ , and  
the



the segment  $PCY$ , found by the table of circular segments, is 107.5185.

The sum of these two is 587.5185 = the generating area  $RPCYW$ .

Whence  $\overline{LL - \frac{1}{3}ll \times l - ac \times 2p} =$   
 $\overline{64 \times 22 - \frac{1}{3} \times 400 \times 20 - 36 \times 587.5185 \times 2 \times 3.14159 \&c. =}$   
 $\overline{4342.7879 \times 6.28318 \&c. = 27286.5411256} = \text{the solidity in inches.}$

Then  $27286.5411256 \div 282 = 96.7608$  ale gallons.

#### PROBLEM XXXII.

*To find the content of the segment of a circular spindle.*

#### RULE.

From 3 times half the length of the spindle subtract the height of the segment, and multiply the remainder into the square of the said height, from  $\frac{1}{3}$  of the product subtract the double of the product of the generating area and central distance; then the remainder drawn into 3.14159 &c. will produce the content.

That is,  $\overline{3EA - AR \times AR^2 - 2FE \times APR \times p} = \text{the solidity of } PAQ.$

#### EXAMPLE.

If the length of the whole spindle be 12, and its greatest diameter 9; what will be the content of a segment of it whose height is 1.

$\frac{AE^2}{EC} = \frac{36}{4\frac{1}{2}} = \frac{72}{9} = 8 = EK$ ,  $KE + EC = 8 + 4\frac{1}{2} = 12\frac{1}{2} = \text{the diameter}$ , and  $6\frac{1}{4} = \text{the radius of the generating circle}$ ;  $FC - CE = 6\frac{1}{4} - 4\frac{1}{2} = 1\frac{3}{4} = EF$  the central distance;  
 $\sqrt{FP^2 - PT^2} = \sqrt{6\frac{1}{4}^2 - 5^2} = \sqrt{\frac{5}{4}^2 \times 5^2 - 4^2} = 3\frac{3}{4} = TF$ ,  $CF - FT = 6\frac{1}{4} - 3\frac{3}{4} = 2\frac{1}{2} = TC$ .  
 By



By the table of circular segments the half segment  $PCT$  is  $8.736234375$ , to which adding  $RT = 5 \times 2 = 10$ , makes  $18.736234375 =$  the area  $RC$ .

And in like manner, the half segment  $ACE$  is  $19.886763281$ . Hence  $ACE - RC = 19.886763281 - 18.736234375 = 1.150528906 =$  the generating area  $APR$ .

Wherefore  $3EA - AR \times \frac{1}{3}AR^2 - 2FE \times ACE \times p =$   
 $17 \times \frac{1}{3} - 3 \frac{1}{2} \times 1.150528906 \times 3.14159 \&c. = 1.639815495$   
 $\times 3.14159 \&c. = 5.1516323 =$  the content required.

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## SECTION II.

*Of conic sections, and of the figures arising from or depending on them.*

### GENERAL DEFINITIONS.

**A** Conic section is the plane surface made by a plane cutting a cone.

According to the different places or positions in which the cone is cut by the plane, there arise five different sections, viz. an isosceles triangle, a circle, an ellipse, a parabola, and an hyperbola; the three last of which only are peculiarly called conic sections, because the two first, the triangle and circle, are figures whose properties are known prior to those of the cone.

If the cutting plane pass through the vertex of the cone, the section will be an isosceles triangle  $ATR$ , whose sides  $RA$ ,  $AT$ , are each equal to the  
3 H
flant



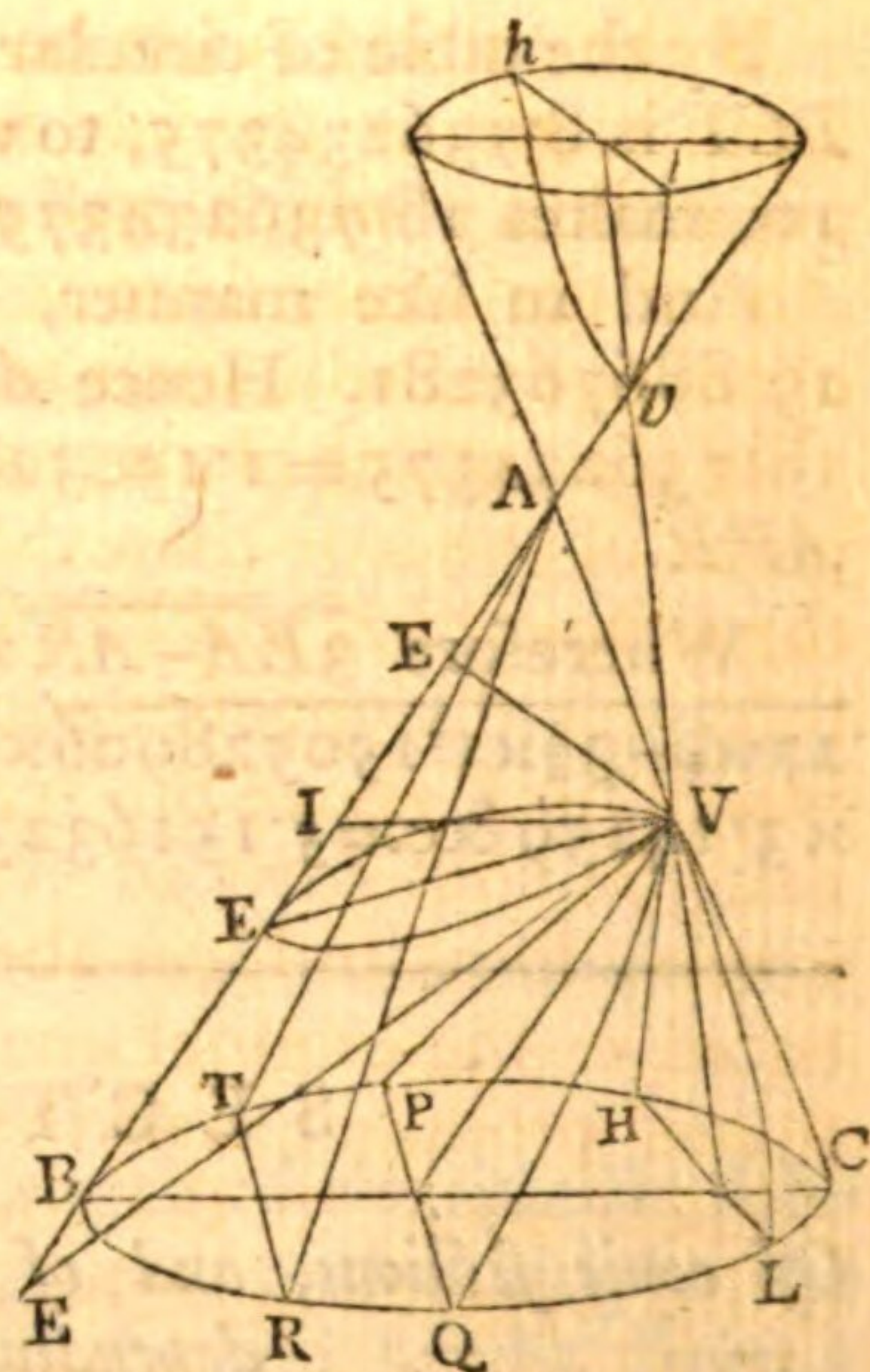
slant height of the cone, and base  $TR$  is a chord of the circular base of the cone; and the cone will be cut into two pyramids  $BTRA$ ,  $CTRA$ , each of the same height with the cone, and whose bases are the segments  $BTR$ ,  $CTR$ , of the circle  $BTCRB$ . If the plane pass through the axe, the cone and its base will be divided equally in two.

If the plane cut the side of the cone below the vertex, the section will be one of the other four figures according to the various inclinations of the plane to the sides of the cone.

If the cutting plane make no angle with the plane of the base, that is, if the plane be parallel to the base, the section  $VI$  will be a circle; for every section of a pyramid parallel to the base, is similar to the base; and will cut off a frustum,  $VCBI$ .

If the plane, cutting the side, make an angle with the plane of the base, the section will be either an ellipse, parabola, or hyperbola, and will cut off a hoof.

If the plane be inclined to the base in a less angle than the side is, the section  $VE$  will be an ellipse, and will, therefore, cut the other side of the cone, produced if necessary, somewhere below the vertex  $A$ .



If



If the cutting plane be inclined to the base in the same angle as the side is, the section  $VPQ$  will be a parabola; which will not meet the opposite side  $AB$  of the cone, since it is parallel to it, by reason of its equal inclination to the base.

If the cutting plane make a greater angle with the plane of the base than the side of the cone doth, the section  $VLH$  will be an hyperbola, and will, therefore, cut the opposite side of the cone, produced upwards, in  $V$  above the vertex. And if the side of the cone be continued in every direction through the vertex, so as to form above the vertex another cone equal to the original one; the section of the cutting plane, produced, and the upper cone will be an hyperbola equal to the lower one; and the two are called opposite hyperbolas.

The points  $V, E, v$ , where the cutting plane meets the upright triangular section  $BCA$ , are called the principal vertexes of the section.—Hence the ellipse and hyperbola have each two vertexes, the parabola only one: or the parabola may also be said to have two, by considering the one as infinitely distant from the other.

A line connecting the principal vertexes is the axe, axis, or transverse diameter of the section.

The center of the section is the middle of the axe.—Hence the center of a parabola is infinitely distant from the vertex.

A diameter is any right line drawn through the center and terminated on each side by the curve, the intersections of the diameter and curve being the vertexes of the diameter.—Hence every diameter of the ellipse and hyperbola have two vertexes; but of the



the parabola only one, unless we consider the other as infinitely distant. And hence, also, all the diameters of a parabola are parallel to each other and infinite.

If a tangent to the curve be drawn through the vertex of any diameter, and another diameter be drawn parallel to the tangent, those diameters are said to be conjugates the one to the other.—The axes, or principal conjugate diameters, are perpendicular to each other.

An abscissa is any part of a diameter terminated at the vertex.

An ordinate to any diameter is a line contained between the diameter and the curve, and is parallel to the conjugate diameter, or to the tangent at the vertex.—The ordinates to the axe are perpendicular to it. And in the ellipse and hyperbola every ordinate hath two abscissas, in the parabola only one.

The parameter of any diameter is a third proportional to that diameter and its conjugate.

The focus is the point of intersection of the axe and an ordinate, to it, which is equal to half the parameter of the axe.—The ellipse and hyperbola have each two focuses, the parabola only one.

A spindle is a body conceived to be generated by the revolution of the section about its double ordinate, and is denominated elliptic, parabolic, or hyperbolic, according as it is generated from the revolution of the ellipse, parabola, or hyperbola.

A spheroid is the body conceived to be generated by the revolution of an ellipse about its axe, and is denominated either prolate (oblong) or oblate according as the revolution is made about the transverse axe or its conjugate. The axe about which  
the

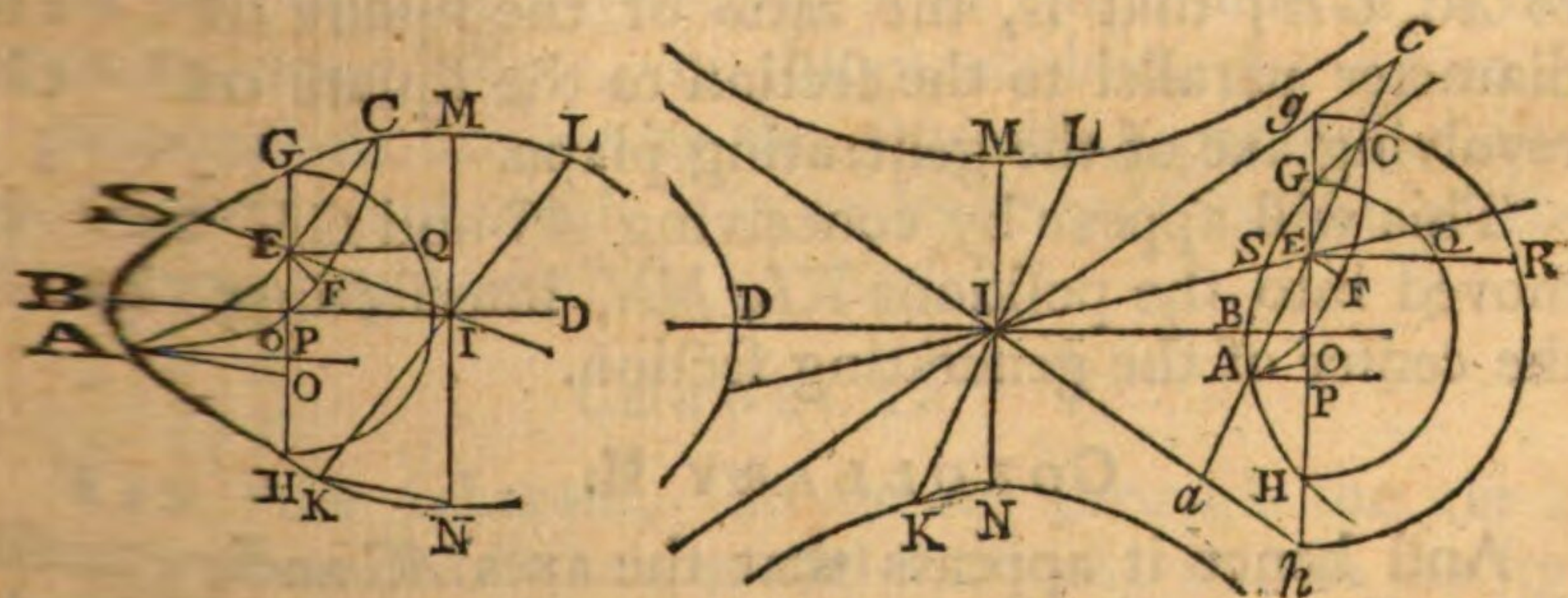


the revolution is made, is the fixed axe, the other is the revolving axe.

A conoid is a solid conceived to be generated from the revolution of the parabola or hyperbola about the transverse axe, and is accordingly denominated either parabolic or hyperbolic. The former of these is, also, called a paraboloid; the latter, an hyperboloid.

## PROPOSITION I.

If any solid formed by the revolution of a conic section about its axe, i. e. a spheroid, paraboloid, or hyperboloid, be cut by a plane in any position; the section will be a conic section, and all the parallel sections will be like and similar figures.



## DEMONSTRATION.

Let  $ABC$  be the generating section, or a section of the given solid through its axe  $BD$ , and perpendicular to the proposed section  $AFC$ , their common intersection being  $AC$ ; let  $GH$  be any other line meeting with the generating section in  $G$  and  $H$ , and cutting  $AC$  in  $E$ ; and erect  $EF$  perpendicular to the plane  $ABC$  and meeting with the proposed plane in  $F$ .

3 I

Then,



Then, if  $AC$  and  $GH$  be conceived to be moved continually parallel to themselves, will the rectangle  $AE \times EC$  be to the rectangle  $GE \times EH$  always in a constant ratio; but if  $GH$  be perpendicular to  $BD$ , the points  $G, F, H$  will be in the circumference of a circle whose diameter is  $GH$ , so that  $GE \times EH$  will be  $= EF^2$ ; wherefore  $AE \times EC$  will be to  $EF^2$  always in a constant ratio; consequently  $AFC$  is a conic section, and every section parallel to  $AFC$  will be of the same kind with and similar to it.  $\text{Q. E. D.}$

## COROLLARY I.

The above constant ratio, in which  $AE \times EC$  is to  $EF^2$ , is that of  $KI^2$  to  $IN^2$  the squares of the diameters of the generating section respectively parallel to  $AC, GH$ ; that is, the ratio of the square of the diameter parallel to the section to the square of the revolving axe of the generating plane.

This will appear by conceiving  $AC$  and  $GH$  to be moved into the positions  $KL, MN$ , intersecting in  $I$ , the center of the generating section.

## COROLLARY II.

And hence it appears that the axes  $AC$  and  $2EF$  (supposing  $E$  now to be the middle of  $AC$ ) of the section will be to each other as the diameter  $KL$  is to the diameter  $MN$  of the generating section.

## COROLLARY III.

If the section of the solid be made so as to return into itself, it will evidently be an ellipse. Which always happens in the spheroid, except when it is perpendicular to the axe, which position is also to be



be excepted in the other solids, the section being always then a circle: in the paraboloid the section is always an ellipse excepting when it is parallel to the axis: and in the hyperboloid the section is always an ellipse when its axis makes with the axis of the solid an angle greater than that made by the said axis of the solid and the asymptote of the generating hyperbola, the section being an hyperbola in all other cases, but when those angles are equal, and then it is a parabola.

## COROLLARY IV.

But if the section be parallel to the fixed axis  $BD$ , it will be of the same kind with and similar to the generating plane  $ABC$ ; that is, the section parallel to the axis, in a spheroid, is an ellipse similar to the generating ellipse; in the paraboloid, the section is a parabola similar to the generating parabola; and in an hyperboloid, it is an hyperbola similar to the generating hyperbola of the solid.

## COROLLARY V.

In the spheroid, the section through the axis is the greatest of the parallel sections; but in the hyperboloid it is the least; and in the paraboloid, all the sections parallel to the axis are equal to one another.

For the axis is the greatest parallel chord line in the ellipse, but the least in the opposite hyperbolas, and all the diameters are equal in a parabola.

## COROLLARY VI.

If the extremities of the diameters  $KL$ ,  $MN$ , be joined by the line  $KN$ , and  $AO$  be drawn parallel to  $KN$



$KN$  and meeting with  $GEH$  in  $O$ ,  $E$  being the middle of  $AC$ , or  $AE$  the semi-axe, and  $GH$  parallel to  $MN$ . Then  $EO$  will be equal to  $EF$  the other semi-axe of the section.

For, by similar triangles,  $KI:IN::AE:EO$ . Or upon  $GH$  as a diameter describe a circle meeting  $E\mathcal{Q}$  perpendicular to  $GH$  in  $\mathcal{Q}$ ; and it is evident that  $E\mathcal{Q}$  will be equal to the semi-diameter  $EF$ .

#### COROLLARY VII.

Draw  $AP$  parallel to the axe  $BD$  of the solid and meeting the perpendicular  $GH$  in  $P$ . And it will be evident that, in the spheroid, the semi-axe  $EF=EO$  will be greater than  $EP$ ; but in the hyperboloid, the semi-axe  $EF=EO$  of the elliptic section will be less than  $EP$ ; and in the paraboloid,  $EF=EO$  is always equal to  $EP$ .

#### SCHOLIUM.

The analogy of the sections of an hyperboloid to those of the cone are very remarkable, all the three conic sections being formed by cutting an hyperboloid in the same positions as the cone is cut.

Thus let an hyperbola and its asymptote be revolved together about the transverse axe, the former describing an hyperboloid and the latter a cone circumscribing it; then let them be supposed to be both cut by a plane in any position; and the two sections will be like, similar, and concentric figures: that is, if the plane cut both the sides of each, the sections will be concentric, similar ellipses; if the cutting plane be parallel to the asymptote or to the side of the cone, the sections will be parabolas; and  
in



in all other positions, the sections will be similar and concentric hyperbolas.

That the sections are like figures, appears from the foregoing corollaries; that they are concentric, will be evident when we consider that  $Cc$  is  $= Aa$ , producing  $AC$  both ways to meet with the asymptotes in  $a$  and  $c$ ; and that they are similar, or have their transverse and conjugate axes proportional to each other, will appear thus: Produce  $GH$  both ways to meet the asymptotes in  $g, b$ , and on the diameters  $GH, gb$ , describe the semi-circles  $GQH, gRb$ , meeting  $EQR$ , drawn perpendicular to  $GH$ , in  $Q$  and  $R$ ;  $EQ$  and  $ER$  being then evidently the semi conjugate axes, and  $EC, Ec$ , the semi-transverse axes of the sections: Now if  $GH$  and  $AC$  be conceived to be moved parallel to themselves,  $AE \times EC$  or  $CE^2$  will be to  $GE \times EH$  or  $EQ^2$  in a constant ratio, or  $\frac{CE}{EQ}$  will be a constant ratio; and since  $cE$  is as  $Eg$ , and  $aE$  as  $Eb$ ,  $aE \times Ec$  or  $cE^2$  will be to  $gE \times Eb$  or  $ER^2$  in a constant ratio, or  $\frac{cE}{ER}$  will be a constant ratio; but at an infinite distance from the vertex  $C$  and  $c$  coincide, or  $EC = Ec$ , as also  $EG = Eg$ , consequently  $EQ = ER$ , and then  $\frac{CE}{EQ}$  will be  $= \frac{cE}{ER}$ ; but as these ratios are constant, if they be equal to each other in one place, they must be always so, and consequently  $CE : Ec :: QE : ER$ .

And this analogy of the sections will not seem strange when we consider that a cone is a species of the hyperboloid, or a triangle a species of the hyperbola, whose axes are infinitely little.



## PROPOSITION II.

If  $SI$  be the femi-diameter belonging to the double ordinate  $AEC$  of the generating plane,  $AEC$  being the diameter of the section  $AFC$  conceived to be moved continually parallel to itself, and if  $x$  denote any part of the diameter  $SI$  intercepted by  $E$  the middle of  $AC$  and any given fixed point taken in  $SI$ ; then will the section  $AFC$  be always as  $A+Bx+Cxx$ ;  $A, B, C$ , being constant quantities;  $B$  in some cases affirmative, and in others negative;  $C$  being affirmative in the hyperbola, negative in the ellipse, and nothing in the parabola; and  $A$  may always be supposed to denote the distance of the given fixed point from the vertex  $S$ .

## DEMONSTRATION.

In any conic section,  $AC^2$  is as  $A+Bx+Cxx$ ; but all the parallel sections are like and similar figures, and similar plane figures are as the squares of their like dimensions; therefore the section  $AFC$  is as  $AC^2$ , that is, as  $A+Bx+Cxx$ . *Q. E. D.*

## COROLLARY.

If the given fixed point, where  $x$  begins, coincide with the vertex  $S$ , then will  $A$  be equal to nothing, and the section will be as  $Bx \pm Cxx$  or as  $x \pm Dxx$  in the hyperbola and ellipse, and as  $Bx$  or as  $x$  in the parabola.

SECT.



## SECTION III.

*Of the ellipse, and the figures generated from it.*

## PROBLEM I.

*To describe an ellipse; having the transverse and conjugate axes given.*

## RULE.\*

1. **D**RAW the transverse  $TR$  and its conjugate  $CO$  bisecting each other perpendicularly in the center  $c$ .

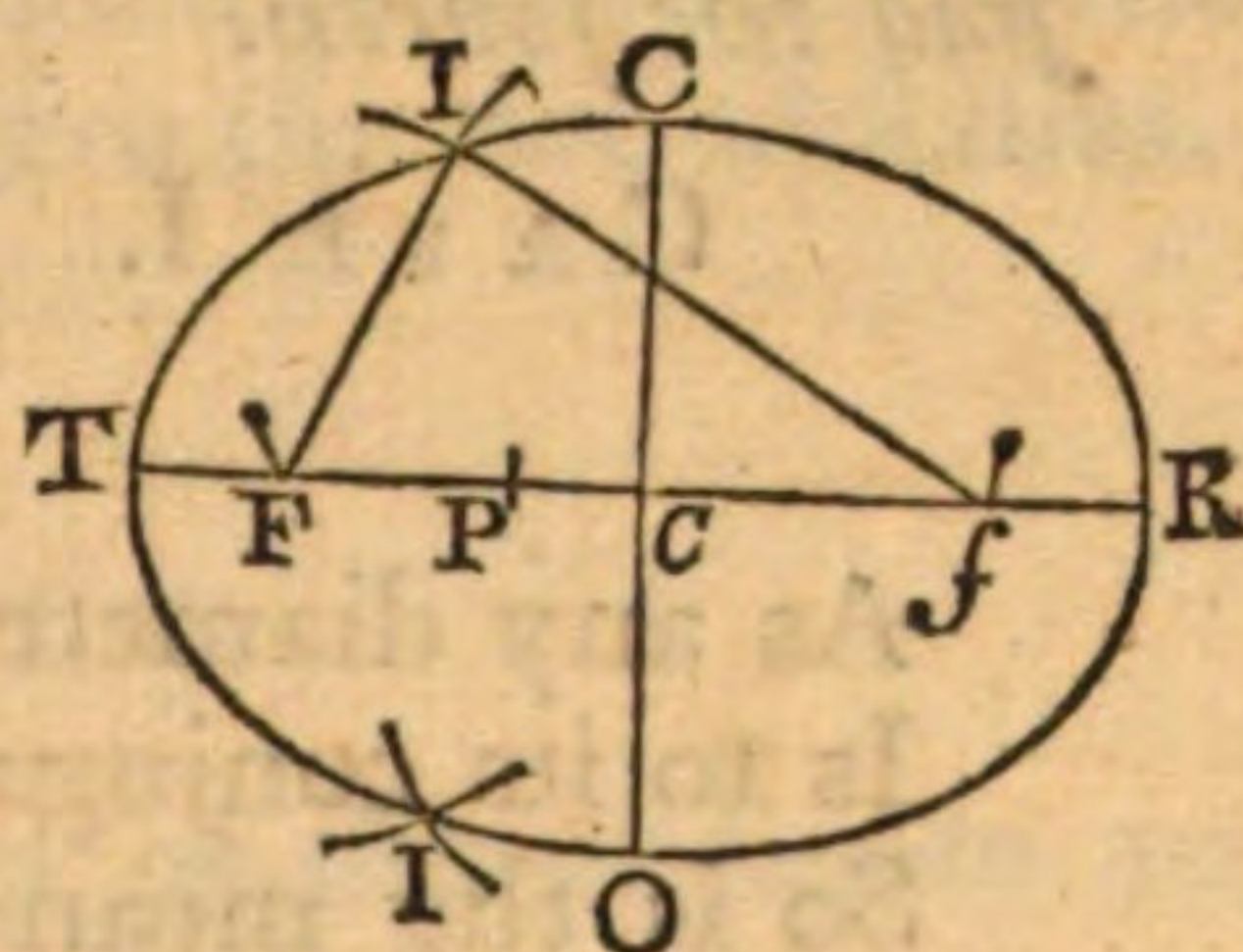
2. With the radius  $Tc$  and center  $C$ , describe an arc cutting  $TR$  in the points  $F, f$ , which will be the focuses of the ellipse.

3. Take any point  $P$  in the transverse, then with the radiuses  $TP, PR$ , and centers  $F, f$ , describe two arcs intersecting in  $I$ ; so will the point  $I$  be in the curve or circumference of the ellipse.

And thus, by assuming several points  $P$  in the transverse, there will be found as many points in the curve as we shall chuse; through all which let the curve line be drawn.

*The same performed from the same principle by means of a string.*

Having found the focuses  $F, f$ , take a thread of the length of the transverse; fasten its ends with pins



\* The truth of this construction will appear by observing that the transverse axis is equal to the sum of two lines drawn from the focuses to meet in any point in the curve.



pins in the focuses; then stretch the thread  $FIf$  and it will reach to  $I$  in the curve: and by moving a pencil round within the thread, keeping it always stretched, it will trace out the curve.

### PROBLEM II.

*In an ellipse, to find any two conjugate diameters, an ordinate to one of them, and its abscissa, one from another; viz. having any three of them given, to find the fourth.*

CASE I. \* *To find the ordinate.*

#### RULE.

As any diameter:

Is to its conjugate::

So is the mean proportional between the two abscissas or segments of the diameter:

To the ordinate.

That is, as  $d : c :: \sqrt{x \times d - x} : \frac{c}{d} \sqrt{x \times d - x} = y$ ; putting  $d$  for the diameter,  $c$  its conjugate, and  $x$  an abscissa to the ordinate  $y$ .

#### EXAMPLE.

If the transverse be 35, the conjugate 25, and the abscissa 7; what is the ordinate.

$$\begin{aligned} \text{Here } \frac{c}{d} \sqrt{x \times d - x} &= \frac{25}{35} \sqrt{7 \times 35 - 7} = \frac{5}{7} \sqrt{7 \times 28} = \frac{5}{7} \sqrt{7^2 \times 4} \\ &= \frac{5 \times 7 \times 2}{7} = 10 = \text{the ordinate.} \end{aligned}$$

#### CASE

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\* The values of the several quantities in the four cases of this problem are found from the general equation expressing the property of the curve, viz.  $dd : cc :: x \times d - x : yy$ .



CASE II. *To find the abscissas.*

## R U L E.

As the conjugate:

Is to the diameter::

So is the square root of the difference of the squares of the ordinate and semi-conjugate:

To the distance between the ordinate and center.

Then that distance being added to and subtracted from the semi-diameter, will give the two abscissas.

That is,  $\frac{1}{2}d \pm \frac{d}{c} \sqrt{\frac{1}{4}cc - yy} = x$ .

## E X A M P L E.

What are the two abscissas of the ordinate 10, the diameters being 35 and 25.

Here  $\frac{1}{2}d \pm \frac{d}{c} \sqrt{\frac{1}{4}cc - yy} = \frac{35}{2} \pm \frac{35}{25} \sqrt{\left(\frac{25}{2}\right)^2 - 10^2} = \frac{35}{2} \pm \frac{7}{2} \sqrt{5^2 - 4^2} = \frac{35 \pm 21}{2} = 28$  and 7 the two abscissas required.

CASE III. *To find the conjugate.*

## R U L E.

As the mean proportional between the two abscissas:

Is to the ordinate::

So is the diameter:

To its conjugate.

That is, as  $\sqrt{x \times d - x} : y :: d : \frac{dy}{\sqrt{dx - xx}} = c$ .

3 L

Ex-



## EXAMPLE.

What is the conjugate to the diameter 35; the abscissa to an ordinate of 10 being 7.

Here  $\frac{dy}{\sqrt{d-xx}} = \frac{35 \times 10}{\sqrt{28 \times 7}} = \frac{350}{14} = 25$  the conjugate required.

CASE IV. *To find the diameter.*

## RULE.

To or from the semi-conjugate, according as the less or greater abscissa is used, add or subtract the root of the difference of the squares of the ordinate and semi-conjugate: Then, as the square of the ordinate is to the product of the conjugate and abscissa, so is the sum or difference above found to the diameter.

That is,  $\frac{cx}{yy} \times \frac{1}{2}c \mp \sqrt{\frac{1}{4}cc - yy} = d$ .

## EXAMPLE.

If an ordinate and its less abscissa be 10 and 7; what is the diameter, supposing the conjugate to be 25.

Here  $d = \frac{cx}{yy} \times \frac{1}{2}c + \sqrt{\frac{1}{4}cc - yy} = \frac{25 \times 7}{10 \times 10} \times \frac{25}{2} + \sqrt{\left(\frac{25}{2}\right)^2 - 10^2} = \frac{7}{4} \times \frac{25+15}{2} = 7 \times 5 = 35$  the transverse.

## PROBLEM III.

*To find the length of the circumference of an ellipse.*

## RULE.



## \* RULE I.

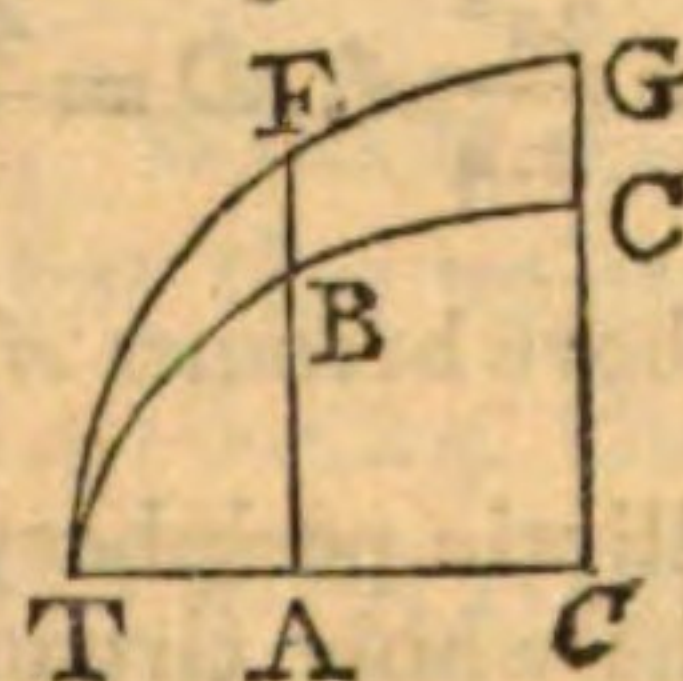
Multiply the circumference of the circumscribing circle into the series  $1 - \frac{d}{2.2} - \frac{3dd}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6} - \frac{3.3.5.5.7d^4}{2.2.4.4.6.6.8.8}$  &c. where  $d$  = the difference between an unit and the square of the quotient of the less axe divided by the greater, and the product will be the periphery of the ellipse sought.

Ex-

## \* DEMONSTRATION.

Let the semi-transverse axe  $Tc$  be called  $a$ ; the semi-conjugate  $cG$ ,  $c$ ; the ordinate  $AB$ ,  $y$ ; and its distance  $cA$  from the center,  $x$ ; also the arc  $CB$ ,  $z$ .

Then since, by the nature of the curve,  $y$  is =  $\frac{c}{a} \sqrt{aa - xx}$ ,  $y$  will be =  $\frac{-cxx}{a\sqrt{aa - xx}}$ ; and consequent-



ly  $z = \int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{x \sqrt{a^2 - x^2}}{a^2} + \frac{a^2 - c^2}{a^2} \int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{x \sqrt{aa - xx}}{a^2} + \frac{a^2 - c^2}{a^2} \int \frac{dx}{\sqrt{aa - xx}}$  (by writing

$d$  for  $\frac{aa - cc}{aa}$  or  $1 - \frac{cc}{aa}$ ) =  $\frac{ax \sqrt{1 - \frac{dxx}{aa}}}{\sqrt{aa - xx}} = \frac{ax}{\sqrt{aa - xx}} \times 1 - \frac{dx^2}{2a^2} - \frac{d^2 x^4}{2.4a^4}$

$- \frac{3d^3 x^6}{2.4.6a^6}$  &c. by throwing the numerator into a series. But the fluent

of  $\frac{ax}{\sqrt{aa - xx}}$  is = the corresponding circular arc  $GF$  described with the center  $c$  and radius  $cT$ , which call  $A$ : then, per page 66 *Cotesii Harmonia Mensurarum*, the fluent comes out  $A - \frac{d}{2a^2} B - \frac{d^2}{2.4a^4} C - \frac{3d^3}{2.4.6a^6} D$

&c. wherein  $B = \frac{aaA - x\sqrt{aa - xx}}{2}$ ,  $C = \frac{3aaB - x^3\sqrt{aa - xx}}{4}$ ,  $D =$

$\frac{5aaC - x^5\sqrt{aa - xx}}{6}$ , &c. for the length of the arc  $CB$ .

But



## EXAMPLE.

Required the periphery of an ellipse whose axes are 24 and 18.

$$\text{Here } 1 - \frac{18^2}{24^2} = 1 - \frac{3^2}{4^2} = \frac{7}{16} = .4375 = d.$$

Then

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But when  $A$  arrives at  $T$ ,  $x$  is  $= a$ , and  $\sqrt{aa-xx} = 0$ ; hence the values of the quantities  $B, C, D$ , &c. become barely  $B = \frac{aa}{2} A$ ,  $C = \frac{3aa}{4} B = \frac{3a^4}{2.4} A$ ,  $D = \frac{5aa}{6} C = \frac{3.5a^6}{2.4.6} A$ , &c. And, consequently, the series above becomes  $Ax : 1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$  &c. for  $\frac{1}{4}$  of the elliptic periphery, where  $A$  is  $\frac{1}{4}$  of the periphery of the circle, or  $=$  the whole circumference of the ellipse when  $A$  is that of the circle. *Q. E. D.*

## COROLLARY I.

Hence the peripheries of similar ellipses are to each other as those of their circumscribed circles, or as any two similar diameters of those ellipses.

## COROLLARY II.

If  $a$  be the less semi-axe, and  $c$  the greater, the fluxion of the arc will be  $\dot{z} = \dot{x} \sqrt{\frac{aa+dx}{aa-xx}}$ ; and if  $a=1$ , and  $c=\sqrt{2}$ , the same fluxion

$$\dot{z} \text{ will be } = \dot{x} \sqrt{\frac{1+xx}{1-xx}} = \dot{x} \frac{1+xx}{\sqrt{1-x^4}} = \frac{\dot{x}}{\sqrt{1-x^4}} + \frac{x^2 \dot{x}}{\sqrt{1-x^4}}. \text{—And if we}$$

write  $v$  for  $xx$ , we shall have  $\dot{z} = \frac{\dot{v}}{2\sqrt{v}} \sqrt{\frac{1+v}{1-v}}$ , as Mr *Landen* found

in page 142 of his *Math. Lucubr.*—Or, by writing  $w^n$  for  $x$ , we shall have universally  $\dot{z} = nw^{n-1} \dot{w} \sqrt{\frac{1+w^{2n}}{1-w^{2n}}}$ . Which theorems are of use in determining some fluents.



Then the 2d term  $A = \frac{d}{4} = .10938$

the 3d —  $B = \frac{3d}{4 \cdot 4} A = .00897$

the 4th —  $C = \frac{3 \cdot 5d}{6 \cdot 6} B = .00164$

the 5th —  $D = \frac{5 \cdot 7d}{8 \cdot 8} C = .00039$

the 6th —  $E = \frac{7 \cdot 9d}{10 \cdot 10} D = .00011$

the 7th —  $F = \frac{9 \cdot 11d}{12 \cdot 12} E = .00003$

the 8th —  $G = \frac{11 \cdot 13d}{14 \cdot 14} F = .00001$

their sum is .12053, which taken from the first term 1, leaves .87947 for the value of the series; and being drawn into  $3 \cdot 1416 \times 24$  the periphery of the circumscribing circle, gives  $66 \cdot 31056$  for that of the ellipse required.

#### \* RULE II.

Multiply  $\frac{1}{2}$  the sum of the two axes by  $3 \cdot 1416$ , and the product will be the circumference nearly.

That is,  $\frac{t+c}{2} \times p =$  the circumference nearly, putting  $t$  for the transverse,  $c$  for its conjugate axe, and  $p$  for  $3 \cdot 1416$ .

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\* It will be evident that this rule will not differ much from the truth, if it be considered that this arithmetical mean between the axes exceeds their geometrical mean, and that the geometrical mean (as will be proved hereafter) is the diameter of a circle equal in area to the ellipse, which circle is of a less ambit than the ellipse, or any other figure, of the same area.



## EXAMPLE.

Let there be here taken the same example as before, in which the axes are 24 and 18.

Then  $\frac{24+18}{2} \times 3.1416 = 21 \times 3.1416 = 65.9736 =$  the circumference nearly.

## \* RULE III.

Multiply the square root of  $\frac{1}{2}$  the sum of the squares of the two axes by 3.1416, and the product will be the circumference nearly.

That is,  $p \sqrt{\frac{tt+cc}{2}} =$  the circumference nearly.

## EXAMPLE.

Taking again the same example.

We have  $\sqrt{\frac{24^2+18^2}{2}} \times 3.1416 = 15\sqrt{2} \times 3.1416 = 21.2132 \times 3.1416 = 66.6433 =$  the circumference nearly.

## RULE

\* Call the infinite series in rule I,  $S$ ; viz.  $S = 1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6} \&c.$  then since  $\sqrt{1-\frac{1}{2}d}$  is  $= 1 - \frac{d}{2.2} - \frac{d^2}{2^3.4} - \frac{3d^3}{2^4.4.6} \&c.$  we shall have  $S - \sqrt{1-\frac{1}{2}d} = -\frac{d^2}{64} - \frac{3d^3}{256} \&c.$  Wherefore rejecting this last remaining series, on account of its smallness, we have  $S = \sqrt{1-\frac{1}{2}d}$  nearly, and consequently the circumference of the ellipse  $C = c\sqrt{1-\frac{1}{2}d}$  nearly,  $c$  being the circumference of the circumscribed circle. Or  $C = pt \sqrt{1-\frac{1}{2}d} = pt \sqrt{1-\frac{1}{2} + \frac{cc}{2tt}} = pt \sqrt{\frac{1}{2} + \frac{cc}{2tt}} = p \sqrt{\frac{tt+cc}{2}}$ , which is the 3d rule,  $p$  being  $= 3.14159 \&c.$

Again,



## RULE IV.\*

Find the square root of  $\frac{1}{2}$  the sum of the squares of the axes as in the last rule, and call it  $A$ .

To 3 times the greater axe add its parameter, divide the sum by 4, and call the quotient  $B$ .

Then multiply the difference between  $B$  and 3 times  $A$  by  $1.5708 (= \frac{3.1416}{2})$ , and the product will be the periphery nearer than by the last rule.

That is,  $\frac{1}{2}p \times 3\sqrt{\frac{tt+cc}{2}} - \frac{3t+P}{4} =$  the periphery,  $P$  being the parameter.

## EXAMPLE.

Taking the same example as in the former rules,

We have  $A = \sqrt{\frac{24^2 + 18^2}{2}} = 15\sqrt{2} = 21.2132$ .

As  $24:18::18:13\frac{1}{2}$  = the parameter, by the definition.

And  $\frac{3 \times 24 + 13\frac{1}{2}}{4} = 21\frac{3}{8} = 21.375 = B$ .

Then  $3A - B \times 1.5708 = 21.2132 \times 3 - 21.375 \times 1.5708 = 42.2646 \times 1.5708 = 66.3892 =$  the circumference nearly.

## RULE

\* Again,  $S - 1 + \frac{1}{4}d = -\frac{3d^2}{64} - \frac{5d^3}{256} \&c.$  but  $3S - 3\sqrt{1 - \frac{1}{2}d} = -\frac{3d^2}{64} - \frac{9d^3}{256} \&c.$  wherefore  $2S - 3\sqrt{1 - \frac{1}{2}d} + 1 - \frac{1}{4}d = -\frac{d^3}{64} \&c.$  hence  $S = \frac{3\sqrt{1 - \frac{1}{2}d} - 1 + \frac{1}{4}d}{2}$  nearly, and consequently  $C = \frac{1}{2}pt \times 3\sqrt{1 - \frac{1}{2}d} - \frac{4-d}{4}$

$= \frac{1}{2}p \times 3\sqrt{\frac{tt+cc}{2}} - \frac{3tt+cc}{4t} = \frac{1}{2}p \times 3\sqrt{\frac{tt+cc}{2}} - \frac{3t+P}{4}$  nearly, where  $P$  is the parameter of the axe. And this is the 4th rule.

Farther,



## RULE V.\*

Find  $A$  and  $B$  as in the last rule.

Divide the difference of the squares of the axes by the square of the greater axe, multiply the square of the quotient by  $\frac{1}{16}$  of the greater axe, and call the result  $C$ .

Then from 5 times  $A$  take 3 times  $B$ , to the remainder add  $C$ , multiply the sum by 1.5708, and the product will give the circumference of the ellipse still nearer.

Or,

From 5 times  $A$  subtract  $\frac{1}{16}$  of the sum of 35 times the greater axe and 14 times its parameter, to the remainder add  $\frac{1}{16}$  of the product arising from the multiplication of the said parameter by the quotient of the square of the less axe divided by that of the greater, and multiply the sum by 1.5708.

That is,  $\frac{1}{2}p \times 5 \sqrt{\frac{tt+cc}{2}} - 3 \times \frac{3t+p}{4} + \frac{tt-cc}{16} \times \frac{t}{16}$ , or  $\frac{1}{2}p \times$   
 $5 \sqrt{\frac{tt+cc}{2}} - \frac{35t+14p}{16} + \frac{pcc}{16tt} =$  the periphery very nearly.

EX-

\* Farther,  $S - 1 + \frac{d}{4} + \frac{3d^2}{64} = -\frac{5d^3}{256}$  &c. or  $\frac{4}{5}S - \frac{4}{5} + \frac{d}{5} + \frac{3d^2}{80} = -\frac{d^3}{64}$ , which taken from the last approximation, and reduced, gives

$S = \frac{1}{2} \times 5 \sqrt{1 - \frac{1}{2}d} - 3 \times \frac{4-d}{4} + \frac{dd}{16}$ ; and consequently  $C = \frac{1}{2}pt \times$

$5 \sqrt{1 - \frac{1}{2}d} - 3 \times \frac{4-d}{4} + \frac{dd}{16} = \frac{1}{2}p \times 5 \sqrt{\frac{tt+cc}{2}} - 3 \times \frac{3t+p}{4} + \frac{tt-cc}{16} \times \frac{t}{16}$

$= \frac{1}{2}p \times 5 \sqrt{\frac{tt+cc}{2}} - \frac{35t+14p}{16} + \frac{pcc}{16tt}$ . Which is the 5th rule. And

thus we may proceed to any degree of accuracy required.



## EXAMPLE.

Taking again the same example as before, we have

$$A = 21.2132,$$

$$B = 21.375,$$

$$C = \frac{24^2 - 18^2}{24} \times \frac{24}{16} = \frac{4^2 - 3^2}{4^2} \times \frac{3}{2} = \frac{7^2 \times 3}{16^2 \times 2} = .2871.$$

Then  $5A - 3B + C \times 1.5708 = 106.066 - 64.125 + .2871 \times 1.5708 = 42.2281 \times 1.5708 = 66.3319 =$  the circumference very near.

Or, to use the latter part of the rule,

$$5A - \frac{35 \times 24 - 14 \times 13\frac{1}{2}}{16} + \frac{13\frac{1}{2} \times 18^2}{16 \times 24^2} \times 1.5708 = 42.2281 \times 1.5078 = 66.3319 \text{ the same as before.}$$

## RULE VI.

Take  $\frac{1}{2}$  the sum of the quantities in the second and third rules, and it will give the circumference very nearly.\*

That is,  $\frac{1}{2}p \times \frac{t+c}{2} + \sqrt{\frac{tt+cc}{2}} =$  the periphery.

## EXAMPLE.

Taking, still, the same example,

The number found by the 2d rule is 65.9736,

And by the 3d rule is 66.6433,

their sum is 132.6169, the half of which is 66.3084 for the circumference, and is nearer to the true number found by rule 1 than any of the other approximations.

3 N

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\* For the one being nearly as much too great as the other is too little,  $\frac{1}{2}$  their sum must be near the truth.

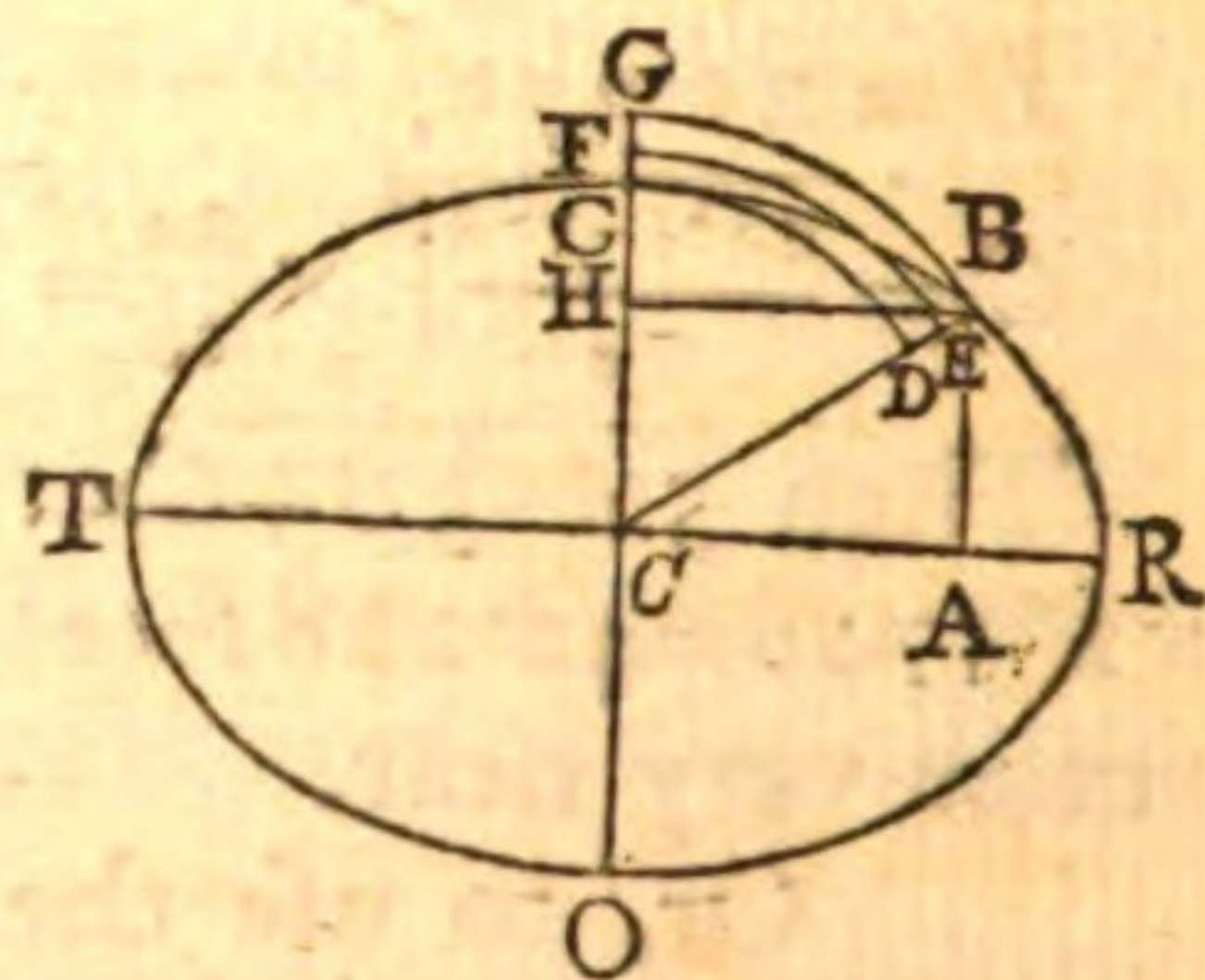


## PROBLEM IV.

*To find the length of any arc of an ellipse.*

## \* RULE I.

If  $a$  denote the semi-transverse  $Tc$ ,  $c$  the semi-conjugate  $cC$ , and  $x$  the distance  $cA$  of the ordinate  $AB$  from the center, then will the arc  $CB$  be

$$= x \times \left( 1 + \frac{c^2}{6a^4} x^2 + \frac{4a^2c^2 - c^4}{40a^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}} x^6 \right) \&c.$$


And the series will be the same if  $x$  be considered as an ordinate to either axis  $2c$ , the other axis being  $2a$ , the arc beginning at the vertex of the axis  $2c$  and terminated by the ordinate  $x$ .

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## \* DEMONSTRATION.

By proceeding as in the investigation of problem 3, we have the fluxion of the curve  $z = x \sqrt{\frac{aa - dxx}{aa - xx}} = x \times \left( 1 + \frac{c^2}{2a^4} x^2 + \frac{4a^2c^2 - c^4}{8a^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{16a^{12}} x^6 \right) \&c.$  restoring  $c$  and exterminating  $d$ ; and the fluent is  $z = x \times \left( 1 + \frac{c^2}{6a^4} x^2 + \frac{4a^2c^2 - c^4}{40a^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}} x^6 \right) \&c.$

## COROLLARY I.

When  $x$  becomes  $= a$ , the rule becomes  $ax : 1 + \frac{c^2}{6a^2} + \frac{4a^2c^2 - c^4}{40a^4} + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^6} \&c.$  for one-fourth of the periphery of the ellipse.

Q. E. D.

## COROLLARY II.

By supposing  $a = c$ , the ellipse will become a circle, and the general series for the elliptic arc will become for the circular arc  $x \times \left( 1 + \frac{x^2}{6a^2} + \frac{3x^4}{40a^4} + \frac{5x^6}{112a^6} \right) \&c. = x \times \left( 1 + \frac{x^2}{2 \cdot 3a^2} + \frac{3x^4}{2 \cdot 4 \cdot 5a^4} + \frac{3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7a^6} \right) \&c.$  the same as was determined at page 88.



## EXAMPLE.

If the two axes be 24 and 18, and the distance  $cA$  be 3; Required the length of the arc  $BC$ .

Here  $a = 12$ ,  $c = 9$ , and  $x = 3$ .

Then the 1st term = — — 1.00000

2d term  $\frac{c^2}{6a^4} x^2 = A =$  — — 0.00586

3d term  $B = 3 \times \frac{4a^2 - c^2}{20a^4} x^2 A =$  — — 0.00019

4th term  $C = \frac{5}{14a^4} \times \frac{8a^4 - 4a^2c^2 + c^4}{4a^2 - c^2} x^2 B =$  — — 0.00001

the sum is 1.00606

And  $3 \times 1.00606 = 3.01818 =$  the arc required.

## \* RULE II.

Find the length of the circular arc  $FE$  intercepted by  $Cc$ ,  $cB$ , and whose radius is  $\frac{1}{2}$  the sum of the lines  $Cc$ ,  $cB$ ; and it will be equal to the elliptic arc  $BC$  nearly.

## EXAMPLE.

Taking here the same example,

We have  $RA = 9$ , and  $AT = 15$ .

Hence  $Tc^2 : cC^2 :: TA \times AR : AB^2 = \frac{9^2 \times 9 \times 15}{12 \times 12} = \frac{9 \times 9 \times 15}{16}$ , and

$$BC = \sqrt{cA^2 + AB^2} = \sqrt{9 + \frac{9 \times 9 \times 15}{16}} = \frac{3}{4} \sqrt{151} = 9.21616.$$

Then

\* For the circular arc  $FE$  is equal to the arithmetical mean between the circular arcs  $BG$ ,  $CD$ , described with the radiuses  $Bc$ ,  $cC$ ; to which it is evident the elliptic arc must be nearly equal, and so much the nearer, the less the arc itself is.—Rule 2 of the last problem is a particular case of this general rule.

This rule will also be the more exact the nearer the axes of the ellipse approach towards an equality.



Then  $\frac{Cc+CB}{2} = \frac{18.21616}{2} = 9.10808 =$  the radius of the circular arc  $EF$ .

But  $\frac{HB}{Bc} = \frac{3}{9.21616} = .325515 =$  the sine of the angle  $CcB$  or arc  $EF$  to the radius 1, to which belong  $18.9968$  degrees.

Wherefore, by rule 1 prob. 6 sect. 1 part 2,  $.01745 \times 18.9968 \times 9.10808 = 3.0192 =$  the circular arc  $EF$ , or elliptic arc  $BC$  nearly.

\* R U L E III.

Divide the difference of the squares of the semi-axes by the square of the greater, and call the quotient  $q$ .

Then divide the difference between the square of the greater semi-axe and the product of  $\frac{1}{3}$  of the square of the distance  $cA$  of the ordinate from the center

mul-

\* DEMONSTRATION.

By the first rule the arc  $A$  is equal to  $xx : 1 + \frac{c^2}{6a^4} x^2 + \frac{4a^2c^2 - c^4}{40a^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}} x^6 \&c.$  but in the investigation of that rule it appears that  $\sqrt{\frac{aa-dxx}{aa-xx}}$  is  $= 1 + \frac{c^2}{2a^4} x^2 + \frac{4a^2c^2 - c^4}{8a^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{16a^{12}} \&c.$  hence it is evident that  $x \sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}} x : 1 + \frac{c^2}{6a^4} x^2 + \frac{4a^2c^2 - c^4}{72a^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{432a^{12}} x^6 \&c.$  and consequently  $A = x \sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}}$  nearly.

Q.E.D.

COROLLARY.

When the two axes are equal to each other, the ellipse will become a circle,  $d$  will be  $= 0$ , and the rule will become  $x \sqrt{\frac{aa}{aa-\frac{1}{3}xx}}$  for the circular arc.



multiplied into  $q$ , by the difference between the square of the said greater semi-axe and  $\frac{1}{3}$  of the square of the said distance; and the root of the quotient multiplied into the said distance will be the arc nearly.

That is,  $x\sqrt{\frac{aa - \frac{1}{3}qxx}{aa - \frac{1}{3}xx}}$  = the arc, where  $a = Tc$  the greater semi-axe,  $x = cA$ , and  $q = \frac{aa - cc}{aa} = 1 - \frac{cc}{aa}$ ,  $c$  being the less semi-axe  $cC$ .

## EXAMPLE.

Taking again the same example,

We have  $1 - \frac{cc}{aa} = 1 - \frac{9}{12^2} = 1 - \frac{3^2}{4^2} = \frac{7}{16} = q$ .

And  $x\sqrt{\frac{aa - \frac{1}{3}qxx}{aa - \frac{1}{3}xx}} = 3\sqrt{\frac{12^2 - \frac{1}{3} \times \frac{7}{16} \times 3^2}{12^2 - \frac{1}{3} \times 3^2}} = 3.017898 =$  the arc required nearly.

## \* RULE IV.

Call the quantity found by the last rule  $B$ .

3 O

Mul-

## \* DEMONSTRATION.

For since  $A = xx : 1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$   
 &c. and  $B = xx : 1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{72a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{432a^{12}}x^6$  &c.  
 we have  $A - B = xx : \frac{4a^2c^2 - c^4}{90a^8}x^4 + \frac{40a^4c^2 - 20a^2c^4 + 5c^6}{756a^{12}}x^6$  &c. but  
 $\frac{4}{3} \times B = xx : 1 + \frac{c^2}{6a^4}x^2 = xx : \frac{4a^2c^2 - c^4}{90a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{540a^{12}}x^6$  &c.  
 therefore  $A - B - \frac{4}{3} \times B = xx : 1 + \frac{c^2x^2}{6a^4} = xx : \frac{8a^4c^2 - 4a^2c^4 + c^6}{210a^{12}}x^6$  &c. and,  
 rejecting the series,  $A = \frac{1}{3} \times 9B - 4xx : 1 + \frac{c^2x^2}{6a^4} =$   
 $\frac{1}{3}xx : 9\sqrt{\frac{aa - \frac{1}{3}dxx}{aa - \frac{1}{3}xx}} - 4xx : 1 + \frac{c^2x^2}{6a^4}$  nearly. Q.E.D.

Co.



Multiply the distance  $cA$  into the less semi-axe, divide the product by the square of the greater, to  $\frac{1}{2}$  of the square of the quotient add 1, multiply the sum into the distance  $cA$ , and call the product  $C$ .

Then from 9 times  $B$  take 4 times  $C$ , divide the remainder by 5, and the quotient will be the arc very near.

$$\text{That is, } A = \frac{9B - 4C}{5} = \frac{1}{5} \times 9x \sqrt{\frac{aa - \frac{1}{3}qxx}{aa - \frac{1}{3}xx}} - 4x \times 1 + \frac{c^2 x^2}{6a^4}.$$

## E X A M P L E.

Taking still the same example,  
The Quantity found by the last rule is  $3.017898 = B$ .

$$\text{And } x \times 1 + \frac{c^2 x^2}{6a^4} = x \times 1 + \frac{9^2 \times 3^2}{6 \times 12^4} = 3 \times 1 + \frac{3}{2 \times 4^4} = 3.017578 = C.$$

$$\text{Then } \frac{9B - 4C}{5} = \frac{27.161082 - 12.070212}{5} = \frac{15.09077}{5} = 3.018154 = \text{the arc very nearly.}$$

## R U L E V.

As the sum of 15 times the parameter of the axe, at whose vertex the arc begins, and a third proportional to the said axe, the abscissa, and the difference between 9 times the said axe and 21 times its parameter; is to the sum of 15 times the parameter and a third proportional to the axe, the abscissa, and the dif-

## C O R O L L A R Y.

When  $a$  is  $= c$ , then the rule becomes  $\frac{1}{5}x \times 9 \sqrt{\frac{aa}{aa - \frac{1}{3}xx}} - 4x \times \frac{6aa + xx}{6aa}$   
for the circular arc,



difference between 19 times the axe and 21 times its parameter; so is the ordinate, to the length of the arc, very nearly.\*

That is,  $\frac{15p + \frac{19C-21p}{C} \times y}{15p + \frac{9C-21p}{C} \times y} \times x = \text{the arc } CB$ , where

$C = \text{the whole axe } CO \text{ at whose vertex the arc commences,}$

\* DEMONSTRATION.

Since  $y$  is  $= c - \frac{c\sqrt{aa-xx}}{a} = \frac{cx^2}{2a^2} + \frac{cx^4}{8a^4} \&c.$  it is evident that a fraction of this form  $x \times \frac{A+B+1.y}{A+By}$ , when expanded in a series, will produce a series of the same form with that expressing the length of the arc in the first rule, and which being put equal to it will afford equations for determining the values of  $A$  and  $B$ .

Thus  $x \times \frac{A+B+1.y}{A+By}$  is  $= x \times \frac{A+B+1 \times : \frac{cx^2}{2a^2} + \frac{cx^4}{8a^4} \&c.}{A+B \times : \frac{cx^2}{2a^2} + \frac{cx^4}{8a^4} \&c.} = x \times : 1 +$

$\frac{cx^2}{2Aa^2} + \frac{Ac-2Bc^2}{8A^2a^4} x^4 \&c.$  which let be put  $= x \times : 1 + \frac{c^2x^2}{6a^4} + \frac{4a^2c^2-c^4}{8a^8} x^4$

$\&c.$  Then, by equating the corresponding terms, we obtain  $\frac{c}{2Aa^2}$

$= \frac{c^2}{6a^4}$ , or  $A = \frac{3a^2}{c} = \frac{3}{2}p$ ; and  $\frac{Ac-2Bc^2}{8A^2a^4} = \frac{4a^2c^2-c^4}{8a^8}$ , hence  $B =$

$\frac{9c^2-21a^2}{10cc} = \frac{9c - \frac{21aa}{c}}{10c} = \frac{9C-21p}{10C}$ , putting  $C = 2c$ , and  $p = \text{its parameter } \frac{2aa}{c}$ .

Consequently

$x \times \frac{A+B+1.y}{A+By}$  is  $= x \times \frac{\frac{3}{2}p + \frac{19C-21p}{10C}y}{\frac{3}{2}p + \frac{9C-21p}{10C}y} = x \times \frac{15p + \frac{19C-21p}{C}y}{15p + \frac{9C-21p}{C}y} \cdot Q.E.D.$

Co-



mences,  $p$  = its parameter =  $\frac{TR^2}{CO}$ ,  $y$  = the abscissa  $CH$ , and  $x$  = the ordinate  $HB$ .

## EXAMPLE.

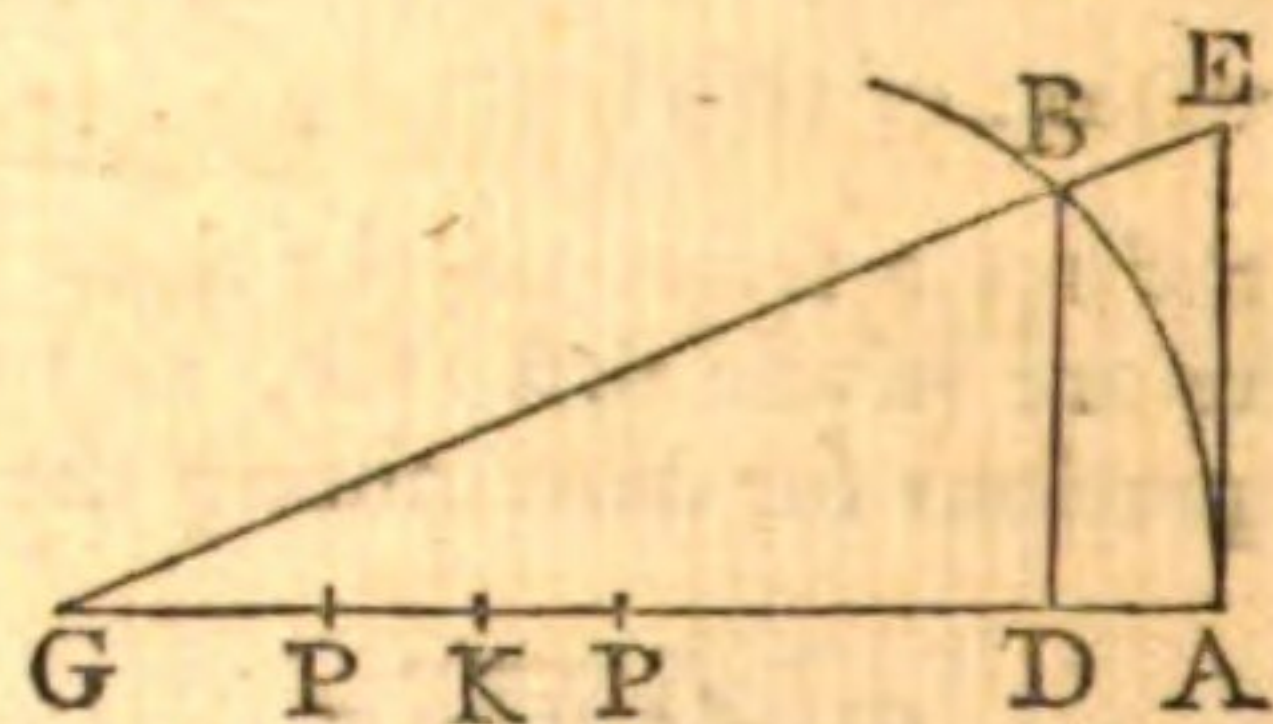
Taking, still, the same example, in which  $c=9$ ,  $a=12$ ,  $C=18$ ,  $p=\frac{12 \times 24}{9}=32$ , and  $x=3$ ; hence  $y = c - \frac{c\sqrt{aa-xx}}{a} = 9 - \frac{9\sqrt{12^2-3^2}}{12} = 9 - \frac{9\sqrt{15}}{4} = 9 \times \frac{4-\sqrt{15}}{4} = .2857874715$ .

Then

## COROLLARY I.

Hence, in the ellipse  $AB$  whose vertex is  $A$ , either axe  $AK$ , and latus rectum  $AP$ ; if there be drawn the ordinate  $BD$ , and  $PG$  be taken =  $\frac{1}{2}AP$

+  $\frac{19AK-21AP}{10AK} \times AD$ , and there be drawn the right line  $GBE$  meeting the tangent  $AE$  in  $E$ ; then will  $AE$  be = to the arc  $AB$ , very nearly.



For, by similar triangles,  $GD : DB :: GA : AE = \frac{AG \times BD}{DG} =$

$$\frac{AP+PG}{AP+PG-AD} \times DB = (\text{by the const.}) \frac{\frac{1}{2}AP + \frac{19AK-21AP}{10AK} \times AD}{\frac{1}{2}AP + \frac{9AK-21AP}{10AK} \times AD} \times DB,$$

which is the arc, nearly, by the above demonstration.

And this proves the truth of the construction given by *Sir I. Newton*, in his letter of June 13, 1676, to Mr *Oldenburg*. But Mr *Jones*, in publishing this letter, hath printed  $AP$  instead of  $AD$ .

## COROLLARY II.

When the ellipse becomes a circle, then  $AK=AP$ , and the above rule will become  $\frac{15AP-2AD}{15AP-12AD} \times DB$  for the length of the circular arc whose diameter is  $AP$ , versed sine  $AD$ , and right sine  $DB$ . And this is nearer the truth than any of the other approximations before given for the circular arc, except that in rule 6 for that purpose, for the example in that rule calculated by this approximation, gives 6.1170598 for the length of the arc.



$$\begin{aligned}
 \text{Then } x \times \frac{15p + \frac{19C-21p}{C}y}{15p + \frac{9C-21p}{C}y} &= 3 \times \frac{15 \times 32 + \frac{19 \times 18 - 21 \times 32}{18}y}{15 \times 32 + \frac{9 \times 18 - 20 \times 32}{18}y} \\
 &= 3 \times \frac{18 \times 15 \times 32 + 19 \times 18 - 21 \times 32 \times y}{18 \times 15 \times 32 + 9 \times 18 - 21 \times 32 \times y} = 3 \times \frac{3 \times 15 \times 32 + 19 \times 3 - 7 \times 16 \times 2857874715}{3 \times 15 \times 32 + 9 \times 3 - 7 \times 16 \times 2857874715} \\
 &= 3 \times 1.006056 = 3.018168 = \text{the length of the arc very nearly.}
 \end{aligned}$$

## PROBLEM V.

\* To find the area of an ellipse.

## RULE I.

Multiply continually together the two axes and the number .7854 for the area of the ellipse.

That is,  $ntc$  = the area, putting  $t$  = the transverse,  $c$  = the conjugate, and  $n = .7854$ . By corollary 4.

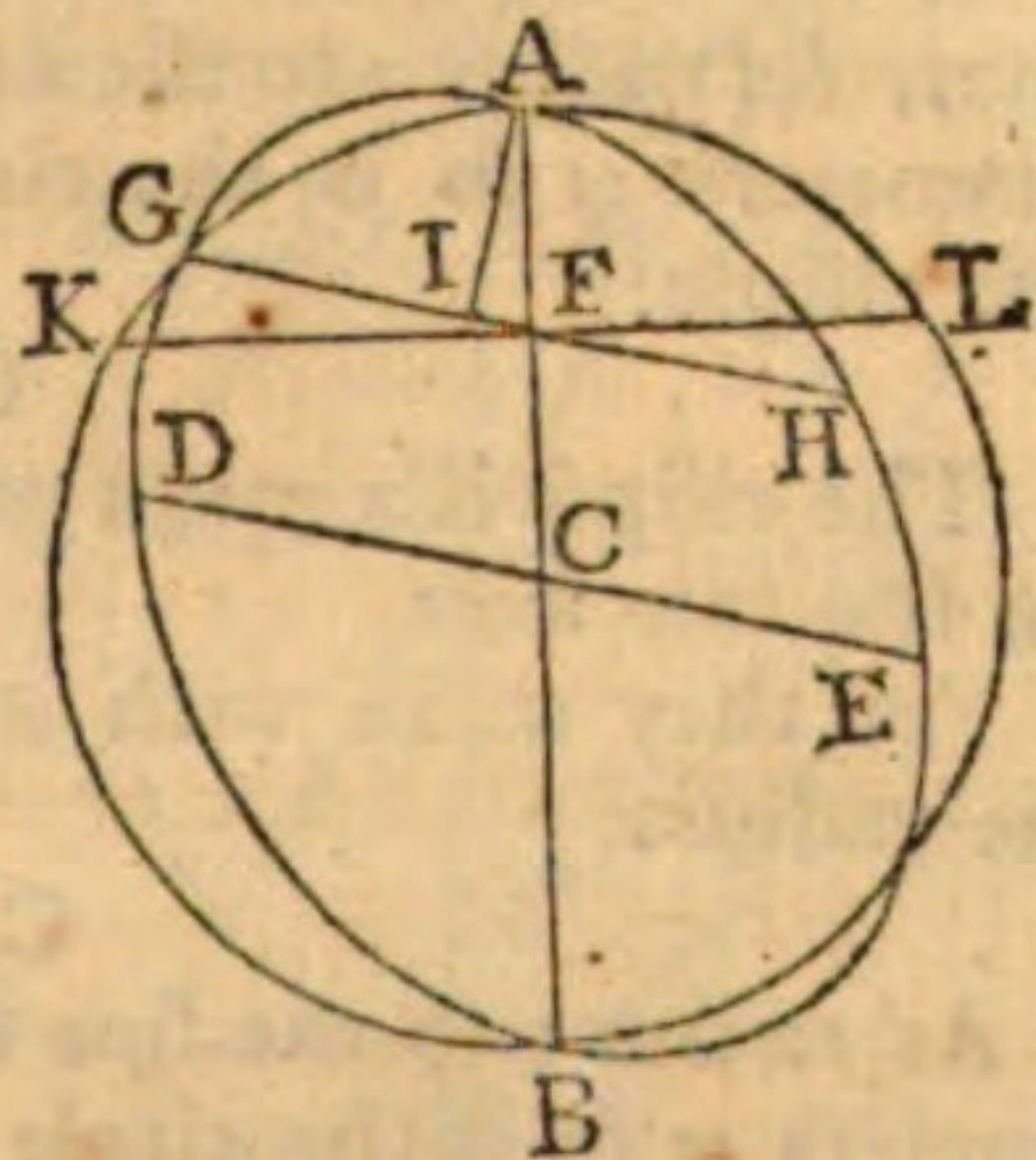
3 P

Ex-

## \* GENERAL INVESTIGATION.

Let  $AB, DE$ , be any two conjugate diameters of the ellipse  $ADBEA$ ,  $GH$  a double ordinate to the diameter  $AB$ ,  $AKBL$  a circle whose diameter is  $AB$ ,  $KL$  a double ordinate of the circle to the diameter  $AB$ , and  $AI$  perpendicular to  $GH$ .

Now whilst  $KL$  by flowing generates the circular segment  $KAL$ ,  $GH$  will describe the elliptic segment  $GAH$ ; but the velocity of  $KL$  is to that of  $GH$  as  $FA$  to  $AI$ ; and by the property of all conjugate diameters, as  $AC : CD :: (\sqrt{AF \times FB} =, \text{ by the nature of the circle,}) KF : FG$ ; wherefore as the circular segment  $KAL$  is to the elliptic segment  $GAH$ , so is  $AC \times AF$  to  $CD \times AI$ , so is  $AC \times \text{radius}$  to  $CD \times \text{fine } LIFA$ , so is  $AC$  to  $CD \times \text{fine } LC$ , putting 1 for the radius.



Co-



## EXAMPLE.

If the axes of an ellipse be 35 and 25, what is the area.

$$.7854 \times 35 \times 25 = 687.225 = \text{the area required.}$$

## RULE

## COROLLARY I.

The whole ellipse and circle, described upon any diameter of it, are to each other as the corresponding segments cut off them by their particular double ordinates  $GH$ ,  $KL$ , passing through the same point of that diameter.

## COROLLARY II.

As radius : sine of the angle made by any two conjugate diameters :: a mean proportional between the two circles described upon those diameters : the ellipse

$$\text{For } \left\{ \begin{array}{l} r \times d : s \times c :: \odot d : \odot \\ r \times c : s \times d :: \odot c : \odot \end{array} \right\} \text{ by cor. 1.}$$

$$\text{Hence } (r^2 cd : s^2 cd ::) r^2 : s^2 :: \odot d \times \odot c : \odot^2,$$

$$\text{And } r : s :: \sqrt{\odot d \times \odot c} : \odot.$$

Where  $d$  and  $c$  are the diameters,  $s$  the sine of the  $\angle$  made by them to the radius  $r$ ,  $\odot$  denotes a circle, and  $\odot$  an ellipse.

## COROLLARY III.

If the conjugate diameters be equal to each other, it will follow that, As radius : to the sine of the angle made by the equal conjugate diameters :: so is the circle described on one of those diameters : to the ellipse.

## COROLLARY IV.

The ellipse is a mean proportional between the two circles described on the two axes.

For they make with each other a right angle, whose sine is = to the radius.

## COROLLARY V.

As radius : to the sine of the angle made by any two conjugate diameters :: so is the circle whose diameter is a mean proportional between the conjugate diameters : to the ellipse.

This follows from corollary 2.

## COROLLARY VI.

The ellipse is equal to a circle whose diameter is a mean proportional between the two axes.

From corollary 4.

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## RULE II.

Multiply continually together any two conjugate diameters, the natural sine of their included angle, and the number  $\cdot 7854$ .

That is,  $dcns =$  the area, putting  $d$  and  $c =$  any two conjugate diameters,  $s =$  sine of their included  $L$ , and  $n = \cdot 7854$ . Ex-

## COROLLARY VII.

As an ellipse : is to the rectangle of its two axes (or to the rectangle of any two conjugate diameters drawn into the sine of their included  $L$ , the radius being 1) :: so is any circle : to the square of its diameter.—Any two like segments or zones of the ellipse and circle are also in the same proportion.

## COROLLARY VIII.

Ellipses, and their like segments, are to one another as the rectangles of their axes, or as the rectangles of any conjugate diameters forming the same angle in each.

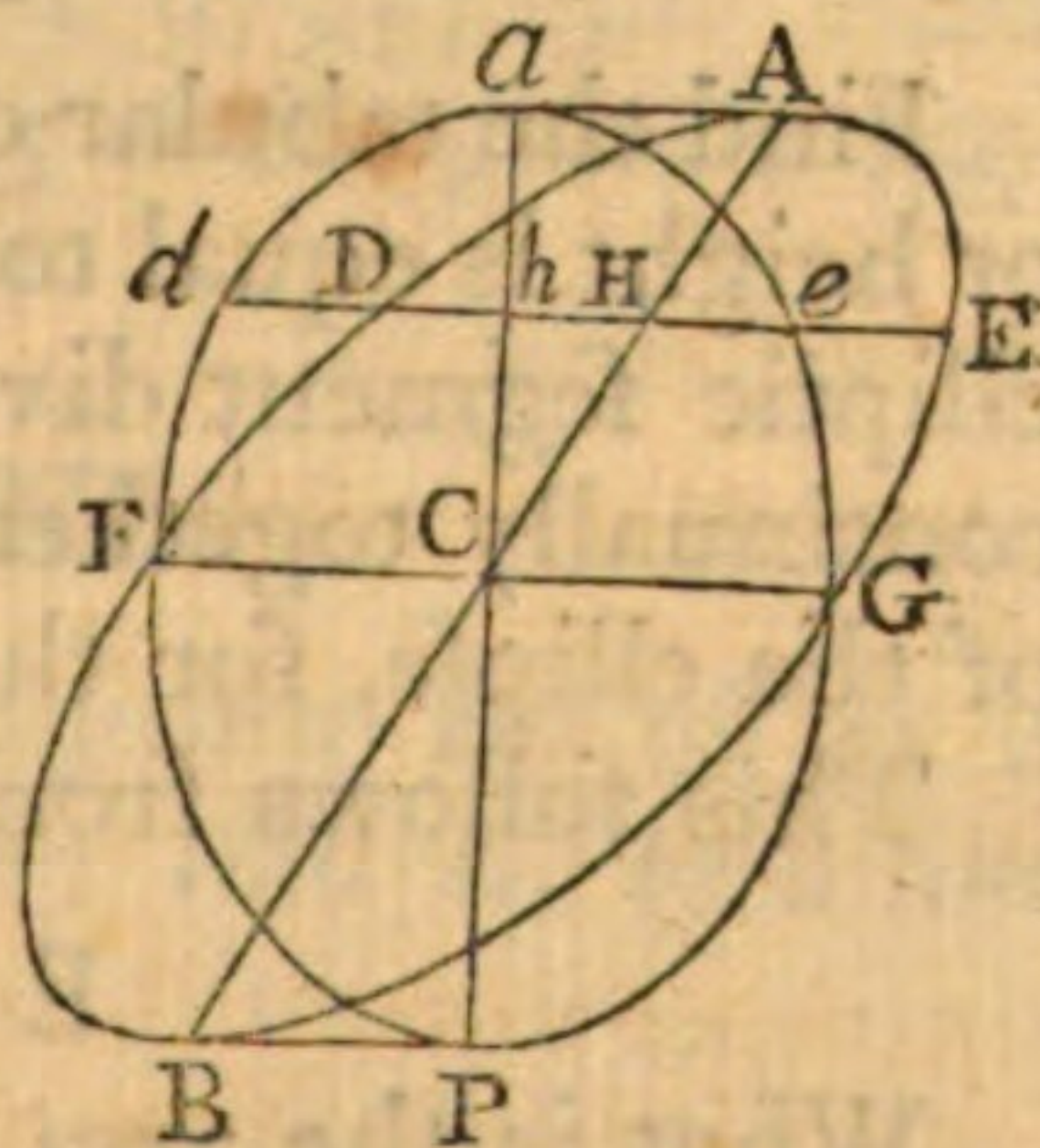
## COROLLARY IX.

Similar ellipses are to one another as the squares of their like diameters.

## COROLLARY X.

From corollary 7 comes also the following construction.

Let  $ADE$  be an oblique segment of the ellipse  $AFBGA$ , cut off by a double ordinate to the diameter  $AB$ ,  $FG$  being the conjugate. Through the center  $C$  draw  $aP$  perpendicular to  $FG$  and meeting, in  $a, P$ , with  $Aa, BP$ , both parallel to  $FG$ ; then about the axes  $aP, FG$ , describe the ellipse  $aFPGa$ , meeting the ordinate, produced, in  $e$  and  $d$ . Then will the right elliptic segment  $dae$  be equal to the oblique segment  $DAE$ , as well as the whole ellipse  $aFPGa$  equal to the ellipse  $AFBGA$ ; moreover their corresponding ordinates  $de$ ,



$DE$ , parallel to the common diameter  $FG$ , are every where equal, as are the like parts or zones contained between any two of such ordinates. And the same may be said of all ellipses contained between the parallels  $Aa, BP$ , infinitely produced : in which property they resemble parallelograms of the same base and between the same parallels.



## EXAMPLE.

If two conjugate diameters of an ellipse be 28 and 32, and their included angle  $77^{\circ} 34\frac{1}{4}'$ ; required its area.

The sine of  $77^{\circ} 34\frac{1}{4}'$  is  $\cdot 9765625$ ; wherefore  $\cdot 9765625 \times 32 \times 28 \times \cdot 7854 = 687\cdot 225 =$  the area.

## PROBLEM VI.

*To find the area of the segment of an ellipse cut off by a double ordinate to either axe, that is, by a line perpendicular to that axe.*

## RULE I.

Find the corresponding segment of the circle described upon the same axe to which the cutting line or base of the segment is perpendicular. Then as this axe: is to the other axe:: so is the circular segment: to the elliptic segment.

This follows from corollary 7 to the last problem.

## RULE II.

Find the tabular circular segment whose versed sine or height is equal to the quotient of the height of the elliptic segment divided by its axe: Then multiply continually together this segment and the two axes of the ellipse, for the area of the segment required.

This follows from the former.

## EXAMPLE.

What is the area of an elliptic segment cut off by a line parallel to and at the distance of  $7\frac{1}{2}$  from the less axe, the axes being 35 and 25.

Here  $17\frac{1}{2} - 7\frac{1}{2} = 10 =$  the height of the segment.

And



And  $10 \div 35 = 2 \div 7 = .2857\frac{1}{7} =$  the tabular versed fine, whose segment is  $.1851669$ .

Then  $.1851669 \times 35 \times 25 = 162.0210375 =$  the area of the less segment.

If the greater segment had been required. Then  $.78539816 - .1851669 = .60023126$ . And  $.60023126 \times 25 \times 35 = 525.2023525 =$  the area of the greater segment.

## EXAMPLE II.

What is the area of the elliptic segment cut off by a double ordinate perpendicular to the conjugate axis at the distance of  $7\frac{1}{2}$  from the center, the axes being 35 and 25.

Here  $12\frac{1}{2} - 7\frac{1}{2} = 5 =$  the altitude of the segment.

And  $5 \div 25 = 1 \div 5 = .2 =$  the tabular versed fine, whose corresponding segment is  $.1118238$ .

Hence  $.1118238 \times 25 \times 35 = 97.845825 =$  the area of the less segment.

Again,  $.78539816 - .1118238 = .67357436$ . And  $.67357436 \times 25 \times 35 = 589.377565 =$  the greater segment.

## PROBLEM VII.

To find the area of an elliptic segment cut off by a double ordinate to any diameter; that is, by a line oblique to the axes.

## RULE.

Divide the abscissa  $AF$  (fig. to prob. 5) of the double ordinate by its diameter  $AB$ , and find the tabular



lar circular segment whose versed sine is the quotient: Then multiply continually together the tabular area and the two axes. Or the tabular area, the diameter  $AB$  to which the base of the segment is a double ordinate, its conjugate diameter  $GH$ , and the sine of their included angle, for the area of the elliptic segment required.\*

## EXAMPLE.

The axes of an ellipse being 35 and 25, it is required to find the area of a segment whose base is a double ordinate to a diameter whose length is 33, it being divided by the double ordinate into the two abscissas 7 and 26.

Here  $FA \div AB = 7 \div 33 = .2121\frac{7}{33} =$  the tabular versed sine; to which corresponds the area  $.12162869$ .

Hence  $.12162869 \times 25 \times 35 = 106.425104 =$  the segment  $GAH$ .

Moreover,  $.78539816 - .12162869 = .66376947$ . And  $.66376947 \times 25 \times 35 = 580.79828625 =$  the greater segment  $GBH$ .

P R O-

## \* DEMONSTRATION.

For, by cor. 7, prob. 5, as  $AB^2$  : to circular segment  $LAK$  :: so is  $(AB \times DE \times s.LC =)$  rectangle of the two axes : to elliptic segment  $GAH$ ; but circular segment  $LAK = AB^2 \times$  tabular circular segment  $(s)$  whose versed sine is  $\frac{FA}{AB}$ ; therefore  $(AB^2 : AB^2 \times s ::) 1 : s ::$  rectangle of the axes to elliptic segment. Q.E.D.



## PROBLEM VIII.

To find the trilineal area  $ABQ$  included by either axe, a line drawn from any point in it, and their intercepted arc.

## RULE.

Draw the ordinate  $DB$  meeting the circle described upon the said axe in  $C$ , and draw  $AC$ ,  $OC$ .

Then, As the above-said axe:

Is to the other axe::

So is the circular trilineal  $ACQ$ :

To the elliptic trilineal  $ABQ$ .\*



Note.

## \* DEMONSTRATION.

For, by the investigation and corollaries to prob. 5, As the axe  $A$ : to the axe  $a$ ::  $CD$ :  $DB$ :: circular segment  $QCD$ : elliptic segment  $QBD$ :: (because of the common base  $AD$ ) triangle  $ACD$ : triangle  $ABD$ :: (by composition) the trilineal  $ACQ$ : trilineal  $ABQ$ .  $Q.E.D.$

## COROLLARY I.

Since, by rule 2 prob. 6 sect. 1 part 2, the circular arc  $QC$  (putting  $CD = y$ , and  $OQ = r$ ) is  $= y \times 1 + \frac{y^2}{3 \cdot 2r^2} + \frac{3y^4}{5 \cdot 2 \cdot 4r^4} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^6}$  &c. we shall have  $\frac{1}{2}ry \times 1 + \frac{y^2}{3 \cdot 2r^2} + \frac{3y^4}{5 \cdot 2 \cdot 4r^4} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^6}$  &c. = the sector  $OCQ$ , which being increased or diminished by  $\frac{1}{2}OA \times DC$  = the triangle  $ACO$ , we shall have  $\frac{1}{2}y \times AQ + \frac{y^2}{3 \cdot 2r} + \frac{3y^4}{5 \cdot 2 \cdot 4r^3} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^5}$  &c. for the general value of the trilineal  $ACQ$ ; and consequently  $\frac{cy}{2r} \times AQ + \frac{y^2}{3 \cdot 2r} + \frac{3y^4}{5 \cdot 2 \cdot 4r^3} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^5}$  &c. = the elliptic trilineal  $ABQ$ ,  $r$  being the radius of the circle, or semi-axe  $OQ$  of the ellipse, and  $c$  the other semi-axe.

And since  $r:c::y:BD=z$ , we shall have  $y = \frac{rz}{c}$ , and consequently the value of the elliptic trilineal  $ABQ$ , expressed in terms of its own ordinate



*Note.* It is evident that the triangle  $ACO$  added to or subtracted from the circular sector  $OCQ$ , will give the circular trilineal  $ACQ$  according as  $AQ$  is greater or less than half the axe. Or the semi-segment  $CDQ$  increased or diminished by the triangle  $ACD$  will give the same.

Ex-

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ordinate  $z$  ( $BD$ ) and semi-axes  $r, c$ , will be  $\frac{1}{2} rz \times : AQ + \frac{rz^2}{3.2c^2} + \frac{3rz^4}{5.2.4c^4} + \frac{3.5rz^6}{7.2.4.6c^6} \&c.$

COROLLARY II.

If  $A$  coincide with  $O$ , the triangle  $AOB$  will vanish, and the trilineal become barely a sector  $OBQ$ , whose value will be  $\frac{1}{2} rz \times : 1 + \frac{z^2}{3.2c^2} + \frac{3z^4}{5.2.4c^4} + \frac{5.3z^6}{7.2.4.6c^6} \&c.$

COROLLARY III.

When  $z = c$ , then the quadrant  $OpQ$  will be  $\frac{1}{2} rc \times : 1 + \frac{1}{3.2} + \frac{3}{5.2.4} + \frac{5.3}{7.2.4.6} \&c.$

COROLLARY IV.

If  $A$  coincide with  $P$ , then  $AQ = QP = 2r$ , and the trilineal  $PBQ$  will be  $= \frac{1}{2} rz \times : 2 + \frac{z^2}{3.2c^2} + \frac{3z^4}{5.2.4c^4} + \frac{5.3z^6}{7.2.4.6c^6} \&c.$

And when  $B$  coincides with  $p$ , then will the trilineal  $PpQ$  be  $= \frac{1}{2} rc \times : 2 + \frac{1}{3.2} + \frac{3}{5.2.4} + \frac{5.3}{7.2.4.6} \&c.$

COROLLARY V.

If from the double of the quadrant in corollary 3, be taken the trilineal in corol. 4, there will remain  $rx : \frac{c-z}{1} + \frac{2c^3-z^3}{6.2c^2} + 3\frac{2c^5-z^5}{10.2.4c^4} + 5.3\frac{2c^7-z^7}{14.2.4.6c^6} \&c.$  for the value of the segment  $PpB$ .

Co.



## EXAMPLE.

Given the axes 35 and 25 of an ellipse, and an ordinate to the transverse 10; required the area included by the transverse and a line drawn from the

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## COROLLARY VI.

If from the value of the sector  $OBQ$  in corollary 2, be taken that of the triangle  $OBQ = \frac{1}{2}rz$ , there will remain  $\frac{1}{2}rz \times : \frac{z^2}{3.2c^2} + \frac{3z^4}{5.2.4c^4} + \frac{5.3z^6}{7.2.4.6c^6}$  &c. for the value of the segment  $BQ$ .

## COROLLARY VII.

When  $B$  coincides with  $p$ , the series in each of the two last corollaries becomes  $\frac{1}{2}rc \times : \frac{1}{3.2} + \frac{3}{5.2.4} + \frac{5.3}{7.2.4.6}$  &c. for the value of each of the segments  $Pp, pQ$ .

## COROLLARY VIII.

If  $A$  be the focus of the ellipse, then  $AQ = r \pm \sqrt{rr - cc}$ , and the trilineal  $ABQ = \frac{1}{2}z \times : r \pm \sqrt{rr - cc} + \frac{rz^2}{3.2c^2} + \frac{3rz^4}{5.2.4c^4} + \frac{3.5rz^6}{7.2.4.6c^6}$  &c.  
 $= \frac{1}{2}rz \times : 1 \pm \sqrt{1 - \frac{cc}{rr}} + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4} + \frac{3.5z^6}{2.4.6.7c^6}$  &c.

## COROLLARY IX.

Putting  $T = ABQ = \frac{1}{2}rz \times : a + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4}$  &c. ( $a$  being  $= 1 \pm \sqrt{1 - \frac{cc}{rr}}$ ) and  $q = \frac{2T}{r} = z \times : a + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4}$  &c. we shall have, by reverting the series,  $z = \frac{q}{a} - \frac{q^3}{6c^2a^4} + \frac{10-9a}{120c^4a^7} q^5 - \frac{280-504a+225a^2}{5040c^6a^{10}} q^7$  &c.  $= \frac{2T}{r} \times : \frac{1}{a} - \frac{1}{6a^4} p^3 + \frac{10-9a}{120a^7} p^5 - \frac{280-504a+225a^2}{5040a^{10}} p^7$  &c. where  $p = \frac{2T}{rc}$ ; which series generally converges very fast, and affords a good direct solution of Kepler's problem.

Suppose, for example, it were required to find the true anomaly of the earth, and its distance from the sun, answering to the mean anomaly



top of the ordinate to cut the transverse in an angle of 40 degrees.

Here, as  $pq = 25 : P\mathcal{Q} = 35 :: DB = 10 : DC = 14$ .

As rad. : tang.  $\angle B (50^\circ) :: BD : DA = 11.917536$ .

Hence  $AD \times \frac{1}{2} DC = 83.422752 = \text{the } \triangle ADC$ .

As  $25 : 35 :: \sqrt{12.5^2 - 10^2} : 10.5 = OD$ , and hence  $QO - OD = 7 = D\mathcal{Q}$ . Then, by the table of circular segments, the semi-segment  $CD\mathcal{Q}$  will come out  $68.4920775$ .

The triangle  $ADC + \text{segment } CD\mathcal{Q} = 151.9143395 = \text{the trilineal } CA\mathcal{Q}$ .

Wherefore as  $35 : 25 :: 151.9143395 : 108.5102425 = \text{the area of the elliptic trilineal } BA\mathcal{Q} \text{ required}$ .

PRO-

anomaly of 1 sign or 30 degrees, the excentricity being .01681 to the mean distance 1.

Here  $PO = r = 1$ ,  $AO = .01681$ ,  $Op = c = \sqrt{1 - .01681^2} = .9998587$ , the area of a quadrant of the ellipse =  $.785398 \&c. \times rc = .7852872$ ;

then, as 3 signs : 1 sign ::  $.7852872 : .2617624 = T$ ; hence  $\frac{2T}{r} = 2T$

=  $.5235248$ , and  $\frac{2T}{rc} = \frac{2T}{c} = \frac{2}{3} \times .78539 \&c. = .5235988 = p$ ; also  $a =$

$1.01681$ , and  $\frac{p^2}{a^3} = .2589869$ .

Then, 1st term  $A = \frac{1}{a} = \quad - \quad - \quad - \quad + .98346790$

2d  $- B = \frac{1}{6} A \times \frac{p^2}{a^3} = \quad - \quad - \quad - .04245088$

3d  $- C = \frac{10-9a}{20} B \times \frac{p^2}{a^3} = \quad - \quad + .00046655$

4th  $- D = \frac{280-504a+225a^2}{42} C \times \frac{p^2}{a^3} = \quad - .00000044$

the sum =  $.94148313$

which drawn into  $\frac{2T}{r} = 2T = .5235248$ , gives  $.4928897 = DB = z$ .

But



## PROBLEM IX.

To find the surface of a spheroid.

## \* RULE I.

For both spheroids.

Let  $f$  denote the fixed axe, or the axe about which the ellipse is conceived to revolve,  $r$  the revolving axe; and put  $p = 3.14159 \text{ \&c.}$   $q = \frac{ff \infty rr}{ff}$ : Then will the surface ( $s$ ) of the spheroid be expressed by either of the following serieses, using the upper or under

But  $OD = \frac{r}{c} \sqrt{cc - zz} = .87005233$ ; hence  $DA = AO + OD = .88686233$ , and as  $AD : DB :: \text{rad.} : \text{tang. } \angle BAD = 29.063904^\circ = 29^\circ 3' 50'' 3'''$  the true anomaly; also, as  $\text{rad.} : \text{sec. } \angle BAD :: DA : AB = 1.0146255$ , the distance of the earth from the sun to the mean distance 1.

## \* DEMONSTRATION.

Put the fixed semi-axe  $Tc = cR = a$  (fig. to prob. 4),  $Cc = cO = b$ ,  $cA = x$ ,  $AB = y$ , arc  $CB = z$ , and  $3.14159 \text{ \&c.} = p$ . Then,  $a : b ::$

$$\sqrt{aa - xx} : \frac{b}{a} \sqrt{aa - xx} = y; \text{ hence } \dot{y} = -\frac{bxx}{a\sqrt{aa - xx}}, \text{ and } \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$= \sqrt{\dot{x}^2 + \frac{b^2 x^2 \dot{x}^2}{a^2 x a^2 - x^2}} = \frac{\dot{x}}{a} \sqrt{\frac{a^4 - x^2 x aa - bb}{aa - xx}}; \text{ and consequently the}$$

$$\text{fluxion of the surface } \dot{s} = 2py\dot{z} = \frac{2pbx}{aa} \sqrt{a^4 - x^2 x aa - bb} =$$

$$\frac{2pbx}{aa} \sqrt{a^4 - c^2 x^2} \text{ (putting } cc = aa - bb) = \frac{2pbx}{aa} \sqrt{\frac{a^4}{cc} - xx} = \frac{2pbx}{aa} x :$$

$$\frac{a^2}{c} - \frac{cx^2}{2a^2} - \frac{c^3 x^4}{2.4a^6} - \frac{3c^5 x^6}{2.4.6a^{10}} \text{ \&c. and the fluent is } s = 2pbx \times :$$

$$1 - \frac{c^2 x^2}{2.3a^4} - \frac{c^4 x^4}{2.4.5a^8} - \frac{3c^6 x^6}{2.4.6.7a^{12}} - \frac{3.5c^8 x^8}{2.4.6.8.9a^{16}} \text{ \&c. and if for } \frac{cc}{aa} \text{ be}$$



under signs according as it is an oblong or oblate spheroid, viz.  $s = prf \times 1 \mp \frac{1}{2.3} q - \frac{1}{2.4.5} q^2 \mp \frac{3}{2.4.6.7} q^3 - \frac{3.5}{2.4.6.8.9} q^4 \&c.$  or  $prf \times 1 \mp \frac{A}{2.3} q - \frac{3B}{4.5} q^2 \mp \frac{3.5C}{6.7} q^3 - \frac{5.7D}{8.9} q^4 \&c.$  where  $A, B, C, \&c.$  are the several preceding terms.

## EXAMPLE.

If the axes be 50 and 40, required the surface of each spheroid.

## I. For the oblong spheroid.

Here  $f = 50$ ,  $r = 40$ , and  $\frac{ff - rr}{ff} = \frac{30 \times 30}{50 \times 50} = \frac{9}{25} = .36 = q.$

Hence

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be written  $dd$ , the series will become  $s = 2pbx \times 1 - \frac{d^2 x^2}{2.3a^2} - \frac{d^4 x^4}{2.4.5a^4} - \frac{3d^6 x^6}{2.4.6.7a^6} - \frac{3.5d^8 x^8}{2.4.6.8.9a^8} = (\text{when } x = a) 2pab \times 1 - \frac{d^2}{2.3} - \frac{d^4}{2.4.5} - \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9} \&c. = 2pab \times 1 - \frac{A}{2.3} d^2 - \frac{3B}{4.5} d^4 - \frac{3.5C}{6.7} d^6 - \frac{5.7D}{8.9} d^8 \&c.$  where, when  $cc (= aa - bb)$  is affirmative, viz. in the

case of an oblong spheroid, all the signs after the first will be negative; but for an oblate spheroid,  $cc$  being negative, the signs after the first will become alternately *plus* and *minus*. Q.E.D.

## COROLLARY I.

The series above will not always converge for an oblate spheroid, viz. when  $d$  is greater than 1, or when  $\frac{bb}{aa}$  is greater than 2, the series will not converge.

## COROLLARY II.

Hence the surfaces of similar spheroids, and also of their like parts, are to each other as the rectangles of their axes, or as the squares of the like dimensions.



Hence 1st term  $A = 1$   
 2d term  $B = \frac{A}{2.3} q = -0.06$   
 3d term  $C = \frac{3B}{4.5} q = -0.00324$   
 4th term  $D = \frac{3.5C}{6.7} q = -0.0004166$   
 5th term  $E = \frac{5.7D}{8.9} q = -0.000072900$   
 6th term  $F = \frac{7.9E}{10.11} q = -0.00001504$   
 7th term  $G = \frac{9.11F}{12.13} q = -0.0000035$   
 8th term  $H = \frac{11.13G}{14.15} q = -0.0000009$

the sum of the negative terms  $= -0.0637489$ , which taken from the first term 1, leaves .936251 for the sum of the series; and this being drawn into  $prf = 3.14159265 \times 50 \times 40 = 6283.18531$ , will produce 5882.6385 for the surface required.

## II. For the oblate spheroid.

Here  $f=40$ ,  $r=50$ , and  $\frac{rr-ff}{ff} = \frac{30 \times 30}{40 \times 40} = \frac{9}{16} = q$ . Then by increasing the values of the terms, found above for the oblong spheroid, in proportion to such power of the ratio of the value of  $q$  in this case to its value in the former, whose exponent is respectively 1 less than the number of each term; viz. in the proportion of the 0, 1, 2, 3, &c. power of  $\frac{9}{25}$  to  $\frac{9}{16}$ , or of 16 to 25, or of 64 to 100; we shall have

3 S

the



$$\text{the 1st term} = 1 \times \frac{100}{64}^0 = 1 \times 1 = +1$$

$$\text{2d term} = .06 \times \frac{100}{64} = +0.09375$$

$$\text{3d term} = .00324 \times \frac{100}{64}^2 = -0.00791$$

$$\text{4th term} = .0004166 \times \frac{100}{64}^3 = +0.00159$$

$$\text{5th term} = .0000729 \times \frac{100}{64}^4 = -0.00044$$

$$\text{6th term} = .0000150 \times \frac{100}{64}^5 = +0.00014$$

$$\text{7th term} = .0000035 \times \frac{100}{64}^6 = -0.00005$$

$$\text{8th term} = .0000009 \times \frac{100}{64}^7 = +0.00002$$

sum of the affirmative terms  $+1.09550$

sum of the negative terms  $-0.00840$

the difference  $1.0871$ ,  
then  $6283.18531 \times 1.0871 = 6830.4507 =$  the surface of the oblate spheroid required.

#### RULE II. *For both spheroids.*

Divide the revolving axe by the fixed axe, and call the quotient  $q$ . Find the difference between 1 and the square of  $q$ , and call the square root of that difference  $s$ .

For an oblong spheroid, multiply  $.01745329$  by the degrees in the arc whose sine is  $s$ , and call the product  $P$ .

For an oblate spheroid, multiply  $2.302585$  by the logarithm of the sum of  $s$  and  $q$ , and call the product  $P$ , also.

Multiply  $P$  into the fixed axe and divide the product by  $s$ ; to the quotient add the revolving axe;



axe ; then the sum multiplied into  $\frac{1}{2}$  the circumference of the greatest circle, viz. into  $3.14159$  &c. and into  $\frac{1}{2}$  the revolving axe, will give the surface required.

That is, putting  $f$  = the fixed axe,

$r$  = the revolving axe,

$q = r \div f,$

$s = \sqrt{1 - qq},$

$p = 3.14159$  &c.

$P = .01745329 \times$  degrees to the sine  $s$ , in the oblong, and =  $2.302585 \times$  log. of  $s+q$  in the oblate spheroid ;

then  $\frac{Pf}{s} + r \times \frac{1}{2} pr =$  the surface of the spheroid.\*

E X A M P L E.

The axes of a spheroid being 50 and 30, required the surface.

I. For the oblong spheroid.

Here  $f = 50$ ,  $r = 30$ ,  $q = \frac{30}{50} = .6$ ,  $s = \sqrt{1 - qq} = \sqrt{1 - .36} = \sqrt{.64} = .8$ , which is the sine of  $53.130105$  degrees.

Then  $53.130105 \times .01745329 = .92729513 = P$ , and  $\frac{50 \times .92729513}{.8} + 30 \times \frac{30 \times 3.14159 \text{ \&c.}}{2} = 4144.8263 =$  the surface of the oblong spheroid.

II. For

\* DEMONSTRATION.

For the fluent of  $\frac{2pbx}{aa} \sqrt{a^4 - c^2 x^2}$  (the fluxion of the surface in the demonstration of the first rule) is, when  $x = a$ ,  $pb \times b + \frac{aa}{c} P$ , where  $P = .017453 \times$  degrees in the arc whose sine is  $\frac{c}{a}$ , when  $a$  is greater than  $b$  ; or  $P = 2.3025 \text{ \&c.} \times$  log. of  $\frac{b+c}{a}$ , when  $a$  is less than  $b$ ,  $c$  being  $= \sqrt{aa \text{ \& } bb}$ . Q. E. D.



## II. For the oblate spheroid.

Here  $f = 30$ ,  $r = 50$ ,  $q = \frac{f}{r} = \frac{3}{5}$ , and  $s = \sqrt{1 - \frac{f^2}{r^2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$ ; hence  $s + q = \frac{4}{5} + \frac{3}{5} = \frac{7}{5} = 1.4$ , whose log. is  $.4771213$ ; and  $P = .4771213 \times 2.302585 = 1.0986124$ .

Then  $\frac{30 \times 1.0986124}{\frac{4}{5}} + 50 \times \frac{50 \times 3.14159 \&c.}{2} = 5868.39918$   
 = the surface of the oblate spheroid.

## R U L E III.

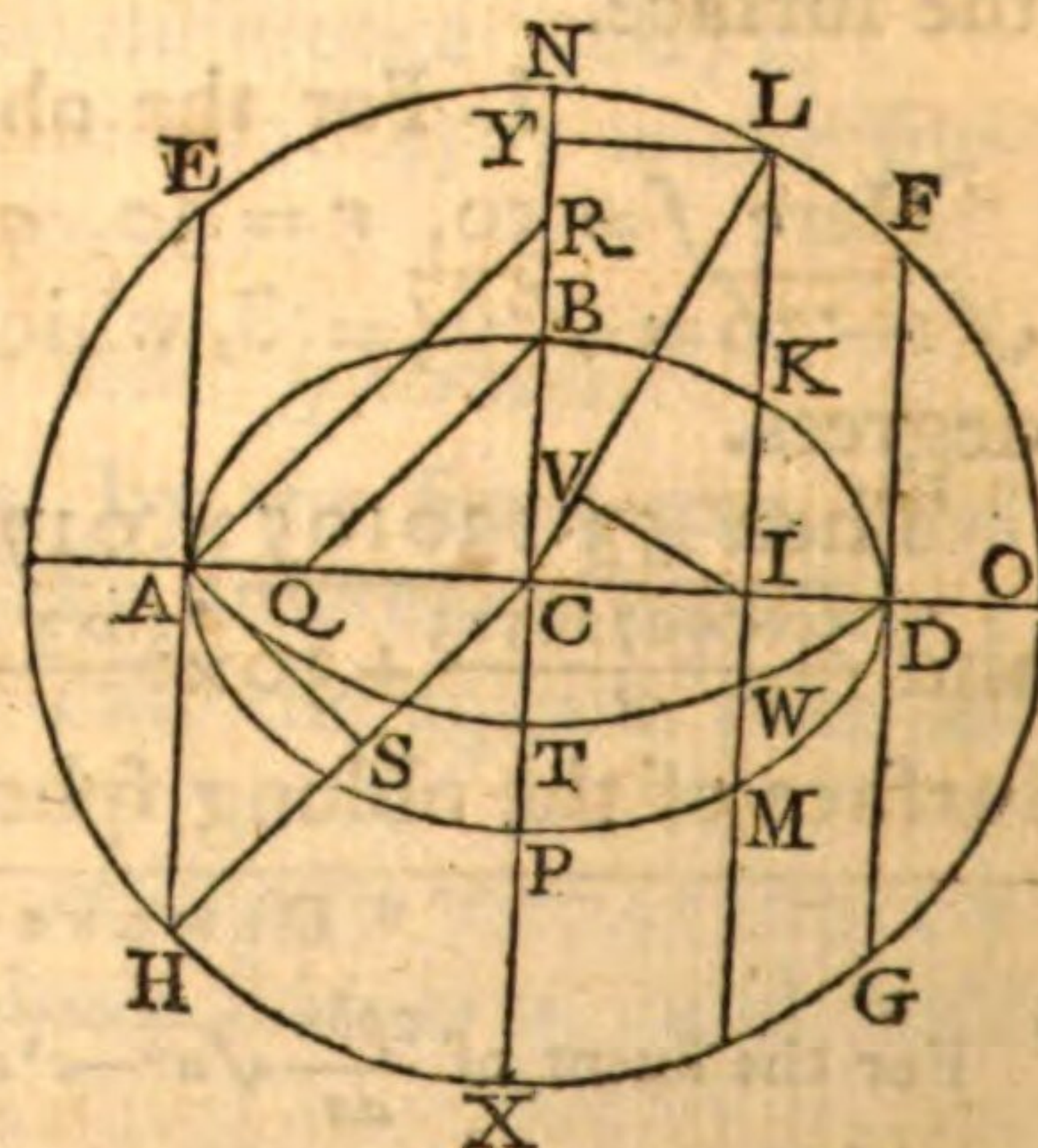
*For the oblong spheroid.*

Find  $q$  and  $s$  as in the last rule.

Find also the area of the circular middle zone whose radius is half the quotient of the fixed axe divided by  $s$ , and the distance of each end from the center is half the fixed axe, and call this zone  $z$ .

Then multiply continually together  $z$ ,  $q$ ,  $s$ , and  $3.14159 \&c.$  for the surface of the oblong spheroid.

That is,  $pqs z =$  the surface of the oblong spheroid  $ABDP$ , where  $z$  is equal to the middle zone  $EFGH$  of the circle whose radius is  $\frac{\frac{1}{2}f}{s}$ .\*



E X-

## \* DEMONSTRATION.

For since the fluxion of the surface  $BKMP$  is  $\frac{2pbcx}{aa} \sqrt{\frac{a^4}{cc} - xx}$ , and that



## EXAMPLE.

Taking again the same example,

We have  $\frac{1}{2}f \div s = \frac{25}{8} = 31.25 =$  the radius, or  $62.5 =$  the diameter of the circle.

Hence  $OC - CD = 31.25 - 25 = 6.25 = DO$ , and  $DO \div 2OC = 6.25 \div 62.5 = .1 =$  the tabular versed sine, to which belongs the area  $.04087528$ ; and this taken from  $.39269908$ , the tabular semi-circle, leaves  $.3518238$  for half the tabular zone.

Whence  $.3518238 \times 2 \times 62.5 \times 62.5 = 2748.6234375 =$  the middle zone  $EFGH = z$ .

Then  $pqs = 3.14159 \&c. \times .6 \times .8 \times 2748.6234375 = 4144.8265 =$  the surface, nearly the same as before.

## RULE IV.

*For the oblong spheroid.*

Between the semi-conjugate  $BC$  (see the figure to the last rule) and the sum of the conjugate  $BP$  and circular arc  $EF$ , find a mean proportional, and it will be the radius of a circle equal to the surface of the spheroid; the circle being described as in the last rule.

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That

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that of the circular area  $NLIC = x \sqrt{\frac{a^4}{cc} - xx}$ , which being to the former in the constant ratio of 1 to  $\frac{2pbc}{aa}$ , their fluents must be continually in the same ratio; and consequently the surface  $BKMP = \frac{2pbc}{aa} \times CNLI = pqs \times 2CNLI = p \times \frac{BC}{CN} \times 2CNLI$ . And the whole surface  $ABDPA = pqs \times EFGH$ . Q. E. D.



That is, multiply continually together  $BC$ ,  $BP + EF$ , and  $3.14159$  &c. for the surface.\*

Ex-

\* DEMONSTRATION.

Apply  $BQ = AC$ , and draw  $AR$  parallel to  $BQ$ ; then is  $AR$  = the radius  $CN$ ; for, by the construction,  $CN$  is  $= \frac{aa}{\sqrt{aa-bb}}$   
 $= \frac{AC^2}{\sqrt{QB^2 - BC^2}} = \frac{AC^2}{CQ} = \frac{AC \times BQ}{CQ} =$  (by similar triangles)  $AR$ .  
 Draw  $CH$ , and  $AS$  perpendicular to it.

Then, by the last rule, the surface  $BAP$  is  $= p \times \frac{BC}{CN} \times 2CNEA$  or  
 $2CXHA = p \times BC \times \frac{2 \text{ sector } CXH + 2 \triangle HAC}{CH} = p \times BC \times \overline{NE + AS}$ .

But, by similar triangles,  $\left\{ \begin{array}{l} HC : CA :: AH : AS \\ AR : BQ :: RC : CB \end{array} \right\}$ ; and since  $CH$  =  $AR$ , and  $CA = BQ$ , by the construction; as also  $AH = RC$ , by reason of the equal hypotenuses  $CH$ ,  $AR$ , and common base  $AC$ ; the fourth terms  $AS$ ,  $CB$ , must, also, be equal to each other. And consequently the surface  $BAP = p \times BC \times \overline{NE + AS} = p \times BC \times \overline{AT + BC}$ ; and that of the whole spheroid  $= p \times BC \times \overline{ATD + BP} = p \times BC \times \overline{ENF + BP}$ , the arc  $ATD$  being described with the center  $R$  and radius  $RA$ ; viz. equal to a circle whose radius is a mean proportional between  $BC$  and  $ATD + BP$ . Q.E.D.

SCHOLIUM.

This construction, for finding a circle equal to the surface of an oblong spheroid, was first given by Mr *Huygens* in his *Horologium Oscillatorium*, prop. 9, but without demonstration.—The construction is here rendered general for any part or zone  $BKMP$  of the spheroid, viz. that its curve surface is equal to a circle whose radius is a mean proportional between  $BC$  and  $\overline{NL + IV}$  or  $\overline{TW + IV}$ ,  $IV$  being perpendicular to  $CL$ . For in the same manner as  $p \times BC \times \overline{NE + AS}$  was found for the surface  $BAP$ , may the surface of any zone  $BKMP$  be found to be  $p \times BC \times \overline{NL + IV}$ .

In imitation of the two rules above for expressing the surface of the oblong spheroid, by means of a circular arc and a circular or elliptic area, I shall here shew the investigation of others for that of



## EXAMPLE.

Taking still the same example,

We have the radius  $CN = \frac{\frac{1}{2}f}{\frac{1}{2}} = \frac{25}{.8} = 31\frac{1}{4}$ ; then as  $31\frac{1}{4} : 25 = CA$  the sine of the arc  $EN :: 1 : \frac{4}{5} = .8 =$  the sine of

of the oblate spheroid, by means of the parabolic arc or hyperbolic area.

Thus if  $x$  be = the abscissa, and  $y$  = the ordinate of a parabola whose parameter is  $n$ ; since  $x = \frac{yy}{n}$ , and  $\dot{x} = \frac{2yy}{n}$ , the fluxion of the

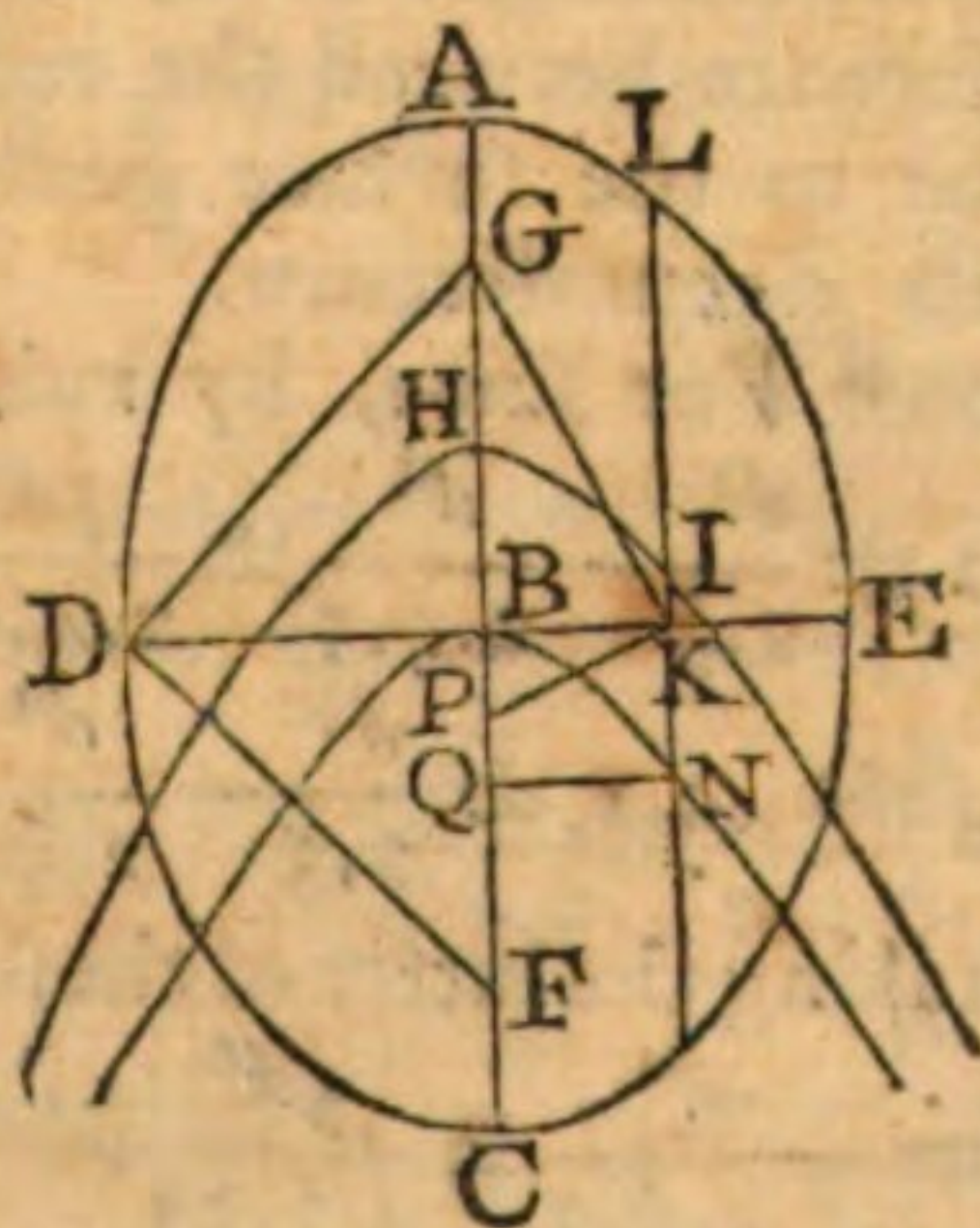
arc will be  $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{4y^2\dot{y}^2}{nn} + \dot{y}^2} = \frac{\dot{y}}{n} \sqrt{nn + 4yy} =$

$\frac{\dot{y}}{m} \sqrt{mm + yy}$  (putting  $m = \frac{1}{2}n = \frac{1}{2}$  the parameter); which being to

that of the oblate surface, viz.  $\frac{2pb\dot{x}}{aa} \sqrt{a^4 + c^2x^2} = \frac{2pb\dot{c}x}{aa} \sqrt{\frac{a^4}{cc} + x^2}$

$= \frac{2pb\dot{x}}{m} \sqrt{mm + xx}$  (putting  $m = \frac{a^2}{c}$ , and supposing  $x = y$ ) in the

constant ratio of 1 to  $2pb$ , their fluents must be in the same ratio; and consequently if  $P$  be put to denote the parabolic arc, then will  $2pbP$  express the value of the oblate surface, which therefore is equal to a circle whose radius is a mean proportional between  $b$  and  $2P$ , or between  $2b$  and  $P$ . And hence this construction. Let  $DE$  be the less or fixed axe of the ellipse,  $AC$  the greater axe, and  $F$  the focus; join  $D$ ,  $F$ , and draw  $DG$  perpendicular to  $DF$ ; bisect  $BG$  in  $H$ ; with the vertex  $H$  and focus  $B$  describe a parabola  $HI$ : Then draw any ordinate  $LK$  to the axe  $DE$  meeting with the parabola in  $I$ ; and the surface generated by the arc  $AL$  will be equal to a circle whose radius is a mean proportional between  $AC$  and the parabolic arc  $HI$ . Or the surface generated by  $AL$  will be equal to the curve surface of a cylinder whose diameter is  $AC$  and height  $HI$ .



Again,



of the similar arc to the radius 1, to which belong  $53.130105$  degrees; hence  $53.130105 \times .01745329 \times 31\frac{1}{4} = 28.977972 =$  the arc  $EN$ ; then  $BP \times BC + EN \times 3.14159 \&c. = 4144.8262 =$  the surface as before, nearly.

### R U L E V.

*For both spheroids.*

Find  $q$  as in the first rule, then to or from 1 add or subtract  $\frac{1}{3}$  of  $q$  according as it is the oblate or oblong spheroid, and call the square root of the sum or difference  $A$ . Or  $A = \sqrt{1 \pm \frac{1}{3}q}$ .\*

Mul-

Again, since  $P = \frac{\frac{2by}{m} \sqrt{mm+yy} - A}{b}$  ( $A$  being the area of an hyperbola whose semi-axes are  $b$  and  $m$ , and  $y$  the ordinate, see cor. 7 rule 1 prob. 6 sect. 5), the surface generated by  $AL$  will be equal to  $2p \times \frac{2by}{m} \sqrt{mm+yy} - A$ ; and hence this construction is evident.

Let  $LK$ , produced, meet, in  $N$ , with the hyperbola  $BN$  whose vertex is  $B$ , center  $A$ , and semi-conjugate axe  $BG$ ; draw  $GK$ , and  $KP$  perpendicular to it, and draw the ordinate  $NQ$ : Then will the difference between the rectangle  $AC \times KP$  and the hyperbolic area  $BNQ$  be to the surface generated by  $AL$ , as the radius of a circle is to its circumference.

\* For since, by rule 1, the value of the surface is  $S = 4abp \times : 1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9} \&c.$  and  $4abp \sqrt{1 \pm \frac{1}{3}d^2} = 4abp \times : 1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.9} \pm \frac{3d^6}{2.4.6.27} - \frac{3.5d^8}{2.4.6.8.81} \&c.$  we shall have, by taking the latter of these quantities from the former,  $S - 4abp \sqrt{1 \pm \frac{1}{3}d^2} = -\frac{d^4}{2.5.9} \pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81} \&c.$  and consequently  $S = 4abp \sqrt{1 \pm \frac{1}{3}d^2}$  nearly, which is rule 5.



Multiply the product of the two axes by 3.14159 &c. and call the product  $P$ .

Then the product of  $A$  and  $P$  will be the surface nearly.

That is,  $pfr \sqrt{1 \pm \frac{1}{3}q}$ .

#### EXAMPLE.

Let there be taken here the same example as at rule I, in which the axes are 50 and 40.

Then, I. *For the oblong spheroid.*

Here  $q = \frac{9}{25}$ , and  $\sqrt{1 - \frac{9}{3 \times 25}} = \sqrt{1 - \frac{3}{25}} = \frac{1}{5} \sqrt{22} = .93808 = A$ . But  $3.14159 \&c. \times 50 \times 40 = 6283.1853 = P$ . Then  $A \times P = 6283.1853 \times .938 = 5893.6278 =$  the surface nearly.

II. *For the oblate spheroid.*

Here  $q = \frac{9}{16}$ , and  $\sqrt{1 + \frac{9}{3 \times 16}} = \sqrt{1 + \frac{3}{16}} = \frac{1}{4} \sqrt{19} = 1.089725 = A$ . Then  $A \times P = 1.0897 \times 6283.1853 = 6846.7869 =$  the surface of the oblate spheroid nearly.

#### \* RULE VI.

*For both spheroids.*

To or from 1 add or subtract, according as it is for the oblate or oblong spheroid,  $\frac{1}{6}$  of  $q$ , and call  $\frac{4}{9}$  of the sum or difference  $B$ .

3 U

Then

\* Again, by transposing the two first terms of the series in the value of  $S$ , we shall have  $S - 4abp \times 1 \pm \frac{d^2}{6} = 4abp \times: - \frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9} \&c.$  or  $\frac{4}{9} \times S - 4abp \times 1 \pm \frac{d^2}{6} = 4abp \times: - \frac{d^4}{2.5.9} \pm \frac{d^6}{4.7.9}$



Then multiply  $P$  by the difference between  $A$  and  $B$ , and  $\frac{2}{5}$  of the product will be the surface nearly.

That is,  $\frac{2}{5} P \times \overline{A-B} = \frac{2}{5} \times pfr \times \sqrt{1 \pm \frac{1}{3}q} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6}q}$   
 $=$  the surface nearly.

#### EXAMPLE.

Taking the same example, we shall have

I. *For the oblong spheroid.*

$$B = \frac{4}{9} \times 1 - \frac{1}{6}q = \frac{4}{9} \times 1 - \frac{9}{6 \times 25} = \frac{4}{9} \times \frac{47}{50} = .417777$$

&c. Then  $A-B = .93808 - .417777 = .5203$ . And  $\frac{2}{5} \times .5203 \times 6283.1853 = 5884.2029 =$  the surface of the oblong spheroid very nearly.

II. *For the oblate spheroid.*

$$B = \frac{4}{9} \times 1 + \frac{1}{6}q = \frac{4}{9} \times 1 + \frac{9}{6 \times 16} = \frac{4}{9} \times \frac{35}{32} = .48611$$

&c. Then  $A-B = 1.0897 - .4861 = .6036$ . And  $\frac{2}{5} \times .6036 \times 6283.1853 = 6826.6806 =$  the surface of the oblate spheroid nearly.

#### RULE VII.

*For both spheroids.*

From the sum or difference of 1 and  $\frac{1}{6}$  of  $q$ , according as it is for the oblate or oblong spheroid,

$$- \frac{5d^8}{4.8.81} \text{ \&c. which being taken from } S - 4abp \sqrt{1 \pm \frac{1}{3}d^2} = - \frac{d^4}{2.5.9}$$

$$\pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81} \text{ \&c. the remainder above found, leaves } \frac{5}{9} S - 4abp \times$$

$$\sqrt{1 \pm \frac{1}{3}d^2} - \frac{4}{9} \times 1 \pm \frac{1}{6}d^2 = \pm \frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81} \text{ \&c. and consequently}$$

$$S = \frac{2}{5} \times 4abp \times \sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9} \times 1 \pm \frac{1}{6}dd = \frac{4}{5}abp \times 9 \sqrt{1 \pm \frac{1}{3}dd} - 4 \times 1 \pm \frac{1}{6}dd$$

more nearly, which is rule 6.



roid, take  $\frac{1}{40}$  of the square of  $q$ , and call  $\frac{8}{27}$  of the remainder  $C$ .\*

Then from  $A$  subtract the sum of  $B$  and  $C$ , multiply the remainder by  $P$ , and  $\frac{27}{7}$  of the product will be the surface of the spheroid still nearer.

That is,  $\frac{27}{7} P \times \overline{A - B - C}$  = the surface nearly, where  $C = \frac{8}{27} \times 1 \pm \frac{1}{8}q - \frac{1}{40}qq$ , and the value of the other letters as before.

## E X A M P L E.

Taking still the same example, in which the axes are 40 and 50;

I. *For the oblong spheroid.*

Here again  $q = \frac{2}{5} = .36$ ,  $P = 6283.1853$ ,  $A = .93808$ ,  $B = .417777$  &c. and  $C = \frac{8}{27} \times 1 - \frac{1}{8}q - \frac{1}{40}qq = .277558$ . Then  $\frac{27}{7} P \times \overline{A - B - C} = 6283.1853 \times .93628 = 5882.8206$  = the surface of the oblong spheroid very near.

II. *For*

\* Farther, by transposing the first three terms of the original series, and reducing the result to make the first term of it equal to that of the last remainder, we shall have

$$\frac{8}{27} \times S - 4abp \times 1 \pm \frac{1}{8}d^2 - \frac{1}{40}d^4 = 4abp \times : \pm \frac{d^6}{2.7.27} - \frac{5d^8}{6.8.81}$$

&c. which being taken from the last remainder,  $\frac{8}{27} S - 4abp \times$

$$\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9} \times 1 \pm \frac{1}{8}dd = \pm \frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81} \text{ \&c. leaves } \frac{7}{27} S$$

$$- 4abp \times \sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9} \times 1 \pm \frac{1}{8}dd - \frac{8}{27} \times 1 \pm \frac{1}{8}d^2 - \frac{1}{40}d^4 =$$

$$- \frac{5d^8}{8.12.81} \text{ \&c. and consequently } S = \frac{27}{7} \times 4abp \times$$

$$\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9} \times 1 \pm \frac{1}{8}dd - \frac{8}{27} \times 1 \pm \frac{1}{8}d^2 - \frac{1}{40}d^4 = \frac{4}{7} abp \times$$

$$27 \sqrt{1 \pm \frac{1}{3}dd} - 12 \times 1 \pm \frac{1}{8}dd - 8 \times 1 \pm \frac{1}{8}d^2 - \frac{1}{40}d^4, \text{ nearer still, and this is rule 7.}$$



II. *For the oblate spheroid.*

Here  $q = \frac{2}{18} = .5625$ ,  $P = 6283.1853$ ,  $A = 1.089725$ ,  $B = .486111$ , and  $C = \frac{8}{27} \times 1 + \frac{1}{6}q - \frac{1}{40}qq = .32173$ . Then  $\frac{27}{7} \times A - B - C \times P = 1.0872 \times 6283.1853 = 6831.079 =$  the surface of the oblate spheroid very nearly.

## PROBLEM X.

*To find the curve surface of the frustum of a spheroid, contained between two planes cutting the spheroid perpendicular to the fixed axe, and one of them passing through the center.*

## RULE I.

*For both spheroids.*

Let  $f$  denote the fixed, and  $r$  the revolving axe; put  $p = 3.14159$  &c. and  $q = \frac{f f \infty r r}{f f}$ , as in the last problem; also  $h =$  the height  $CA$  of the frustum (see the figure to rule 1 of prob. 4) and  $z = \frac{4 q h h}{f f}$ ; then will the value of the surface be expressed by this series  $p r h \times : 1 \pm \frac{1}{2.3} z - \frac{1}{2.4.5} z^2 \pm \frac{3}{2.4.6.7} z^3 - \frac{3.5}{2.4.6.8.9} z^4$  &c. or  $p r h \times : 1 \pm \frac{1}{2.3} A z - \frac{3}{4.5} B z \pm \frac{3.5}{6.7} C z - \frac{5.7}{8.9} D z$  &c. where  $A, B, C$ , &c. are the several preceding terms, and the upper or under signs to be used according as it is the oblate or oblong spheroid.\*

E x-

\* See the investigation of rule 1 of the last problem.



## EXAMPLE I.

What is the surface of the frustum of a spheroid whose height is 15; the diameter of the greater end being 40, and that of the less 32?

Here  $CO = 40 =$  the revolving axe,  $2AB = 32$ , and  $BH = cA = 15$ ; hence  $HC = Cc - AB = 4$ , and  $HO = Cc + AB = 36$ ; and consequently, by case 3 prob. 2, we have as  $\sqrt{CH \times HO} = 12 : HB = 15 :: CO = 40 : TR = \frac{15 \times 40}{12} = 50 =$  the fixed axe, which shews that the frustum is that of an oblong spheroid.

Then, as in the example to rule 1 of the last problem,  $f = 50$ ,  $r = 40$ ,  $q = \frac{2}{25} = .36$ . But  $h = 15$ , and  $z = \frac{4qhb}{ff} = \frac{4 \times 9 \times 15 \times 15}{25 \times 50 \times 50} = .1296$ .

Hence the 1st term  $A = \quad \quad \quad + 1$ .

$$2d \quad B = \frac{1}{2.3} Az = -0.0216$$

$$3d \quad C = \frac{3}{4.5} Bz = -0.0004199 +$$

$$4th \quad D = \frac{3.5}{6.7} Cz = -0.0000195 -$$

$$5th \quad E = \frac{5.7}{8.9} Dz = -0.0000012 +$$

$$6th \quad F = \frac{7.9}{10.11} Ez = -0.0000001 -$$

the sum of the negative terms is  $-0.0220407$ ,

which taken from the first term  $1.0000000$ ,

leaves  $0.9779593$

for the value of the infinite series; which being drawn into  $prb = 3.14159 \&c. \times 40 \times 15 = 1884.955592$ , produces  $1843.4098512$  for the surface required.

3 X

Ex-



## EXAMPLE II.

It is required to find the curve surface of the frustum  $2cABC$  of a spheroid whose height is 16; the diameter of the greater end being 50 and that of the less 30.

Here the revolving axe  $TR = 50$ ,  $HC = AB = 16$ , and  $BH = Ac = 15$ ; hence  $AR = Rc - cA = 10$ , and  $AT = Tc + cA = 40$ ; then, by case 3 prob. 2, we shall have as  $\sqrt{AR \times AT} = 20 : AB = 16 :: TR = 50 : CO = \frac{50 \times 16}{20} = 40$  = the fixed axe, which indicates the frustum to be that of an oblate spheroid.

Wherefore  $f = 40$ ,  $r = 50$ ,  $q = \frac{2}{15}$ ,  $h = 16$ , and  $z = \frac{4qhb}{ff} = \frac{4 \times 9 \times 16 \times 16}{16 \times 40 \times 40} = \frac{36}{100} = .36$ , which being the same as the converging quantity  $q$  in the example to rule 1 of the last prob. the several terms of the series must be the same as there found, viz.

|                         |                                   |
|-------------------------|-----------------------------------|
| the 1st term            | $A = +1$                          |
| 2d                      | $B = +0.06$                       |
| 3d                      | $C = -0.00324$                    |
| 4th                     | $D = +0.0004166-$                 |
| 5th                     | $E = -0.000072900$                |
| 6th                     | $F = +0.00000150+$                |
| 7th                     | $G = -0.00000035-$                |
| 8th                     | $H = +0.00000009-$                |
| 9th                     | $I = -0.00000002+$                |
| sum of the affir. terms | $= +1.0604325$                    |
| sum of the neg. terms   | $= -0.0033166$                    |
| their difference        | $= 1.0571159$ the value of<br>the |



the series; which being drawn into  $prb = 3.14159$  &c.  $\times 50 \times 16 = 2513.27412287$ , will produce 2656.822036 for the surface of the oblate frustum required.

## R U L E II.

*For both spheroids.*

1. Multiply the square root of the difference of the squares of the axes by the height of the frustum, divide the product by the square of the fixed axe, and call the double of the quotient  $q$ .

That is,  $q = \frac{2b\sqrt{ff\infty rr}}{ff}$ ,  $b$  being the height of the frustum,  $f$  the fixed, and  $r$  the revolving axe.

2. In the oblong spheroid, call the square root of the difference between 1 and the square of  $q$ ,  $A$ . But in the oblate spheroid, let  $A$  be the root of the sum of 1 and the square of  $q$ .

That is,  $A = \begin{cases} \sqrt{1 - qq} & \text{in the oblong,} \\ \sqrt{1 + qq} & \text{in the oblate.} \end{cases}$

3. In the oblong spheroid, let the product of .01745329 and the degrees whose sine is  $q$  be called  $B$ . But in the oblate spheroid, let  $B$  be the product of 2.30258509 and the logarithm of the sum of  $q$  and the root of the sum of 1 and the square of  $q$ .

That is,  $B = \begin{cases} .0174 \text{ \&c. } \times \text{degrees whose sine is } q & \text{in the oblong,} \\ 2.302 \text{ \&c. } \times \log. \text{ of } q + \sqrt{1 + qq} & \text{in the oblate.} \end{cases}$

4. Divide



4. Divide  $B$  by  $q$ , to the quotient add  $A$ , multiply the sum by the continual product of the height, the revolving axe, and the number 3.14159 &c. and the half of the last product will be the surface required.

That is,  $\frac{1}{2}prb \times \overline{A + \frac{B}{q}} = \text{the surface.}^*$

#### EXAMPLE I.

Let there be taken here the first example to the last rule, in which  $f$  is = 50,  $r = 40$ , and  $b = 15$ .

Then  $\frac{2b\sqrt{ff-rr}}{ff} = \frac{30 \times 30}{50 \times 50} = \frac{9}{25} = .36 = q$ . And  $A = \sqrt{1 - qq} = \sqrt{1 - \frac{9}{25}} = \frac{4\sqrt{34}}{25} = .9329523$ . Also the sine ( $q =$ ) .36 answers to 21.1001965 degrees, which being drawn into .01745329, will produce .3682678 =  $B$ .

Hence  $\overline{A + \frac{B}{q}} \times \frac{1}{2}prb = .9779593 \times 1884.955592 = 1843.4098512 = \text{the surface, the same as before.}$

#### EXAMPLE II.

Let there be taken the oblate frustum in the second example to the preceding rule, in which the axes are 50 and 40, and the height 16.

Here then  $f = 40$ ,  $r = 50$ , and  $b = 16$ ; hence  $\frac{2b\sqrt{rr-ff}}{ff} = \frac{32 \times 30}{40 \times 40} = \frac{3}{5} = .6 = q$ ; and  $\sqrt{1 + qq} = \sqrt{1 + \frac{9}{25}} = \frac{1}{5}\sqrt{34} = 1.16619038 = A$ ; likewise  $q +$

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\* For this is the fluent of the fluxion of the surface in the investigation of the rules in the last problem.



$q + \sqrt{1 + qq} = 1.76619038$ , whose logarithm is  $.24703757$ , which being drawn into  $2.30258509$  will produce  $.56882503 = B$ .

Then  $A + \frac{B}{q} \times \frac{1}{2}prb = 1.057116 \times 2513.27412287 = 2656.822287 =$  the surface, nearly the same as before.

## R U L E III.

*For the oblong spheroid.*

1. Divide the square of the fixed axe by the root of the difference of the squares of the axes, and call the quotient  $d$ .

That is,  $d = \frac{ff}{\sqrt{ff - rr}}$ .

2. Find the area of the circular zone whose diameter is  $d$ , and its height from the center equal to the height  $b$  of the frustum of the spheroid, and call the area of that zone  $z$ .

3. Then as the diameter of the circle is to the circular zone, so is the circumference of the greatest circle of the spheroid to the surface of its frustum.

That is, as  $d : z :: pr : \frac{prz}{d} =$  the surface of the frustum.\*

## E X A M P L E.

It is required to find the surface of the frustum of an oblong spheroid whose height is 15, the axes being 50 and 40.

Here  $\frac{ff}{\sqrt{ff - rr}} = \frac{50 \times 50}{30} = \frac{250}{3} = 83\frac{1}{3} = d$  the diameter of the circle. 3 Y And,

\* This is the same with rule 3 to the last problem, only expressed in other terms.



And, by means of the table of circular areas, the area of the zone will be found to be  $\cdot 17603268 \times \frac{250}{3}$ .

Also  $pr = 3\cdot 14159 \text{ \&c.} \times 40 =$  the greatest circumference of the spheroid.

Then, as  $\frac{250}{3} : \cdot 17603268 \times \frac{250}{3} :: 3\cdot 14159 \text{ \&c.} \times 40 : \cdot 17603268 \times 3\cdot 14159 \text{ \&c.} \times 40 \times \frac{250}{3} = 1843\cdot 4099 =$  the surface, nearly the same as before.

#### R U L E IV.

*For the oblong spheroid.*

Let  $BKMP$  be the frustum whose surface is sought, [figure to rule 3 of the last prob.]  $NLO$  a quadrant of the circle mentioned in the last problem; draw  $CL$ , and  $IV$  perpendicular to it.

Then the surface of the frustum will be equal to the circle whose radius is a mean proportional between  $BC$  and  $NL + IV$ . Or the surface will be equal to the continual product of  $NL + IV$ ,  $BC$ , and  $3\cdot 14159 \text{ \&c.}$ \*

#### E X A M P L E.

Let there be taken here the same example as before, in which the axes are 50 and 40, and the height 15.

Then since, by the example to the last rule,  $CL = CN = \frac{125}{3}$ , we shall have as  $CL = \frac{125}{3} : LY = 15 ::$   
I =

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\* This is proved at rule 4 to the last problem.



$1 = \text{radius} : \frac{2}{3} = .36 = \text{the sine of } 21.1001965 \text{ degrees, which are those contained in the arc } NL;$   
 and therefore  $.01745329 \times 21.1001965 \times \frac{125}{3} = 15.3444958 = NL.$  And, by similar triangles,  
 (since  $IL = \sqrt{LC^2 - CI^2} = \sqrt{\left(\frac{125}{3}\right)^2 - 15^2} = \frac{20\sqrt{34}}{3} = 38.8730126$ ) as  $CL = \frac{125}{3} : LI :: CI = 15 : IV = 13.9942845.$

And hence  $NL + IV \times 3.14159 \text{ \&c.} \times BC = 29.3387803 \times 3.14159 \text{ \&c.} \times 20 = 1843.409933 = \text{the surface, nearly the same as before.}$

## R U L E V.

*For both spheroids.*

The 5th rule to the last problem will serve, if instead of  $q$  we use  $z$ , as found in rule I of this problem, and  $b$  the height instead of  $f$  the fixed axe.

That is,  $prb \sqrt{1 \pm \frac{1}{3}z} = \text{the surface nearly.}$

## E X A M P L E I.

Taking the same example as in the last rule, we have  $r = 40$ ,  $b = 15$ , and  $z$  (found in the first example to rule I)  $= .1296.$

Consequently  $prb \sqrt{1 - \frac{1}{3}z} = 1884.955592 \times .97816 = 1843.7879 = \text{the surface nearly.}$

## E X A M P L E II.

If we take the oblate frustum, whose height is 16, and the axes 40 and 50; we shall have

$r =$



$r=50$ ,  $b=16$ , and  $z$  (found in the second example to rule I)  $= .36$ .

Then  $prb \sqrt{1 + \frac{1}{3}z} = 2513.27412287 \times 1.0583 = 2659.797 =$  the surface nearly.

### R U L E VI.

*For both spheroids.*

This rule will be the same as the 6th rule to the last problem, using  $z$  instead of  $q$ , and  $b$  for  $f$  as in the last rule.

That is,  $\frac{2}{5}P \times \overline{A-B} = \frac{2}{5}prb \times \sqrt{1 \pm \frac{1}{3}z} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6}z} =$  the surface nearly.

### E X A M P L E I.

Taking the 1st example to the last rule, we have  $P = 1884.955592$ ,  $A = \sqrt{1 - \frac{1}{3}z} = .97816157$ , and  $B = \frac{4}{9} \times \overline{1 - \frac{1}{6}z} = .43484444$ .

Then  $\frac{2}{5}P \times \overline{A-B} = 1884.955592 \times .97797 = 1843.4298 =$  the surface nearly.

### E X A M P L E II.

Taking here the 2d example to the last rule, we have  $P = prb = 2513.27412287$ ,  $A = \sqrt{1 + \frac{1}{3}z} = 1.0583005$ , and  $B = \frac{4}{9} \times \overline{1 + \frac{1}{6}z} = .47\frac{1}{9}$ .

Then  $\frac{2}{5}P \times \overline{A-B} = 2513.27412287 \times 1.0569409 = 2656.3821 =$  the surface nearly.

### R U L E



## R U L E VII.

*For both spheroids.*

This rule, also, will be the same with the 7th to the last problem, using  $z$  and  $b$  instead of  $q$  and  $f$ .

That is,  $\frac{27}{7} P \times \overline{A-B-C} = \frac{27}{7} p r b \times \sqrt{1 \pm \frac{1}{3}z - \frac{4}{9} \times 1 \pm \frac{1}{6}z - \frac{7}{27} \times 1 \pm \frac{1}{6}z - \frac{1}{40}zz} =$  the surface very nearly.

## E X A M P L E I.

Taking, still, the same example of the oblong spheroid, we shall have, as before,  $P = 1884.955592$ ,  $A = .97816157$ , and  $B = .43484444$ ; also  $C = \frac{8}{27} \times 1 - \frac{1}{6}z - \frac{1}{40}zz = .28977188$ .

Then  $\frac{27}{7} P \times \overline{A-B-C} = 1884.955592 \times .97796 = 1843.411 =$  the surface very nearly.

## E X A M P L E II.

Let there be taken, again, the oblate frustum, in which, as before,  $P = 2513.27412287$ ,  $A = 1.0583005$ , and  $B = .47\frac{1}{9}$ ; also  $C = \frac{8}{27} \times 1 + \frac{1}{6}z - \frac{1}{40}zz = .31311407$ .

Then  $\frac{27}{7} P \times \overline{A-B-C} = 2513.27412287 \times 1.05714 = 2656.8825 =$  the surface very nearly.

## S C H O L I U M.

It is evident that the double of the frustum will give the middle zone; and that the frustum being added to or taken from half the spheroid, will give the greater or less segment.



## PROBLEM XI.

\* To find the solidity of a spheroid.

## RULE I.

Multiply continually together the fixed axe, the square of the revolving axe, and the number  $\cdot 52359877$  &c. ( $= \frac{1}{6}$  of  $3\cdot 14159$  &c.), and the last product will be the content required.

That

## \* DEMONSTRATION.

Put  $f = BI$  the fixed semi-axe, [fig. to prop. 1 sect. 2]  $r = IM$  the revolving semi-axe of the spheroid,  $a = SI$  any semi-diameter of the section  $NBM$ ,  $b = IK$  its semi-conjugate,  $y = AE$  an ordinate to the diameter  $SI$ , or a semi-axe of the elliptic section  $AFC$  parallel to  $KL$ , and  $z = EF$  its other semi-axe, also  $x = EI$ ,  $s =$  the sine of the angle  $SEA$  or of the angle  $SIK$  to the radius 1, and  $p = 3\cdot 14159$  &c.

Then, by the property of the ellipse  $KSL$ ,  $aa : bb :: aa - xx :$   
 $bb \times \frac{aa - xx}{aa} = yy$ ; and, by prop. 1 sect. 2,  $b : r :: y : \frac{ry}{b} = z :$   
 but the fluxion of the solid  $KACL$  is  $psyzx =$  (by writing for  
 $z$  its value  $\frac{ry}{b}$ )  $\frac{psryyx}{b} =$  (by substituting for  $yy$  its value  $bb \times$   
 $\frac{aa - xx}{aa}$ )  $psrxx \times \frac{aa - xx}{aa} =$  (by putting for  $abs$  its value  $rf$ )  
 $pfrrx \times \frac{aa - xx}{aaa}$ ; and hence the fluent or  $pfrrx \times \frac{aa - \frac{1}{3}xx}{aaa}$   
 will be the value of the frustum  $KACL$ ; which, when  $EI$  or  $x$   
 becomes  $IS$  or  $a$ , gives  $\frac{2}{3}pfrr$  for the value of the semi-spheroid  
 $KSL$ ; or the whole spheroid  $= \frac{1}{6}pFRR$ , putting  $F$  and  $R$  for the  
 whole fixed and revolving axes. Q.E.D.

## COROLLARY I.

From the foregoing demonstration it appears that the value of  
 the general frustum  $KACL$  is expressed by  $pfrrx \times \frac{aa - \frac{1}{3}xx}{aaa}$ .

And



That is,  $\frac{1}{6}pttc$  = the oblate, and  $\frac{1}{6}ptcc$  = the oblong spheroid, where  $p = 3.14159$  &c.  $t$  = the transverse, and  $c$  = the conjugate axe of the generating ellipse.

## RULE

And if for  $fr$  be substituted its value  $abs$ , it will also be expressed by  $pbrsx \times \frac{aa - \frac{1}{3}xx}{aa}$ .

Also, if for  $aa$  be put its value  $\frac{bbxx}{bb - yy}$ , the last expression will become  $prrsx \times \frac{2bb + yy}{3b}$  or  $pssx \times \frac{2br + \frac{ryy}{b}}{3}$ ; which, by writing  $z$  instead of its value  $\frac{ry}{b}$ , gives  $pssx \times \frac{2br + yz}{3}$  for the value of the frustum, viz. The sum of the area of the less end and twice that of the greater, drawn into one-third of the altitude or distance of the ends.—And out of this last expression may be expunged any one of the four quantities  $b, r, y, z$ , by means of the proportion  $b : r :: y : z$ .

When the ends of the frustum are perpendicular to the fixed axe, then  $a$  is  $= f$ , and the value of the frustum becomes  $prrsx \times \frac{ff - \frac{1}{3}xx}{ff}$  for the value of the frustum whose ends are perpendicular to the fixed axe, its altitude being  $x$ .

And when the ends of the frustum are parallel to the fixed axe,  $a$  is  $= r$ , and the expression for such a frustum becomes  $pfxx \times \frac{rr - \frac{1}{3}xx}{r}$ .

## COROLLARY II.

If to or from  $\frac{2}{3}pfr$ , the value of the semi-spheroid, be added or subtracted  $pfrsx \times \frac{aa - \frac{1}{3}xx}{aaa}$ , the value of the general fru-

stum  $KACL$ , there will result  $pfrhb \times \frac{a - \frac{1}{3}b}{aaa}$  for the value of a general segment, either greater or less than the semi-spheroid, whose height, taken upon the diameter passing through its vertex and center of its base, is  $b = a \pm x$ .

When



## R U L E II.

Multiply the area of the generating ellipse by  $\frac{2}{3}$  of the revolving axe, and the product will be the content of the spheroid.

That

When  $a$  coincides with  $f$ , the above expression becomes  $prrbb$   $\times \frac{f - \frac{1}{3}b}{ff}$  for the value of a segment whose base is perpendicular to the fixed axe.—And here if we put  $R$  for the radius of the segment's base, and for  $rr$  its value  $\frac{RRff}{2fb - bb}$ , the expression for the said segment will become  $\frac{1}{3}pRRb \times \frac{3f - b}{2f - b}$ .

And when  $a$  coincides with  $r$ , the general expression will become  $pfbb \times \frac{r - \frac{1}{3}b}{r}$  for the value of the segment whose base is parallel to the fixed axe.—And if we put  $F, R$ , for the two semi-axes, of the elliptic base of this segment, respectively corresponding or parallel to  $f, r$ , the semi-axes of the generating ellipse, when parallel to the base of the segment, and for  $\frac{f}{r}$  and  $r$  substitute their values  $\frac{F}{R}$  and  $\frac{RR + bb}{2b}$ , the said frustum will be expressed by  $\frac{1}{3}pFb \times \frac{3RR + bb}{2R}$ , in which the dimensions of itself only are concerned.

## COROLLARY III.

A semi-spheroid is equal to  $\frac{2}{3}$  of a cylinder or to double a cone of the same base and height, or they are in proportion as the numbers 3, 2, 1. For the cylinder is  $= 4nfr r = \frac{12}{3}nfr r$ , the semi-spheroid  $= \frac{8}{3}nfr r$ , and the cone  $= \frac{4}{3}nfr r$ .

## COROLLARY IV.

When  $f = r$ , the spheroid becomes a sphere, and the expression  $\frac{8}{3}fnrr$  for the semi-spheroid becomes  $\frac{8}{3}nr^3$  for the semi-sphere, as in prob. 24 sect. 1.—And in like manner,  $f$  and  $r$  being supposed equal to each other in the values of the frustums and segments of



That is,  $\frac{2}{3}tA =$  the oblate, and  $\frac{2}{3}cA =$  the oblong spheroid; where  $A$  is the area of the ellipse.—And this rule is evidently taken from the former.

4 A

Ex-

of a spheroid, in the preceding corollaries, will give the values of the like parts of a sphere.

COROLLARY V.

All spheres and spheroids are to each other as the fixed axes drawn into the squares of the revolving axes.

COROLLARY VI.

Any spheroids, and spheres, of the same revolving axe, as also their like or corresponding parts cut off by planes perpendicular to the said common axe, are to one another as their other or fixed axes. This follows from the foregoing corollaries.

COROLLARY VII.

But if their fixed axes be equal, and their revolving axes unequal, the spheroids and spheres, with their like parts terminated by planes perpendicular to the common fixed axe, will be to each other as the squares of their revolving axes.

COROLLARY VIII.

An oblate is to an oblong spheroid generated from the same ellipse, as the longer axe of the ellipse is to the shorter. For, if  $T$  be the transverse axe and  $C$  the conjugate, the oblate spheroid will be  $= \frac{2}{3}\pi T^2 C$ , and the oblong  $= \frac{2}{3}\pi C^2 T$ ; and these quantities are in the ratio of  $T$  to  $C$ .

COROLLARY IX.

And if about the two axes of an ellipse be generated two spheres and two spheroids, the four solids will be continual proportionals whose common ratio will be that of the two axes of the ellipse; that is, as the sphere upon the greater axe, or greater sphere, is to the oblate spheroid, so is the oblate spheroid to the oblong spheroid, so is the oblong spheroid to the less sphere, and so is the transverse axe to the conjugate. For these four bodies will be as  $T^3$ ,  $T^2C$ ,  $TC^2$ ,  $C^3$ , where each term is to the consequent one as  $T$  to  $C$ .



## EXAMPLE.

Required the content of an oblate and of an oblong spheroid, the axes being 50 and 30.

First,  $50 \times 30 \times .78539816 = 1178.09724 =$  the area of the ellipse.

Then,  $1178.09724 \times \frac{2}{3} \times 30 = 23561.9448 =$  the oblong spheroid.

And  $1178.09724 \times \frac{2}{3} \times 50 = 39269.908 =$  the oblate one.

## PROBLEM XII.

*To find the content of the frustum of a spheroid, its ends being perpendicular to one of the axes, and one of them passing through the center.*

## RULE I.

To the area of the less end add twice that of the greater, multiply the sum by the altitude of the frustum, and  $\frac{1}{3}$  of the product will be the content.—By corollary 1 to the last problem.

That is,  $2D^2 + d^2 \times \frac{1}{3}an =$  the frustum whose ends are perpendicular to the fixed axe, where  $D$  is the diameter of the greater end,  $d$  is that of the less,  $a$  the altitude, and  $n = .785398 \&c.$

And  $2TC + tc \times \frac{1}{3}an =$  the frustum whose ends are parallel to the fixed axe, where  $T$  and  $C$  are the transverse and conjugate axes of the greater end, and  $t$  and  $c$  those of the less end.

*Note.* It is evident that the double of the frustum will give the content of the zone, or spheroidal cask.

Ex-



## EXAMPLE I.

There is a cask in the form of the middle frustum or zone of an oblong spheroid, the bung diameter is 30, the head diameter 18, and the length of the cask is 40 inches; what is the content in ale and wine gallons.

Here  $D = 30$ ,  $d = 18$ , and  $a = 40$ .

Wherefore  $2D^2 + d^2 \times \frac{1}{3}an = 2 \times 30^2 + 18^2 \times .2618 \times 40 = 2124 \times 10.472 = 22242.528 =$  the content in inches.

Then, since the gallon ale measure contains 282 cubic inches, and the wine gallon 231,

We have  $\frac{22242.528}{282} = 78.874$  ale gallons.

And  $\frac{22242.528}{231} = 96.288$  wine gallons.

## EXAMPLE II.

If a vessel, in the form of the middle frustum of an oblate spheroid, have the diameter of each end 40, in the middle 50, and its length 18 inches; what is its content in ale and wine gallons.

Here  $2D^2 + d^2 \times \frac{1}{3}an = 2 \times 50^2 + 40^2 \times .2618 \times 18 = 118800 \times .2618 = 31101.84$  cubic inches.

Then  $\frac{31101.84}{282} = 110.29$  ale gallons,

And  $\frac{31101.84}{231} = 134.64$  wine gallons.

## EXAMPLE III.

In the frustum of an oblong spheroid, the greater end is the generating ellipse whose axes are 50  
and



and 30, the axes of its less end 40 and 24, and its height nine inches; required the content in winchester bushels.

Here  $\overline{2TC + tc} \times \frac{1}{3}an = 2 \times 50 \times 30 + 40 \times 24 \times .2618 \times 9 = 9330.552$  cubic inches.

But 268.8 inches = a gallon, or 2150.4 = a corn bushel.

And therefore  $\frac{9330.552}{2150.4} = 4.339$  bushels.

#### EXAMPLE IV.

In the frustum of an oblate spheroid the greater end is the generating ellipse whose axes are 50 and 30, and the height is 20 inches; required the solidity.

Here, by the nature of the ellipse, as  $50 : 30 :: \sqrt{25 + 20 \times 25 - 20} = \sqrt{45 \times 5} = 15 : \frac{15 \times 30}{50} = 9$  = the semi-conjugate axe of the less end.

And, by prop. 1 sect. 2, as  $30 : 50 :: 9 : 15$  = the semi-transverse axe of the less end.

Then  $\overline{2TC + tc} \times \frac{1}{3}na = 2 \times 50 \times 30 + 30 \times 18 \times .2618 \times 20 = 118 \times 30 \times 20 \times .2618 = 70800 \times .2618 = 18535.44$  cubic inches.

#### RULE II.

From 3 times the square of the semi-axe perpendicular to the ends of the frustum subtract the square of the height of the frustum, then multiply the difference by  $\frac{1}{3}$  of the height, and the product by 3.14159 &c. and call the last product  $P$ .—  
Then

1. If



1. *If the ends be parallel to the fixed axe,* As the revolving axe is to the fixed axe, so will  $P$  be to the content of the frustum.

2. *When the ends are perpendicular to the fixed axe,* As the square of the fixed axe is to the square of the revolving axe, so is  $P$  to the content of the frustum.

That is,  $\frac{3ff-bb}{3ff} \times phrr$  will be the frustum whose ends are perpendicular. And  $\frac{3rr-bb}{3r} \times phf$  = that whose ends are parallel to the fixed axe;  $f$  being the semi-fixed and  $r$  the semi-revolving axe,  $h$  the height, and  $p = 3.14159$  &c.

## EXAMPLE I.

If the axes of an oblong spheroid be 50 and 30, required the content of a frustum whose ends are perpendicular to the fixed axe, and one of them passing through the center, the height of the frustum being 20 inches.

Here  $\frac{3ff-bb}{3ff} \times phrr = \frac{3 \times 25^2 - 20^2}{3 \times 25^2} \times 3.14159$  &c.  
 $\times 20 \times 15^2 = 1475 \times 3.14159$  &c.  $\times \frac{1}{5} = 11121.23799$   
 = the content required.

## EXAMPLE II.

If from the same spheroid be cut a frustum whose height is 9, the ends being parallel to the fixed axe, and one of them passing through the center, what will be its content.

Here  $\frac{3rr-bb}{3r} \times phf = \frac{3 \times 15^2 - 9^2}{3 \times 15} \times 9 \times 25 \times 3.14159$   
 &c. =  $594 \times 5 \times 3.14159$  &c. =  $9330.53018$  = the content required.



## EXAMPLE III.

If from an oblate spheroid whose axes are 50 and 30, a frustum whose height is 9 and whose ends are perpendicular to the fixed axe be cut, what will be its solidity.

Here  $\frac{3ff - bb}{3ff} \times phrr = \frac{3 \times 15^2 - 9^2}{3 \times 15^2} \times 9 \times 25^2 \times 3.14159 = 594 \times \frac{25}{3} \times 3.14159 \text{ \&c.} = 15550.88363$   
= the content required.

## EXAMPLE IV.

In the same spheroid it is required to find the content of a frustum whose height is 20, the ends being parallel to the fixed axe.

Here  $\frac{3rr - bb}{3r} \times phf = \frac{3 \times 25^2 - 20^2}{3 \times 25} \times 20 \times 15 \times 3.14159 \text{ \&c.} = 1475 \times 4 \times 3.14159 \text{ \&c.} = 18535.39665$   
= the content required.

## PROBLEM XIII.

*To find the content of a spheroidal cask, not full, standing upon its end, the axe being perpendicular to the horizon. That is, of the frustum of an oblong spheroid, the ends being perpendicular to the axe, but neither of them passing through the center.*

## RULE.

Multiply the difference of the squares of the bung and head diameters by 4 times the square of the difference between the height of the liquor and half the length of the cask, and divide the product



duct by the square of the length of the cask; subtract the quotient from 3 times the square of the bung diameter, and multiply the remainder by the aforesaid difference between the height of the liquor and  $\frac{1}{2}$  the length of the cask; then the product multiplied by .261799 &c. ( $= \frac{1}{3}$  of .78539 &c.) will produce the quantity by which the cask is more or less than half full; and which, therefore, being added to or taken from half the content of the cask, will give the content of the part filled.

That is,  $2b^2 + b^2 \times \frac{1}{2}l \pm 3b^2 - \frac{4dd}{ll} \times b^2 - b^2 \times d \times \frac{1}{3}n =$  the content of the part filled, called the ullage, using the upper or under signs according as the cask is more or less than half full; where  $b$  and  $b =$  the bung and head diameters,  $l =$  the length of the cask,  $n = .785398$  &c. and  $d = \frac{1}{2}l \sin w$ ,  $w$  being the wet part of  $l$  or the height of the liquor in the cask.\*

Ex-

## \* DEMONSTRATION.

For, if  $D$  be the diameter at the surface of the liquor, by the property of the ellipse  $b^2 - b^2 : \frac{1}{4}l^2 :: b^2 - D^2 : d^2$ , and hence  $D^2 = b^2 - \frac{4dd}{ll} \times b^2 - b^2$ .

But, by the last problem,  $2b^2 + b^2 \times \frac{1}{6}ln =$  half the content of the cask, and  $3b^2 - \frac{4dd}{ll} \times b^2 - b^2 \times \frac{1}{3}dn =$  the frustum by which the part filled is more or less than half the cask.

Consequently  $2b^2 + b^2 \times \frac{1}{2}l \pm 3b^2 - \frac{4dd}{ll} \times b^2 - b^2 \times d \times \frac{1}{3}n =$  the part filled.



## EXAMPLE I.

If a spheroidal cask whose head and bung diameters are 18 and 30 inches, and length 40 inches, be filled to the height of 30 inches; how many ale and wine gallons are in it.

Here  $b = 30$ ,  $h = 18$ ,  $l = 40$ ,  $w = 30$ , and  $d = \frac{1}{2}l \cap w = 30 - 20 = 10$ .

Then  $3b^2 - \frac{4dd}{ll} \times b^2 - h^2 \times d = 2700 - \frac{400}{1600} \times 48 \times 12$   
 $\times 10 = 2700 - 144 \times 10 = 25560$ .

And  $2b^2 + h^2 \times \frac{1}{2}l = 1800 + 18 \times 18 \times 20 = 42480$ .

Consequently  $42480 + 25560 \times \frac{1}{3}n = 68040 \times .2617993878 = 17812.8303458 =$  the content in inches; and being divided by 282 and 231, gives 63.166065 ale, and 77.111819 wine gallons.

## EXAMPLE II.

If the height of the liquor in the same cask be only 10 inches, required the ullage.

Here  $d = \frac{1}{2}l \cap w = 20 - 10 = 10$  the same as before, and therefore the part which in the last example was added must here be subtracted; so that  $42480 - 25560 \times \frac{1}{3}n = 16920 \times .2617993878 = 4429.6456415$  the content in inches = 15.707963 ale, and 19.175955 wine gallons.

PRO-



## PROBLEM XIV.

*To find the solidity of the segment of a spheroid whose base is perpendicular to one of the axes.*

## RULE I.

From 3 times the semi-axis perpendicular to the base of the segment, take the height of the segment, multiply the remainder into  $\frac{1}{3}$  of the square of the height, then multiply the product by 3.14159 &c. and call the last product  $\mathcal{Q}$ .

Divide the axis which is parallel to the base by the other axis, and call the quotient  $q$ .

Then for the segment whose base is perpendicular to the fixed axis, multiply  $\mathcal{Q}$  into the square of  $q$ ; and for the other segment, multiply  $\mathcal{Q}$  into  $q$ .

That is,  $\mathcal{Q}qq = \frac{3f-b}{3ff} \times prrbh =$  the segment whose base is perpendicular to the fixed axis;—And  $\mathcal{Q}q = \frac{3r-b}{3r} \times pfbh =$  the other segment whose base is parallel to the fixed axis; where  $b$  is the height of the segment, and the other symbols as in the last problem.

## EXAMPLE I.

If from an oblong spheroid, whose axes are 50 and 30, be cut a segment whose base is perpendicular to the fixed axis, its height being 5; required the content of it.

Here  $\frac{3f-b}{3ff} \times prrbh = \frac{3 \times 25 - 5}{3 \times 25^2} \times 15^2 \times 5^2 \times 3.14159$   
&c.  $= 70 \times 3 \times 3.14159$  &c.  $= 659.73445 =$  the content required.



## EXAMPLE II.

If from the same spheroid be cut a segment whose height is 6, what will its content be, supposing its base to be parallel to the fixed axe.

Here  $\frac{3r-b}{3r} \times pfbh = \frac{3 \times 15 - 6}{3 \times 15} \times 25 \times 6^2 \times 3.14159$   
 &c. =  $39 \times 20 \times 3.14159$  &c. =  $2450.44226$  = the content required.

## EXAMPLE III.

Required the solidity of the segment of an oblate spheroid whose axes are 50 and 30, the height of the segment being 6, and its base perpendicular to the fixed axe.

Here  $\frac{3f-b}{3ff} \times prrbh = \frac{3 \times 15 - 6}{3 \times 15 \times 15} \times 25 \times 25 \times 6 \times 6 \times 3.14159$   
 &c. =  $1300 \times 3.14159$  &c. =  $4084.07044$  = the content required.

## EXAMPLE IV.

To find the content of a segment of the same spheroid, its base being parallel to the fixed axe, and its height 5.

Here  $\frac{3r-b}{3r} \times pfbh = \frac{3 \times 25 - 5}{3 \times 25} \times 15 \times 5 \times 5 \times 3.14159$   
 &c. =  $70 \times 5 \times 3.14159$  &c. =  $1099.55742$  = the content required.

## RULE II.

*For the segment whose base is parallel to the fixed axe, having given its height and the diameters of its end.*

To the square of the height add 3 times the square of the less or greater semi-axe of the base  
 according



according as it is the segment of an oblong or oblate spheroid, divide the sum by the same semi-axe, and multiply the quotient by the other semi-axe, the product by the height, and this product by ( $\frac{1}{8}$  of 3.14159 &c.) .52359 &c. will give the content of the segment.

That is,  $\frac{3A^2 + b^2}{6A} \times pBb =$  the content of the segment, where  $b$  is its height,  $A$  the less or greater semi-diameter of the base according as the segment is that of an oblong or oblate spheroid,  $B$  is the other semi-diameter, and  $p = 3.14159$  &c.

## EXAMPLE I.

It is required to find the content of the segment of an oblong spheroid, whose base is parallel to the fixed axe, its height being 6, and the axes of its elliptic base 40 and 24.

Here  $\frac{3AA + bh}{6A} \times pBb = \frac{3 \times 12^2 + 6^2}{6 \times 12} \times 20 \times 6 \times 3.14159$   
&c.  $= 13 \times 60 \times 3.14159$  &c.  $= 2450.44226 =$  the content required.

## EXAMPLE II.

Required the content of the segment of an oblate spheroid, whose base is parallel to the fixed axe, its height being 5, and the diameters of its base 18 and 30.

Here  $\frac{3A^2 + b^2}{6A} \times pBb = \frac{3 \times 15^2 + 5^2}{6 \times 15} \times 9 \times 5 \times 3.14159$   
&c.  $= 70 \times 5 \times 3.14159$  &c.  $= 1099.55742 =$  the content required.

RULE



## R U L E III.

*For a segment whose base is perpendicular to the fixed axe, having given the height, the diameter of its base, and a diameter in the middle between its base and vertex.*

To the square of the diameter of the base add 4 times the square of the diameter in the middle, or the square of twice this diameter; multiply the sum into the height, and the product multiplied by ( $\frac{1}{8}$  of .78539 &c.) .13089969 &c. will give the content.

That is,  $\overline{D^2 + 4d^2} \times \frac{1}{8}nb =$  the content of the segment;  $D$  being the diameter of the base,  $d =$  the diameter in the middle,  $b =$  the height, and  $n = .785398$  &c.\*

E x-

## \* DEMONSTRATION.

By rule I the segment is  $= 4nrrbh \times \frac{3f-b}{3ff}$ . But, by the nature of the ellipse,  $D$  is  $= \frac{2r\sqrt{2fb-hb}}{f}$ , and  $d = \frac{2r\sqrt{fb-\frac{1}{4}bb}}{f}$ ; hence  $f = b \times \frac{4dd-DD}{8dd-4DD}$ , and  $\frac{r}{f} = \frac{\sqrt{4dd-2DD}}{2b}$ ; which values being substituted in the above value of the segment, gives  $\frac{1}{8}nb \times \overline{DD + 4dd}$  for the value of the segment in terms of  $D$ ,  $d$ , and  $b$ .

## S C H O L I U M.

This theorem I have investigated in order to express the value of this segment independent of the axes of the spheroid. For this purpose I was under a necessity of introducing another dimension of the segment, because it is not determinable from its base and height alone as the other segment was.



## EXAMPLE I.

What is the content of the segment of an oblong spheroid, whose base is perpendicular to the fixed axe, its height being 5, the diameter of its base 18, and its middle diameter  $3\sqrt{19}$ .

Here  $\overline{DD} + 4dd \times \frac{1}{6}nb = \overline{18^2 + 6\sqrt{19}^2} \times \frac{5}{6} \times .785398 \&c. = 3^2 + 19 \times 30 \times .785398 \&c. = 210 \times 3.14159 \&c. = 659.73445 = \text{the content required.}$

## EXAMPLE II.

Required the content of the spheroidal segment whose height is 6, its base diameter 40, and diameter in the middle 30, the base being perpendicular to the fixed axe of the spheroid.

Here  $\overline{DD} + 4dd \times \frac{1}{6}hp = \overline{40^2 + 60^2} \times \frac{6}{6} \times .785398 \&c. = \overline{2^2 + 3^2} \times 20^2 \times .785398 = 1300 \times 3.14159 \&c. = 4084.07044 = \text{the content required.}$

## PROBLEM XV.

*To find the content of the second segment of a spheroid.*

## RULE.

As a sphere is to the spheroid, so is any part of the sphere to the like part of the spheroid.\*

4 D

PRO-

## \* DEMONSTRATION.

For the like parts of any quantities are as the wholes. And that the second segments,  $FGH$ ,  $fgb$ , are like parts of the sphere and spheroid is evident from the nature of the figures.

Co-

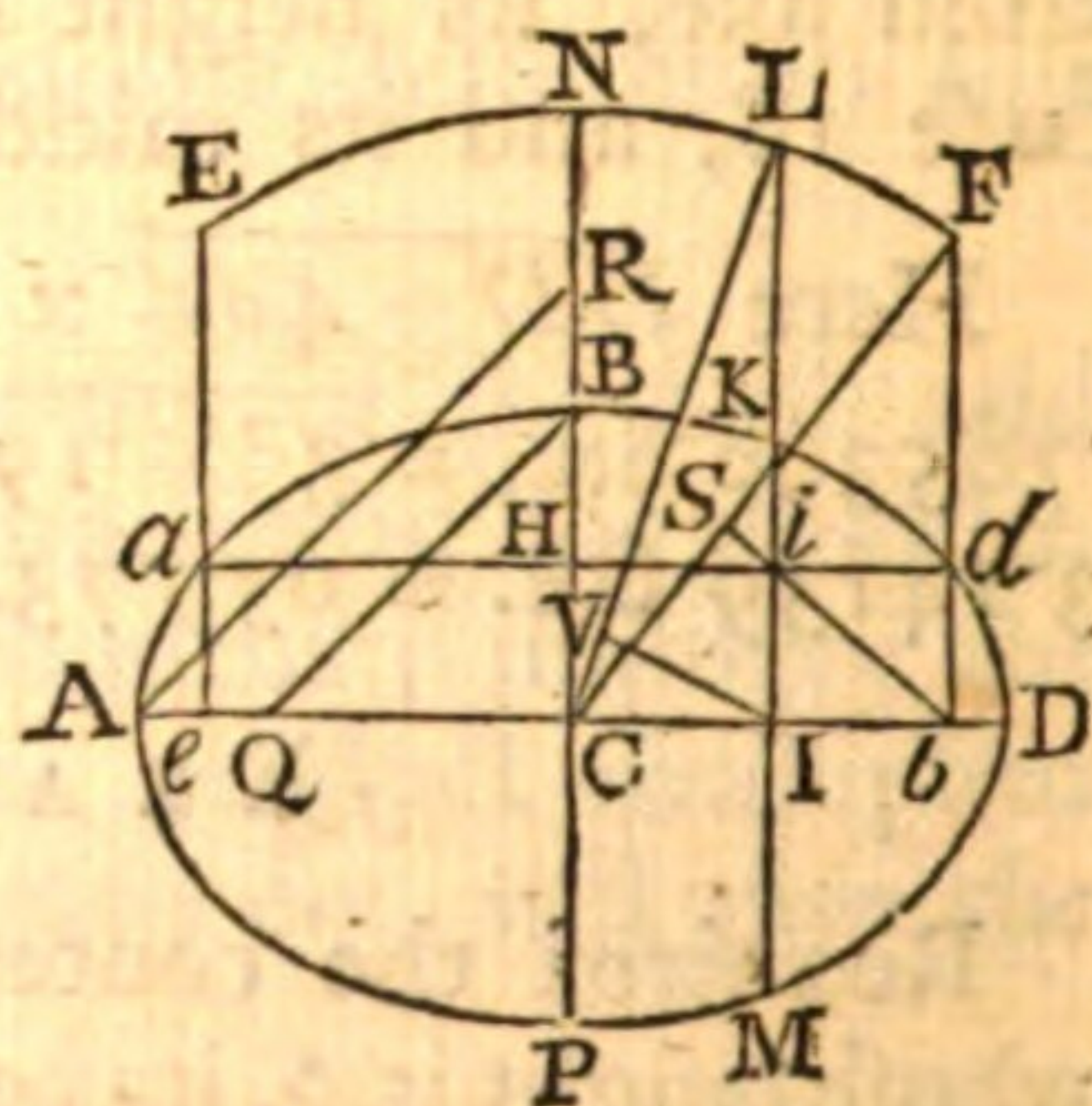


## PROBLEM XVI.

*To find the surface of an elliptic spindle, or of any frustum or segment of it.*

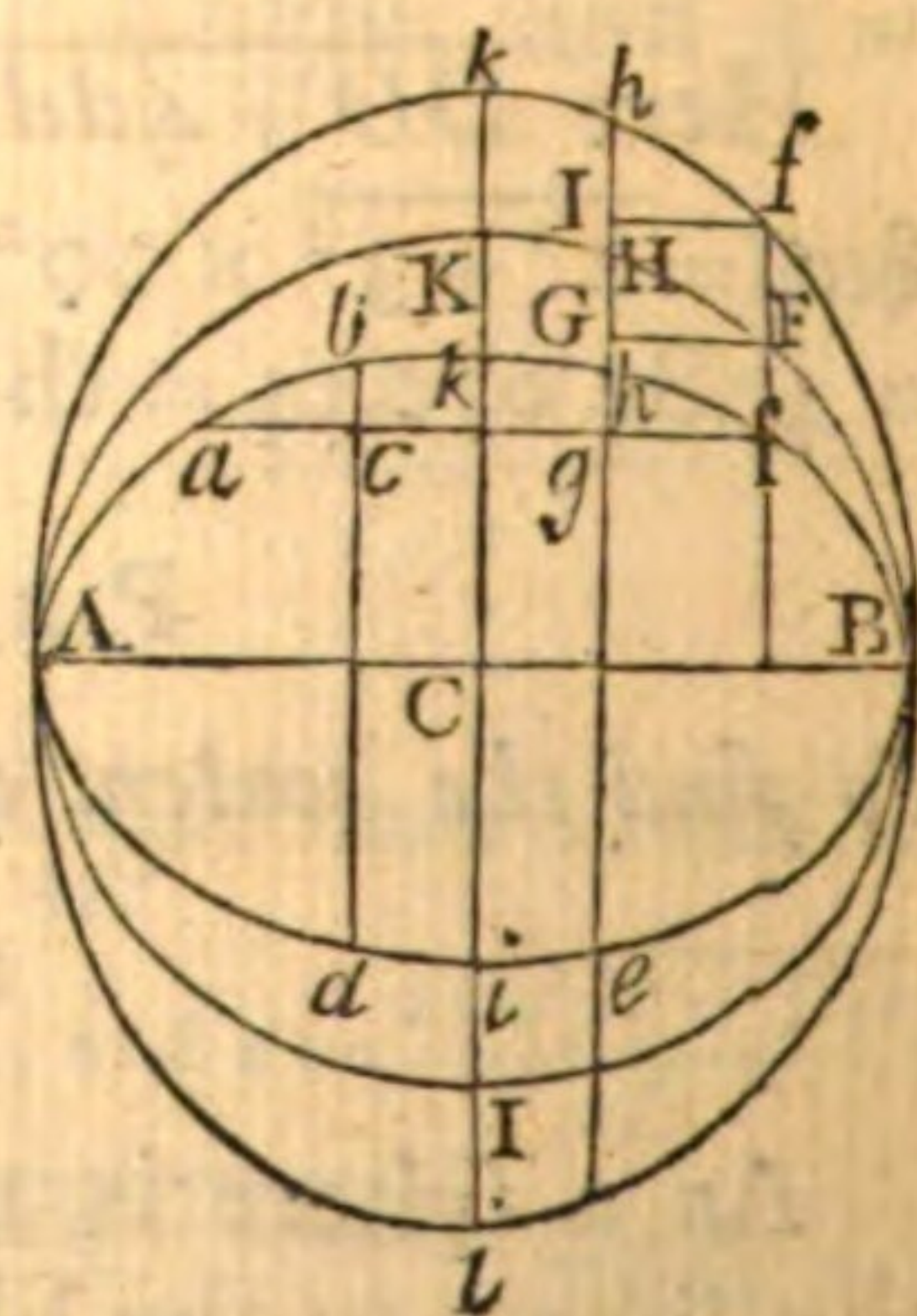
## GENERAL RULE.

From the spheroidal surface generated by any arc  $BK$  of the ellipse, take the product of the same arc  $BK$  and the circumference of the circle whose diameter is equal to twice  $CH$  the distance of



## COROLLARY I.

And if the sphere  $AKBI$  and spheroid  $AkBi$  have one axe  $AB$  common; then the whole solids and the second segments  $FGH$ ,  $fgb$ , will be to each other as the squares of the other axes  $KI$ ,  $ki$ , if the common axe  $AB$  be the fixed one; or they will be to each other as the other axes  $KI$ ,  $ki$ , simply, if the common axe be the revolving one. For, if  $A$  be the axe of the sphere,  $F$  and  $R$  the fixed and revolving axes of the spheroid, the bodies will be to each other as  $A^3$  to  $FR^2$ ; hence if  $A = F$ , they will be as  $A^2$  to  $R^2$ ; but if  $A = R$ , they will be as  $A$  to  $F$ .



## COROLLARY II.

Hence may be found the true quantity of liquor in a spheroidal cask, not full, whose axe is parallel to the horizon.

For if from the segment  $akf$  be taken the double of the second segment  $fgb$ , there will remain the part  $bbgc$ ; which taken from the whole cask  $bbed$  will leave the part  $cdeg$ .

And after the same manner may be found the quantity of liquor in a spheroidal cask partly filled and standing a-tilt with its axe inclined to the horizon.



of the centers of the spheroid and spindle, and the remainder will be the surface of the spindle generated by the arc  $BK$  about  $ad$  parallel to  $AD$  the fixed axe of the spheroid.\*

Ex-

## \* DEMONSTRATION.

For the fluxion of the spindular surface is  $= \dot{z}$  the fluxion of the arc  $BK$  drawn into  $iK \times 2p = 2pz \times KI - Ii$ , and the fluent is equal to the spheroidal surface  $BKMP - 2pz \times Ii$  or  $CH$ .

## COROLLARY I.

The spindular surface generated by  $BK$  is equal the spheroidal surface  $BKMP$  — the surface of a sphere whose axe is a mean proportional between  $BK$  and  $2CH$ . For  $2pz \times CH =$  that spheric surface. — Or  $2pz \times CH =$  a circle whose radius is  $\sqrt{2CH \times BK}$ .

## COROLLARY II.

When  $CH$  is equal nothing, the spindle becomes barely a spheroid. And when  $H$  falls below  $C$ , the surface of the sphere must be added. — What hath been hitherto done agrees to the surface generated by an arc about a line parallel to either axe of the ellipse.

## COROLLARY III.

From  $B$  the extremity of the less axe of the ellipse, apply  $BQ = AC$  the semi-transverse, and parallel thereto draw  $AR$  meeting  $CB$ , produced, in  $R$ ; with the center  $C$  and the radius  $AR$  describe the arc  $EF$  meeting the parallels  $aE, CB, IK, bd$  in  $E, L, N, F$ : Draw  $CL$  and  $CF$ , and perpendicular to them  $IV$  and  $bS$ . — Then the circle whose radius is equal to a mean proportional between the sum and difference of two lines whereof the one is a mean proportional between  $BC$  and  $IV + LN$ , and the other a mean proportional between  $BK$  and  $2CH$  will be equal to the spindular surface generated by  $BK$  about  $ab \parallel$  the transverse axe. For the spheroidal surface is equal to a circle whose radius is a mean proportional between  $BC$  and  $IV + LN$  by rule 4 to prob. 9, and  $2p \times BK \times CH$  is equal to a circle whose radius is a mean proportional between  $BK$  and  $2CH$ ; but the spindular surface is equal to the difference of those two quantities, and the difference of two circles is equal to a circle whose radius is a mean proportional between the sum and difference of the radiuses of the two circles, therefore, &c. — And, in the same manner,

the



## EXAMPLE I.

Given the axes of an ellipse 50 and 40, to find the surface of the spindle generated from an arc of that ellipse, the length of the spindle being 30.

Here  $AD = 50$ ,  $BP = 40$ ,  $ad = be = 30$ ,  $Ae = AC - Ce = 25 - 15 = 10$ , and  $eD = DC + Cb = 25 + 15 = 40$ . Then as  $AC : CB :: \sqrt{Ae \times eD} : ea = CH = \frac{CB \sqrt{Ae \times eD}}{AC} = \frac{20 \sqrt{10 \times 40}}{25} = \frac{4 \times 20}{5} = 16 =$  the distance of the centers of the ellipse and spindle.

By example 1 to problem 10, the spheroidal surface generated by the arc  $ad$  is  $3686.8197024$ , and by prob. 4, the length of the elliptic arc  $ad$  is  $30.852$ .

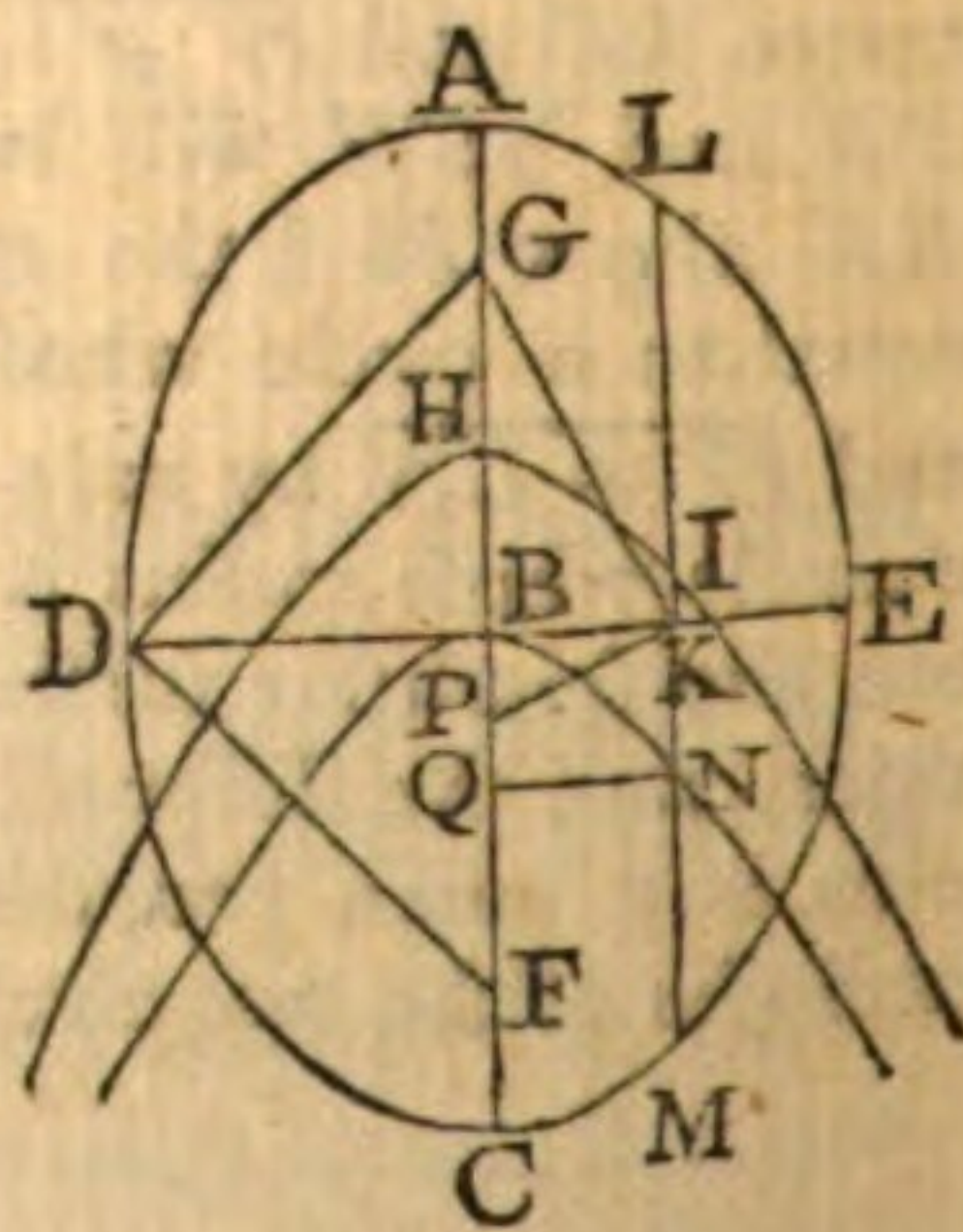
Wherefore  $3686.8197 - 30.852 \times 3.1416 \times 32 = 3686.8197 - 3101.5872 = 585.2325 =$  the surface of the spindle.

E x-

the surface of half the spindle generated by the arc  $Bd$  is equal to a circle whose radius is a mean proportional between the sum and difference of two lines, the one of which is a mean proportional between  $BC$  and  $bS + NF$ , and the other a mean proportional between  $Bd$  and  $2CH$ .

## COROLLARY IV.

In like manner, for the surface of the spindle whose axe is parallel to the less axe of the ellipse: having constructed the annexed figure as in page 259, the spindular surface generated by the arc  $CM$  about  $NQ$  will be equal to a circle whose radius is a mean proportional between the sum and difference of two lines whereof the one is a mean proportional between  $AC$  and  $HI$ , and the other a mean proportional between  $CM$  and  $2BQ$ .—This, like the last corollary, will be evident by comparing what is done above with what is done in page 259.





## EXAMPLE II.

Required the surface of the frustum of a spindle generated from an arc of the same ellipse as in the last example, the height of the frustum being 15, and the central distance 10.

Here the arc  $E\mathcal{Q}$  generating the frustum  $E\mathcal{Q}IC$  is equal to the arc in the last; and consequently that arc is  $= 15.426$ , and the spheroidal frustum  $E\mathcal{Q}FD = 1843.40985$ .

Then  $1843.4098 - 15.426 \times 3.1416 \times 20 = 1843.4098 - 969.246 = 874.163 =$  the surface of the frustum  $E\mathcal{Q}IC$  required.

## PROBLEM XVII.

*To find the solidity of an elliptic spindle.*

## RULE I.

1. Divide the square of the perpendicular axe  $DE$  by 3 times the square of the parallel axe  $AB$ , and multiply the quotient by the cube of  $FG$  the axe or length of the spindle, and call the product  $P$ .

2. Find the area of the elliptic segment  $FDG$  from which the spindle is generated, multiply this area by 4 times  $CH$  the central distance, and call the product  $\mathcal{Q}$ .

3. Multiply  $1.57079$  &c. by the difference between  $P$  and  $\mathcal{Q}$ , and the product will be the content of the spindle  $FDGN$ .

4 E

That







## EXAMPLE.

The axes of an ellipse being 50 and 30, required the solidity of a spindle generated from an arc

## COROLLARY II.

If  $H$  coincide with  $C$ ,  $b$  will vanish,  $a$  will be  $= c$ , and the theorems will become the same with those before found for the spheroid.

## COROLLARY III.

Putting  $D = DN$  the greatest, and  $d = KO$  the least diameter of the frustum,  $b = HI$  its height,  $C = CH$  the central distance,  $s =$  the elliptic semi-segment  $KDM$ , and  $n = .78539$  &c. Then, in the foregoing theorems,  $b = x$ ,  $b = C$ ,  $d = C + \frac{1}{2}D$ , area  $HIKD = s + \frac{1}{2}db$ ,  $c = b \times \frac{C + \frac{1}{2}D}{\sqrt{\frac{1}{4}DD + DC - \frac{1}{4}dd - dC}}$ , and  $a = b \sqrt{\frac{\frac{1}{4}DD + DC}{\frac{1}{4}DD + DC - \frac{1}{4}dd - dC}}$ .

which values being substituted in the theorems above, give  $\frac{1}{3}nb \times$

$2DD + dd - 8C \times -D + d + \frac{3S}{b}$  for the value of the frustum, or half a cask in the form of the middle zone of an elliptic spindle; and

$\frac{1}{3}nl \times 2DD - 8C \times -D + \frac{3S}{l}$  for half the spindle when  $S =$  the area  $DFH$ , and  $l = HF$ .

## COROLLARY IV.

But in real practice, such as cask gauging, none of these rules can be used, because that we have not given either the axes of the ellipse or the central distance; and to accommodate rules to that purpose we must introduce another dimension of a frustum besides its length and greatest and least diameters; Thus, putting  $m = PQ$  the diameter through  $R$  the middle of the length  $HI$ , and the other letters as before. Then, by the property of the ellipse,  $C + \frac{1}{2}D|^2 - C + \frac{1}{2}d|^2 : 4 :: C + \frac{1}{2}D|^2 - C + \frac{1}{2}m|^2 : 1$ , hence  $4 \times C + \frac{1}{2}m|^2 - 3 \times C + \frac{1}{2}D|^2 =$

$C + \frac{1}{2}d|^2$ , and  $C = \frac{1}{4} \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}$ . And the last two theorems

become  $\frac{1}{3}nb \times 2D^2 + d^2 - 2 \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m} \times -D + d + \frac{3S}{b}$

for the frustum  $DKON$ , and  $\frac{1}{3}nl \times 2D^2 - 2 \times \frac{3D^2 - 4m^2}{-3D + 4m} \times -D + \frac{3S}{l}$

for the semi-spindle  $DFN$ .



arc of it about its chord parallel to, and at the distance of, 9 from the transverse axe.

Here  $15 - 9 = 6 = DH$  = the height of the generating segment; and, by the property of the ellipse,  $DC:CA::2\sqrt{DH \times HE}:\frac{2AC\sqrt{DH \times HE}}{CD} = \frac{50\sqrt{6 \times 24}}{15}$   
 $= \frac{10 \times 12}{3} = 40 = FG$  the base of the elliptic segment, or length of the spindle; also, by prob. 6, the area of the segment is  $167.7345$ .

Then  $\frac{3.14159 \text{ \&c.}}{2} \times \frac{30 \times 30}{3 \times 50 \times 50} \times 40^3 - 4 \times 9 \times 167.7345$   
 $= 18 \times 3.14159 \text{ \&c.} \times 213\frac{1}{3} = 167.7345 = 2578.55 =$   
 the content required.

#### R U L E II.

Divide 3 times the generating segment by the length of the spindle, from the quotient subtract the greatest diameter of the spindle, multiply the remainder by 4 times the central distance, and subtract the product from the square of the greatest diameter; then the difference multiplied by  $\frac{1}{3}$  of the length of the spindle, and the product by  $1.57079 \text{ \&c.}$  will give the content of the spindle.

That is,  $\frac{2}{3}nl \times D^2 - 4c \times -D + \frac{3s}{4} =$  the spindle, where  $D = DN$  the greatest diameter, and the rest of the symbols as in the last rule.—By corollary 3.

#### E X A M P L E.

Required the content of an elliptic spindle whose length is 40, greatest diameter 12, and the central distance 9.

Here



Here  $\frac{1}{2}^2 = 6 =$  the height of the generating segment, which is therefore the same as before, the area being 167.7345.

Then  $3.14159 \&c. \times \frac{40}{6} \times 12^2 - 4 \times 9 \times -12 + \frac{3 \times 167.7345}{40}$   
 $= 3.14159 \&c. \times 80 \times 12 - 1.74024 = 2578.56 =$  the content required.

## R U L E III.

From three times the square of the greatest diameter take 4 times the square of the diameter in the middle between the greatest diameter and the end. And from 4 times the said middle diameter take 3 times the said greatest diameter. Divide the former difference by the latter, and  $\frac{1}{4}$  of the quotient will be the central distance.

That is,  $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} =$  the central distance,  $D$  being the greatest diameter and  $m$  the middle diameter.

Then proceed as in the last rule.—This is proved in corollary 4.

## E X A M P L E.

If, as in the example to the last rule, the greatest diameter be 12, and the length 40, required the content supposing the diameter at  $\frac{1}{4}$  of the length to be  $6 \times \sqrt{21-3} = 9.49546$ .

Here  $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} = \frac{1}{4} \times \frac{3 \times 12^2 - 4 \times 6^2 \times \sqrt{21-3}}{4 \times 6 \times \sqrt{21-3} - 3 \times 12}$   
 $= \frac{12}{4} \times \frac{3-21+6\sqrt{21-3}}{2\sqrt{21-3}-6-3} = 3 \times \frac{6\sqrt{21-3}-27}{2\sqrt{21-3}-9} = 3 \times 3 = 9 =$  the central distance, the same as in the last example; and therefore the content will be 2578.56, as before.

4 F

P R O.



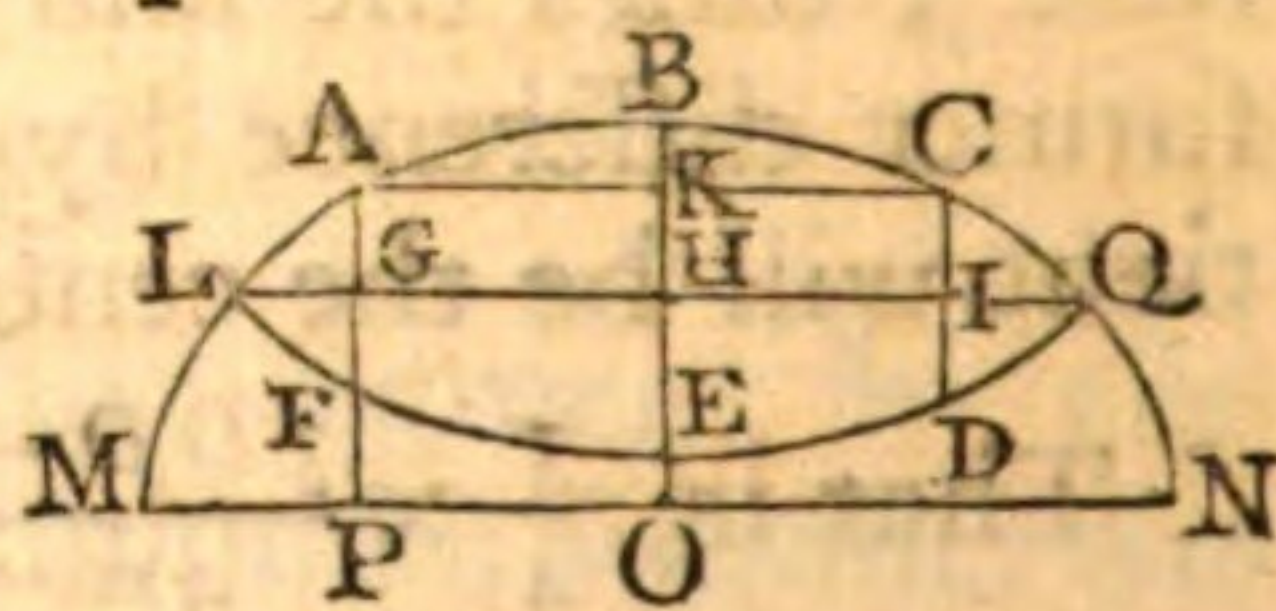
## PROBLEM XVIII.

*To find the solidity of the frustum of an elliptic spindle, or the content of a cask in form of the middle zone of such a spindle.*

## RULE I.

1. Divide the square of the less axe of the ellipse by the square of the greater, multiply the quotient by  $\frac{1}{3}$  of the length of the frustum or of half the length of the cask, multiply the product by the difference between the square of the said half length of the cask and 3 times the square of half the length of the whole spindle, and call this product  $P$ .

That is,  $P = ccl \times \frac{3LL - ll}{3tt}$ ,  
 $t$  being the transverse  $MN$ ,  
 and  $c$  the conjugate axe  $2BO$   
 of the ellipse,  $l = GH$  half the  
 length of the cask, and  $L = LH$  half the length of  
 the whole spindle.



2. Multiply  $OH$  the central distance by  $ABHG$  the generating area, and call double the product  $\mathcal{Q}$ .

That is,  $\mathcal{Q} = 2OH \times ABHG$ .

3. Then the difference between  $P$  and  $\mathcal{Q}$  multiplied by  $3.14159$  &c. will give the content of the frustum  $ABEF$ , or half the cask  $ACDF$ .

That is,  $\overline{P - \mathcal{Q}} \times p = ABEF = \frac{1}{2} ACDF$ .—By the demonstration of the last problem.

## NOTE.

It is evident that the sum or difference of two frustums will give the ullage of a cask standing with its axe perpendicular to the horizon.

EX-



## EXAMPLE.

The axes of an ellipse being 40 and  $66\frac{2}{3}$ ; it is required to find the content of the frustum of a spindle generated by an arc of the ellipse; the length of the frustum being 20, and the central distance 4.

Here, as  $40 : 66\frac{2}{3} :: \sqrt{16 \times 24} : 13\frac{1}{3} \sqrt{6} = HL =$  half the length of the spindle.

And  $ccl \times \frac{3LL - ll}{3lt} = 40 \times 40 \times 20 \times \frac{3 \times 6 \times 13\frac{1}{3} \times 13\frac{1}{3} - 20 \times 20}{3 \times 66\frac{2}{3} \times 66\frac{2}{3}}$   
 $= 24 \times 280 = 6720 = P.$

But the elliptic segment  $ABK$  is  $= 54\frac{1}{2}$ , to which adding the rectangle  $AH = 20 \times 12 = 240$ , makes  $294\frac{1}{2} =$  the generating area  $ABHG$ ; which being multiplied by  $2OH = 8$ , produces  $2356 = Q.$

Then  $P - Q \times p = 6720 - 2356 \times 3.14159 \&c. = 13709.91033 =$  the content required.

## RULE II.

Divide 3 times the elliptic segment whose chord is the length of the cask by the said length, to the quotient add the least or head diameter, and from the product subtract the greatest or bung diameter, and multiply the remainder by 8 times the central distance; then take the product from the sum of the square of the head diameter and 2 times the square of the bung diameter, multiply the difference by the length, and the product drawn into  $.261799 \&c.$  ( $= \frac{1}{3}$  of  $.785398 \&c.$ ) will be the content of the cask.

That is,  $2DD + dd - 8c \times -D + d + \frac{3s}{l} \times \frac{1}{3}nl =$   
 the content of the cask, where  $D = BE$  the bung  
 diameter



diameter,  $d = AF$  the head diameter,  $c = HO$  the central distance,  $l = GI$  the length,  $s =$  the segment  $ABC$ , and  $n = .785398$  &c.—This is proved at corollary 3 to the last problem.

#### EXAMPLE.

The bung and head diameters, of a cask which is the middle zone of an elliptic spindle, being 32 and 24, and its length 40 inches; required the content in ale and wine gallons, supposing the distance of the centers of the ellipse and spindle to be 4 inches.

Here  $D = 32$ ,  $d = 24$ ,  $l = 40$ ,  $c = 4$ , and the segment  $ABC = s = 109$ .

Then  $2DD + dd - 8c \times -D + d + \frac{3s}{l} \times \frac{1}{3}nl =$   
 $2624 - 5.6 \times 40 \times .261799 \text{ \&c.} = 27419.8219$  cubic inches.

Which being divided by 282, and 231, we have 97.2334 ale gallons, and 118.7005 wine gallons, for the content required.

#### RULE III.

From the sum of the square of the least or head diameter and 3 times the square of the greatest or bung diameter take 4 times the square of the diameter equidistant from the two former, and from 4 times the said middle diameter take the sum of the said least and 3 times the greatest diameter; then divide the former difference by the latter, and  $\frac{1}{4}$  of the quotient will be the central distance, with which proceed as in the 2d rule.

That



That is,  $\frac{1}{4} \times \frac{3DD + dd - 4mm}{-3D - d + 4m} =$  the central distance,  $m$  being the middle diameter, and  $D$  and  $d$  the other diameters as before.—By corollary 4 to the last problem.

## EXAMPLE.

If the bung and head diameters be 32 and 24, and the length 40 inches, as in the last example; required the content, supposing the middle diameter to be  $4\sqrt{91} - 8$ .

Here  $\frac{3DD + dd - 4mm}{-3D - d + 4m} \times \frac{1}{4} = \frac{3 \times 32^2 + 24^2 - 64 \times \sqrt{91} - 2^2}{-3 \times 32 - 24 + 16 \times \sqrt{91} - 2} \times \frac{1}{4} = \frac{-76 + 8\sqrt{91}}{-19 + 2\sqrt{91}} = 4 = c$  the central distance, the same as in the last example; and since the other parts are all the same, the content must likewise be the same.

## PROBLEM XIX.

*To find the content of the segment of an elliptic spindle.*

## RULE.

1. Multiply the square of the altitude of the segment by the square of the less axe of the ellipse and divide the product by the square of the greater axe, multiply the quotient by the difference between  $\frac{1}{2}$  the length of the whole spindle, and  $\frac{1}{3}$  of the altitude of the segment, and call the product  $P$ .

That is,  $P = aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt}$ , where  $c$  is the conjugate and  $t$  the transverse axe of the ellipse,  $a$  is the altitude of the segment, and  $l$  the length of the whole spindle.

4 G

2. Mul-



2. Multiply the double of the generating area by the distance of the centers of the ellipse and spindle, and call the product  $\mathcal{Q}$ .

That is,  $\mathcal{Q} = 2CA$ , where  $C = OH$ , and  $A$  = the area  $LAG$ .

3. Then the difference between  $P$  and  $\mathcal{Q}$  drawn into  $3.14159$  &c. will be the content of the segment  $LAH$  of the spindle.

That is,  $\overline{P - \mathcal{Q}} \times p = aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt} - 2AC \times p =$   
the content of the segment.—By the demonstration of problem 17.

#### EXAMPLE.

The axes of an ellipse being 50 and 30, required the content of the segment of a spindle whose height is 10, the central distance being 9.

Here  $t = 50$ ,  $c = 30$ ,  $a = 10$ , and  $C = 9$ .

Then, by the nature of the ellipse, as  $30 : 50 ::$   
 $(\sqrt{15 + 9} \times \sqrt{15 - 9} = \sqrt{24 \times 6} =) 12 : \frac{5 \times 12}{3} = 20 = \frac{1}{2}l$   
 $= LH$  the half of the length of the spindle.

And, by the same, as  $50 : 30 :: (\sqrt{NP} \times \sqrt{PM} =$   
 $\sqrt{25 + 10} \times \sqrt{25 - 10} = \sqrt{35 \times 15} =) 5\sqrt{21} : \frac{3 \times 5\sqrt{21}}{5}$   
 $= 3\sqrt{21} = 13.71495534 = PA$ . Hence  $BO - PA =$   
 $1.285 = BK$ .

But  $\frac{KB}{2BO} = \frac{1.285}{30} = .0428\frac{1}{3}$ , and  $\frac{HB}{BO} = \frac{6}{30} = \frac{1}{5} = .2$ ;  
the tabular numbers answering to which two quotients, in the table of circular segments, are  $.01166678$   
and  $.1118238$ ; hence  $50 \times 30 \times .1118238 - .01166678$   
 $= 1500 \times .10015702 = 150.23553 =$  the area  $LB\mathcal{Q}$ ;  
from



from the half of which taking the rectangle  $AH = AG \times GH = 4.714955 \times 10 = 47.149$ , leaves  $27.968765 = A$  for the generating area  $LAG$ .

Then  $aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{11} - 2AC \times p =$   
 $\frac{100 \times 900 \times \frac{20 - 3\frac{1}{3}}{2500} - 18 \times 27.968765 \times 3.14159 \&c. =$   
 $36 \times \frac{50}{3} - 18 \times 27.968765 \times 3.14159 \&c. = 18 \times$   
 $5.364568 \times 3.14159 \&c. = 303.35917 = \text{the content}$   
 of the segment  $LA F$  required.

### PROBLEM XX.

*To find the content of an universal spheroid, or a solid conceived to be generated by the revolution of a semi-ellipse about its diameter, whether that diameter be one of the axes of the ellipse or not.*

#### RULE I.

Divide the square of the product of the axes of the ellipse by the axe of the solid or the diameter about which the semi-ellipse is conceived to revolve, multiply the quotient by .5236, and the product will be the content required.

That is,  $\frac{T^2 C^2}{d} \times .5236 = \text{the content}$ ,  $T$  and  $C$  being the transverse and conjugate axes of the ellipse, and  $d$  the axe of the solid.

#### RULE II.

The continual product of .5236, the diameter about which the revolution is made, the square of its conjugate diameter, and the square of the sine  
 of



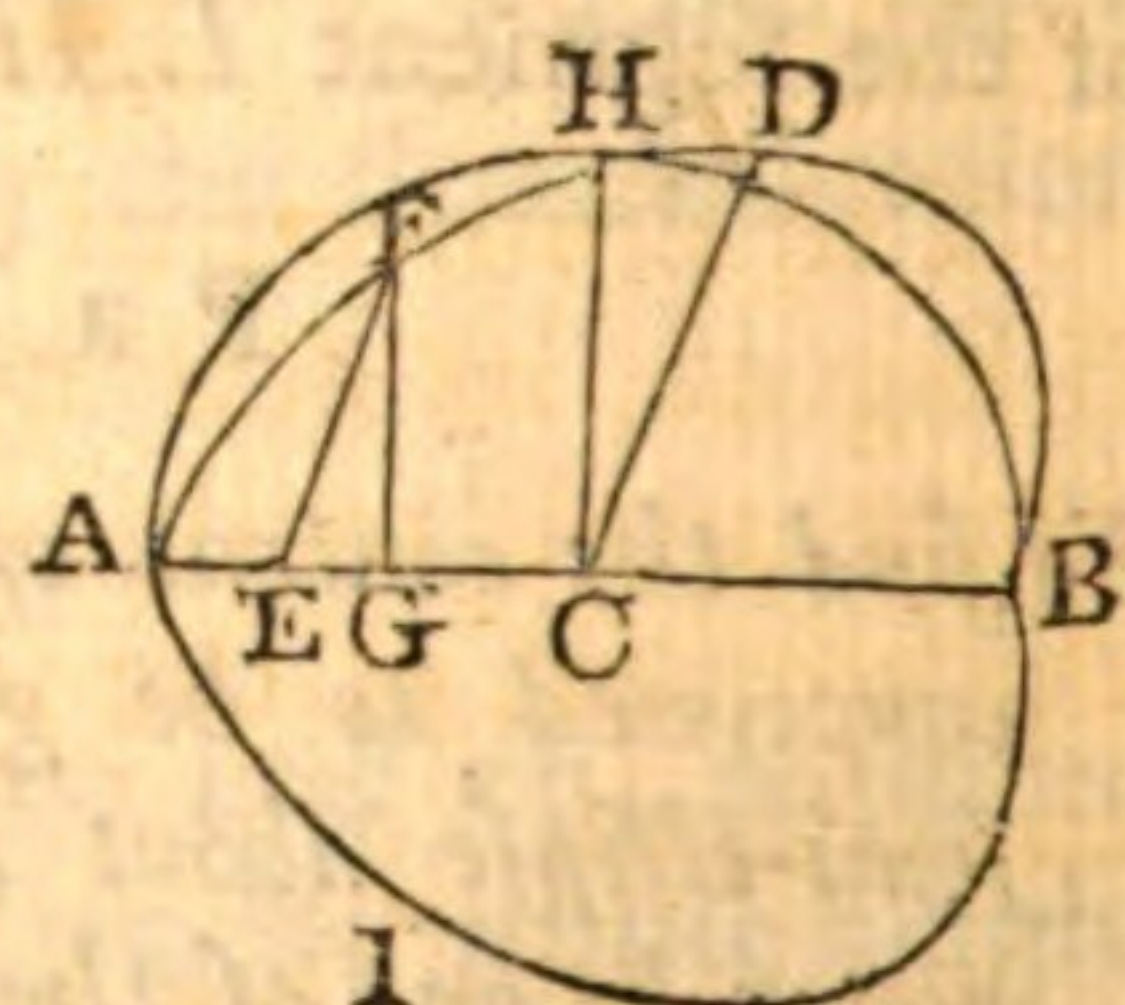
of the angle made by those diameters (the radius being 1) will be the content.

That is,  $dccss \times .5236 =$  the content,  $c$  being the conjugate diameter to  $d$ , and  $s$  the sine of the angle made by the diameters.\*

Ex-

\* DEMONSTRATION.

Let  $ADBIA$  be a section of the solid through its axe  $AB$ ; draw  $DC$  the semi-conjugate diameter to  $AB$ , as also the ordinate  $FE$  parallel thereto, and let fall the perpendicular  $FG$ .



Put  $T$  and  $C$  for the transverse and conjugate axes of the ellipse,  $d =$  the diameter  $AB$ , or axe of the solid,  $c = 2CD = \sqrt{T^2 + C^2 - d^2}$  its conjugate diameter,  $a = \frac{TC}{dc} =$  sine of the  $\angle DCB$

or  $\angle FEG$ , and  $b =$  its cosine,  $x = AE$ , and  $p = 3.14159 \&c.$

Then  $d : c :: \sqrt{dx - xx} : \frac{c\sqrt{dx - xx}}{d} = EF$ , and

$$\left\{ \begin{array}{l} EF \times a = \frac{ac\sqrt{dx - xx}}{d} = FG \\ EF \times b = \frac{bc\sqrt{dx - xx}}{d} = GE \end{array} \right\}, \text{ and hence } AG = GE + EA$$

$$= x + \frac{bc\sqrt{dx - xx}}{d}; \text{ then } p \times FG^2 \times GA = paaccx \times$$

$$dx - xx + \frac{\frac{1}{2}d - x}{d} \times bc\sqrt{dx - xx}$$

$\frac{\quad}{dd} =$  the fluxion of the solid, the

fluent of which is  $paacc \times \frac{\frac{1}{2}dx^2 - \frac{1}{3}x^3 + \frac{bc}{3d} \times \overline{dx - xx}^{\frac{3}{2}}}{dd} =$  the measure of the part generated by  $AFG$ ; and when  $x = d$ , it becomes  $\frac{1}{6}pda^2c^2 = \frac{pT^2C^2}{6d}$  for the value of the whole solid.

Co-



## EXAMPLE I.

If the axes of an ellipse be 50 and 30, and it be cut in two by a diameter whose length is 40; required the content of the solid generated by one of the halves about that diameter.

By rule 1,  $\frac{50 \times 50 \times 30 \times 30}{40} \times .5236 = 225000 \times .1309 = 29452.5 = \text{the content required.}$

## EXAMPLE II.

The diameter of an ellipse about which it revolves being 40, its conjugate diameter  $30\sqrt{2}$ , and the sine of the angle made by those diameters  $\frac{5}{8}\sqrt{2}$ ; required the content of the solid formed by the revolution of the ellipse.

By rule 2, we have  $40 \times 30 \times 30 \times 2 \times \frac{5}{8} \times \frac{5}{8} \times 2 \times .5236 = 225000 \times .1309 = 29452.5 = \text{the content the same as before.}$

4 H

SEC-

## COROLLARY I.

If  $d = T$ , the rule becomes  $\frac{1}{8}pTC^2$  for the oblong spheroid. And if  $d = C$ , it will be  $\frac{1}{8}pCT^2$  for the oblate spheroid. Also if  $T$ ,  $C$ , and  $d$ , be all equal, the rule will be  $\frac{1}{8}pd^3$  for the sphere. Which are the same with the rules before found for the same bodies.

## COROLLARY II.

Draw  $CH$  perpendicular, and  $DH$  parallel to  $AB$ , and about the axes  $AB$  and  $2CH$  describe the semi-ellipse  $AHB$ ; then the spheroid generated by the revolution of the semi-ellipse  $AHB$  about  $AB$  will be equal to the solid generated by the semi-ellipse  $AFDB$  about the same axis  $AB$ .

For  $2CH = ac$ , and therefore the solid  $AFBI$  or  $\frac{1}{8}pda^2c^2 = \frac{1}{8}pd \times 2CH^2 = \text{the spheroid whose axes are } d \text{ and } 2CH.$

And so the solids generated by all semi-ellipses upon the same base and between the same parallels, are all equal to each other.



## SECTION IV.

*Of parabolic Lines, Areas, Surfaces, and Solidities.*

## PROBLEM I.

*To construct a parabola; having given any ordinate PQ to the axe, and its abscissa VP.*

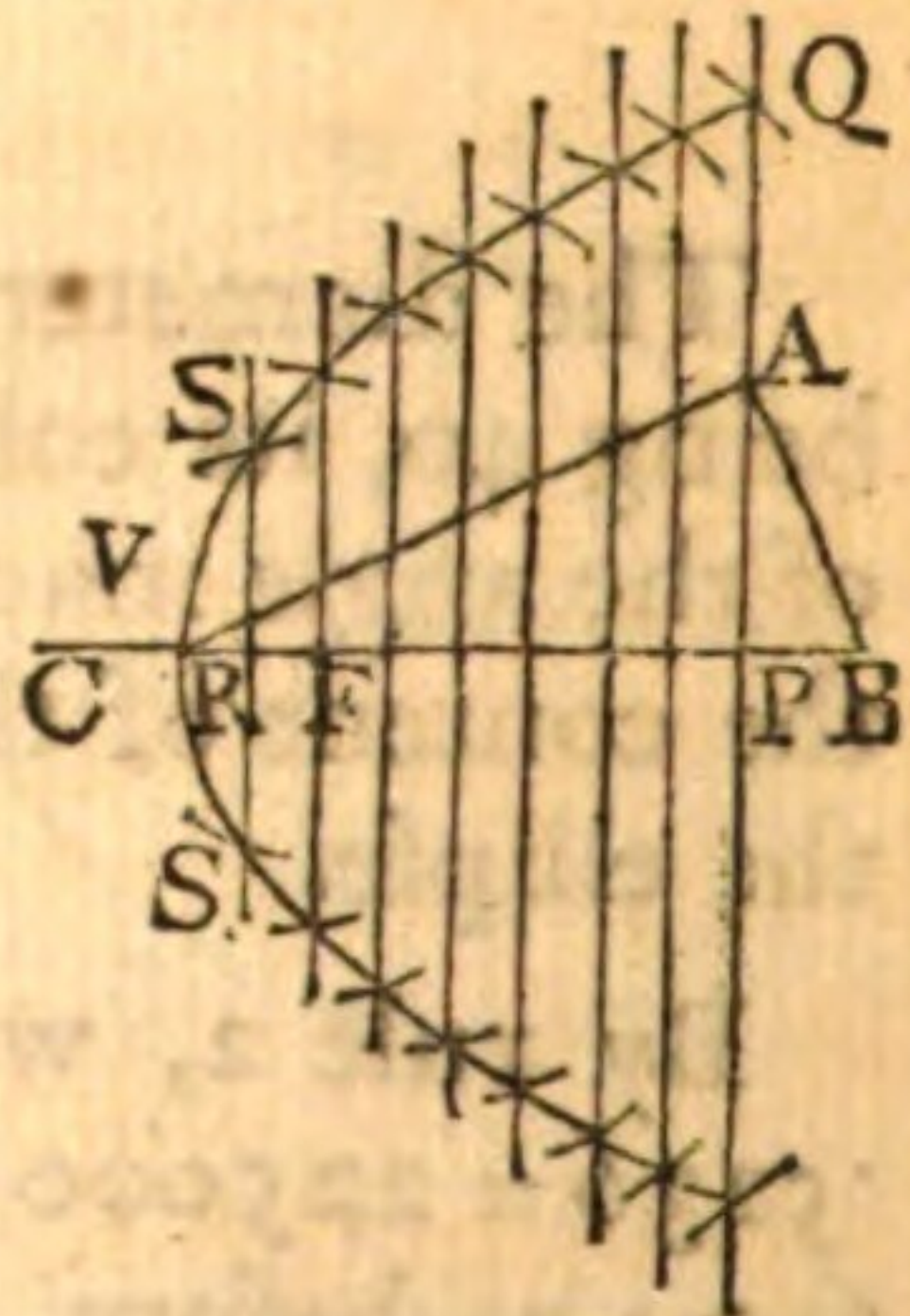
1. **F**IND the focus  $F$  thus:  
Bisect  $PQ$  in  $A$ ; draw  $AV$ , and  $AB$  perpendicular to it; take  $VF = PB$ , and  $F$  will be the focus.

*Arithmetically.*

Divide the square of the ordinate by 4 times the abscissa, and the quotient will be the focal distance  $VF$ .

2. In the axe, produced without the vertex  $V$ , take  $VC = VF$ ; draw several double ordinates  $SRS$ ; with the radiuses  $CR$  and center  $F$  describe arcs cutting the corresponding ordinates in the points  $S$ .

Draw a curve through all the points of intersection, and it will be the parabola required.



## PROBLEM II.

*Of any abscissa  $x$ , its ordinate  $y$ , and parameter  $p$ ; having two given, to find the third.*

## CASE I.

*To find the parameter.*

**RULE.**



## R U L E.

Divide the square of the ordinate by its abscissa, and the quotient will be the parameter. Or, take a third proportional to the abscissa and ordinate, for the parameter.

That is,  $yy \div x = p$ .

## E X A M P L E.

If the abscissa be 9, and its ordinate 6; required the parameter.

Here  $6 \times 6 \div 9 = 36 \div 9 = 4 =$  the parameter.

## C A S E II.

*To find the abscissa.*

## R U L E.

Divide the square of the ordinate by the parameter, and the quotient will be the abscissa.

That is,  $yy \div p = x$ .

## E X A M P L E.

If the ordinate be 6, and the parameter 4; required the abscissa.

Here  $6 \times 6 \div 4 = 36 \div 4 = 9 =$  the abscissa.

## C A S E III.

*To find the ordinate.*

## R U L E.

Multiply the parameter by the abscissa, and the square root of the product will be the ordinate.

That is,  $\sqrt{px} = y$ .

Ex-



## EXAMPLE.

The abscissa being 9, and the parameter 4; required the ordinate.

Here  $\sqrt{9 \times 4} = \sqrt{36} = 6 = \text{the ordinate.}$

## PROBLEM III.

*Of any two abscissas A, B, taken upon the same diameter, and their two ordinates a, b; having any three given to find the fourth.*

## RULE.

The abscissas are to one another as the squares of their ordinates. That is, As any abscissa is to the square of its ordinate, so is any other abscissa to the square of its ordinate; and the contrary. Or, as the root of an abscissa is to its ordinate, so is the root of another abscissa, to its ordinate.

$$\text{Hence } \begin{cases} \sqrt{A} : \sqrt{B} :: a : a\sqrt{\frac{B}{A}} = \frac{a\sqrt{AB}}{A} = b. \\ \sqrt{B} : \sqrt{A} :: b : b\sqrt{\frac{A}{B}} = \frac{b\sqrt{AB}}{B} = a. \end{cases}$$

$$\text{And } \begin{cases} aa : bb :: A : \frac{Abb}{aa} = B. \\ bb : aa :: B : \frac{Baa}{bb} = A. \end{cases}$$

## EXAMPLE I.

If an abscissa of 9 correspond to an ordinate of 6, required the ordinate whose abscissa is 16.

Here  $\sqrt{9} : \sqrt{16} :: 6 : 6\sqrt{\frac{16}{9}} = \frac{6 \times 4}{3} = 8 = \text{the ordinate.}$

Ex-



## EXAMPLE. II.

Required the abscissa corresponding to the ordinate 6, the ordinate belonging to the abscissa of 16 being 8.

Here  $8 \times 8 : 6 \times 6 :: 16 : \frac{16 \times 6 \times 6}{8 \times 8} = 9 = \text{the abscissa.}$

## SCHOLIUM.

The demonstration of the three preceding problems are omitted here, as they properly belong to, and are to be found in, all treatises of conics.

## PROBLEM IV.

*To find the length of the curve or arc of a parabola cut off by a double ordinate to the axe.*

## \* RULE I.

Divide the ordinate by  $\frac{1}{2}$  the parameter, and call the quotient  $q$ .

4 I

Add

## \* DEMONSTRATION.

Putting  $z =$  any curve beginning at the vertex,  $y =$  the ordinate to the axe at the extremity of the curve,  $x =$  its abscissa, and  $a = \frac{1}{2}$  the parameter of the axe, the equation to the curve is  $2ax = yy$ ;

hence  $2a\dot{x} = 2y\dot{y}$ , and  $\dot{x}^2 = \frac{y^2 \dot{y}^2}{aa}$ ; consequently  $\dot{z} = \sqrt{\dot{y}\dot{y} + \dot{x}\dot{x}} =$

$\sqrt{\dot{y}^2 + \frac{y^2 \dot{y}^2}{aa}} = \frac{\dot{y}\sqrt{aa + yy}}{a}$ , and the corrected fluents give  $z =$

$\frac{y\sqrt{aa + yy}}{2a} + \frac{1}{2}a \times \text{hyp. log. of } \frac{y + \sqrt{aa + yy}}{a} = \frac{1}{2}aq\sqrt{1 + qq} +$

$\frac{1}{2}a \times \text{hyp. log. of } q + \sqrt{1 + qq}$ , writing  $q$  for  $\frac{y}{a}$ . And the double of this quantity will give the value of the double curve as in the rule.







*Note 1.* If the common logarithm of any number be multiplied by 2.302585093, the product will be the hyperbolic logarithm of the same number.

2. If the value of  $s$  run into decimals, it will be much easiest found by a trigonometrical canon; for  $s$  is the secant of the arc whose tangent is  $q$ , the radius being 1.

EX-

If there be taken  $d = y$ , we shall obtain  $C = HC - \frac{A}{y}$ ; and to find the distance of the ordinate  $y$  from the center of the hyperbola, we shall have  $a : y (d) :: \sqrt{aa + yy} = \frac{y\sqrt{aa + yy}}{a}$  = the said dis-

tance, and which, therefore, is  $= HC$ . Whence this construction. — In the axe produced take  $BI = HC$ ; with the center  $I$ , semi-transverse  $IK =$  the ordinate  $BC$ , and semi-conjugate  $KL =$  the semi-parameter  $FG$  of the parabola, let be described the hyperbola  $KC$ , which will pass through  $C$ ; and let the rectangle  $CBMN$  be equal the hyperbolic area  $KCB$ : Then will  $IM$  be equal the parabolic curve  $AC$ .

When the abscissa and transverse axe of an hyperbola are given or constant, not only the ordinate but the area also is as the conjugate axe, and therefore the quotient arising from the division of the area by the ordinate, is a constant quantity; and consequently the parabolic curve  $AC = C = HC - \frac{B}{z}$ , where  $B$  is the area and  $z$  the ordinate of any hyperbola whose center is  $I$ , vertex  $K$ , and abscissa  $KB$ . From hence arises the following general construction, given by Mr HUYGENS in his *Horolog. Oscillat.* but without demonstration. — With the center  $I$  and vertex  $K$ , as before, and any conjugate taken at pleasure, describe the hyperbola  $KO$  meeting  $BC$ , produced if necessary, in  $O$ ; and, making the rectangle  $OM =$  the hyperbolic area  $KOB$ ,  $MI$  will be equal to the parabolic curve  $AC$  as before.

#### COROLLARY III.

When  $y = a = FG$ , then  $z = \frac{1}{2}a \times \sqrt{2} + \text{hyp. log. of } 1 + \sqrt{2} = \frac{1}{2}a \times 2.2955871 = 2.2955871 \times AF = 1.1477935 \times FG =$  the curve  $AG$ ;  $F$  being the focus.

#### COROLLARY IV.

The lengths of similar parts of parabolic curves will be as their parameters, or ordinates, or abscissas. For  $q$  will be the same in each.



## EXAMPLE.

Required the length of the curve of a parabola cut off by a double ordinate, to the axe, whose length is 12, the abscissa being 2.

Here  $x = 2$ , and  $y = 6$ .

Hence  $a = \frac{yy}{2x} = \frac{36}{4} = 9$ ,  $q = \frac{y}{a} = \frac{6}{9} = \frac{2}{3}$ , and  $s = \sqrt{1+qq} = \sqrt{1+\frac{4}{9}} = \sqrt{\frac{13}{9}} = \frac{1}{3}\sqrt{13} = 1.2018504$ .—Or, by the table of tangents and secants, the secant corresponding to the tangent  $\frac{2}{3}$  or .666 &c. is  $1.2018504 = s$ .

Then,  $\frac{2}{3} + 1.2018504 = 1.868517$ , whose common logarithm is .271497, which being multiplied by 2.302585093 produces .62514495 for its hyperbolic logarithm; also  $\frac{2}{3} \times 1.2018504 = .8012336$ ; and the sum of these two is  $.6251449 + .8012336 = 1.4263785$ .

Wherefore  $9 \times 1.4263785 = 12.8374065 =$  the length of the curve required.

## \* RULE II.

Putting  $y$  to denote the ordinate, and  $q$  the quotient arising from the division of the ordinate by half

## \* DEMONSTRATION.

By the last, the fluxion of the curve was  $\dot{z} = \dot{y} \sqrt{1 + \frac{yy}{aa}} =$  (by extracting the root)  $\dot{y} \times : 1 + \frac{y^2}{2a^2} - \frac{y^4}{2.4a^4} + \frac{3y^6}{2.4.6a^6} \&c.$  and, by taking the fluents and writing  $q$  for  $\frac{y}{a}$ , we obtain  $z = y \times : 1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} \&c. = y \times : 1 + \frac{1.1}{2.3} q^2 A - \frac{1.3}{4.5} q^2 B + \frac{3.5}{6.7} q^2 C \&c. =$  the length of the curve from the vertex to the ordinate, the double of which will be that of the double curve. Q.E.D.

Co-



half the parameter, or from the division of twice the abscissa by the ordinate; the length of the double curve will be denoted by the infinite series  $2y \times$ :  
 $1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} - \frac{3.5q^8}{2.4.6.8.9} \&c.$  or by  $2y \times$ :  
 $1 + \frac{1}{2.3} q^2 A - \frac{1.3}{4.5} q^2 B + \frac{3.5}{6.7} q^2 C - \frac{5.7}{8.9} q^2 D \&c.$  where  
 $A, B, C, \&c.$  denote the 1st, 2d, 3d, &c. terms.

*Note,* This series will converge no longer than till  $q = 1$ , that is, when the ordinate to the curve, whose length is required, meets the axe in the focus; for if the ordinate  $y$  be beyond the focus, it will be greater than the semi-parameter, and consequently  $q$  greater than 1.

## EXAMPLE.

Let there be taken the same example as before, in which the abscissa is 2 and ordinate 6.

4 K

Then

## COROLLARY.

The hyperbolic logarithm of  $q + \sqrt{1 + qq}$  is  $= q \times : 1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7} \&c.$  For, by the last rule,  $\frac{2z}{a}$  — hyp. log. of  $q + \sqrt{1 + qq}$  is  $= q \sqrt{1 + qq} = q \times : 1 + \frac{q^2}{2} - \frac{q^4}{2.4} + \frac{3q^6}{2.4.6} \&c.$  and, by this rule,  $\frac{2z}{a}$  is  $= q \times : 1 + \frac{q^2}{3} - \frac{q^4}{4.5} + \frac{3q^6}{4.6.7} \&c.$  but by taking the former of these from the latter, we have  $q \times : 1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7} + \frac{3.5.7q^8}{2.4.6.8.9} \&c. = \text{hyp. log. of } q + \sqrt{1 + qq}.$



Then  $\frac{2 \times 2}{6} = \frac{2}{3} = q$ , which being used for it in the general series, and the affirmative and negative terms collected, will appear as below:

|                                                                                                                                                                                                                                                                                    |                                                                                                                                                                                                                         |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $  \begin{array}{r}  A = 1.00000000 \\  B = 0.07407407 \\  D = 78385 \\  F = 4311 \\  H = 368 \\  K = 38 \\  \hline  + 1.0749051 \text{ sum of the affirmative terms} \\  - 0.0051211 \text{ negative} \\  \hline  \text{Dif. } 1.069784 \text{ sum of the series}  \end{array}  $ | $  \begin{array}{r}  C = 0.00493827 \\  E = 16935 \\  G = 1215 \\  I = 117 \\  L = 13 \\  \hline  - 0.0051211 \\  \hline  12 = 2y \\  12.837408 = \text{length of the curve nearly the same as before.}  \end{array}  $ |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

### R U L E III.

To the square of the ordinate add  $\frac{4}{3}$  of the square of the abscissa, and the root of the sum will be the length of the single curve nearly, the double of which will be that of the curve on both sides of the abscissa nearly.

That is,  $\sqrt{yy + \frac{4}{3}xx} = c$  the length of the single curve nearly,  $y$  being the ordinate and  $x$  the abscissa.\*

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\* DEMONSTRATION of this and Rule IV.

By the last rule, the curve is  $c = y \times : 1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.5}q^4 + \frac{3}{2.4.6.7}q^6$  &c. but  $\sqrt{1 + \frac{1}{3}qq} = 1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.9}q^4 + \frac{3}{2.4.6.27}q^6$  &c.



## EXAMPLE.

Taking again the same example, in which  $x = 2$ , and  $y = 6$ , we shall have  $c = \sqrt{yy + \frac{4}{3}xx} = \sqrt{36 + \frac{16}{3}} = 6.4291 =$  the single curve, the double of which is  $12.8582 =$  the length of the curve nearly.

## RULE IV.\*

To the square of the ordinate add  $\frac{2}{3}$  of the square of the abscissa, and divide the sum by the ordinate; then subtract 4 times this quotient from 9 times the length of the single curve as found from the last rule, and  $\frac{1}{5}$  of the remainder will be the length of the single curve extremely near.

That is,  $c = 9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{4}{3}xx}{y} \times \frac{1}{5}$  very nearly.

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&c. hence  $\frac{c}{y} - \sqrt{1 + \frac{4}{3}qq} = -\frac{1}{2.5.9}q^4 + \frac{5}{4.7.27}q^6$  &c. and, supposing  $q$  not greater than 1 and rejecting the series,  $c$  will be  $= y\sqrt{1 + \frac{4}{3}qq} = \sqrt{yy + \frac{4}{3}xx}$  nearly. Which is rule 3.

Again, from the 1st series,  $\frac{c}{y} - 1 - \frac{1}{8}qq = -\frac{1}{2.4.5}q^4 + \frac{1}{4.4.7}q^6$  &c.

or  $\frac{4}{9} \times \frac{c}{y} - 1 - \frac{1}{8}qq = -\frac{1}{2.5.9}q^4 + \frac{3}{4.7.27}q^6$  &c. and, from above,

$\frac{c}{y} - \sqrt{1 + \frac{4}{3}qq} = -\frac{1}{2.5.9}q^4 + \frac{5}{4.7.27}q^6$  &c. hence, by subtracting,

$\frac{c}{y} - \sqrt{1 + \frac{4}{3}qq} - \frac{4}{9} \times \frac{c}{y} - 1 - \frac{1}{8}qq = \frac{1}{2.7.27}q^6$  &c. and, consequently, the remaining series being very small, we shall obtain  $c =$

$\frac{1}{5}y \times 9\sqrt{1 + \frac{4}{3}qq} - 4 \times 1 + \frac{1}{8}qq = \frac{1}{5} \times 9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{4}{3}xx}{y}$  very nearly. Which is rule 4.\*

And in this manner we may proceed to any degree of accuracy required.



## EXAMPLE.

Taking, still, the same example; we shall have  $4 \times \frac{yy + \frac{2}{3}xx}{y} = 4 \times \frac{36 + \frac{8}{3}}{6} = 25\frac{7}{9}$ ; but by the last rule  $c = 6.4291$ , hence  $9 \times 6.4291 = 57.8619$ ; and  $\frac{57.8619 - 25\frac{7}{9}}{5} = 6.4168 = c$ , the double of which is 12.8336.

## NOTE.

It must be observed, that, as these two approximations are derived from the 2d rule, they must be used only in such cases in which that might be applied, viz. those in which the abscissa doth not exceed half the ordinate.

## PROBLEM V.

*To find the area included by the curve of a parabola and any right line, called the base of the segment or area.*

## RULE.\*

Take  $\frac{2}{3}$  of its circumscribed parallelogram for the area required.

## NOTE.

## \* DEMONSTRATION.

Put  $y$  = an ordinate to any diameter,  $x$  = its abscissa, and  $p$  = the parameter of that diameter; then the equation to the curve will be  $px = yy$ , and, putting  $s$  = the sine of the angle made by the abscissa and ordinate, the fluxion of the area  $a$  will be  $= syx = \frac{sv^2y}{3}$ ; hence the area  $a$  is  $= \frac{2sy^3}{3p} = \frac{2}{3}syx$ ; that is,  $\frac{2}{3}$  of the parallelogram having the same base and altitude,

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## NOTE.

The base of the circumscribed parallelogram is the same with the base of the segment, their altitudes are, likewise, the same; and, therefore, the parabolic area will be equal to  $\frac{2}{3}$  of its altitude multiplied by its base; that is  $= \frac{2}{3}ab$ ; putting  $a$  to denote its altitude, and  $b$  its base.——It may, farther, be observed, that if the base be perpendicular to the diameter of the figure, then the altitude  $a$  will be the same with the abscissa of the figure; otherwise, the altitude is equal to the abscissa drawn into the natural sine of the angle made by the abscissa and ordinate or base, the radius being 1. And this is to be observed in every other figure.

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## COROLLARY I.

Hence it is evident that all parabolas  $ADB$ ,  $ACB$ , of equal bases  $AB$ , and of equal altitudes or between the same parallels  $AB$ ,  $DC$ , are equal to one another.

## COROLLARY II.

Any common sections  $FG$ ,  $fg$ , of the parabolas  $ADB$ ,  $ACB$ , are equal to each other.

For  $(EC : Ch :: ED : DH ::) AB^2 : fg^2 :: AB^2 : FG^2$ ; but  $AB = AB$ , therefore  $fg = FG$ .

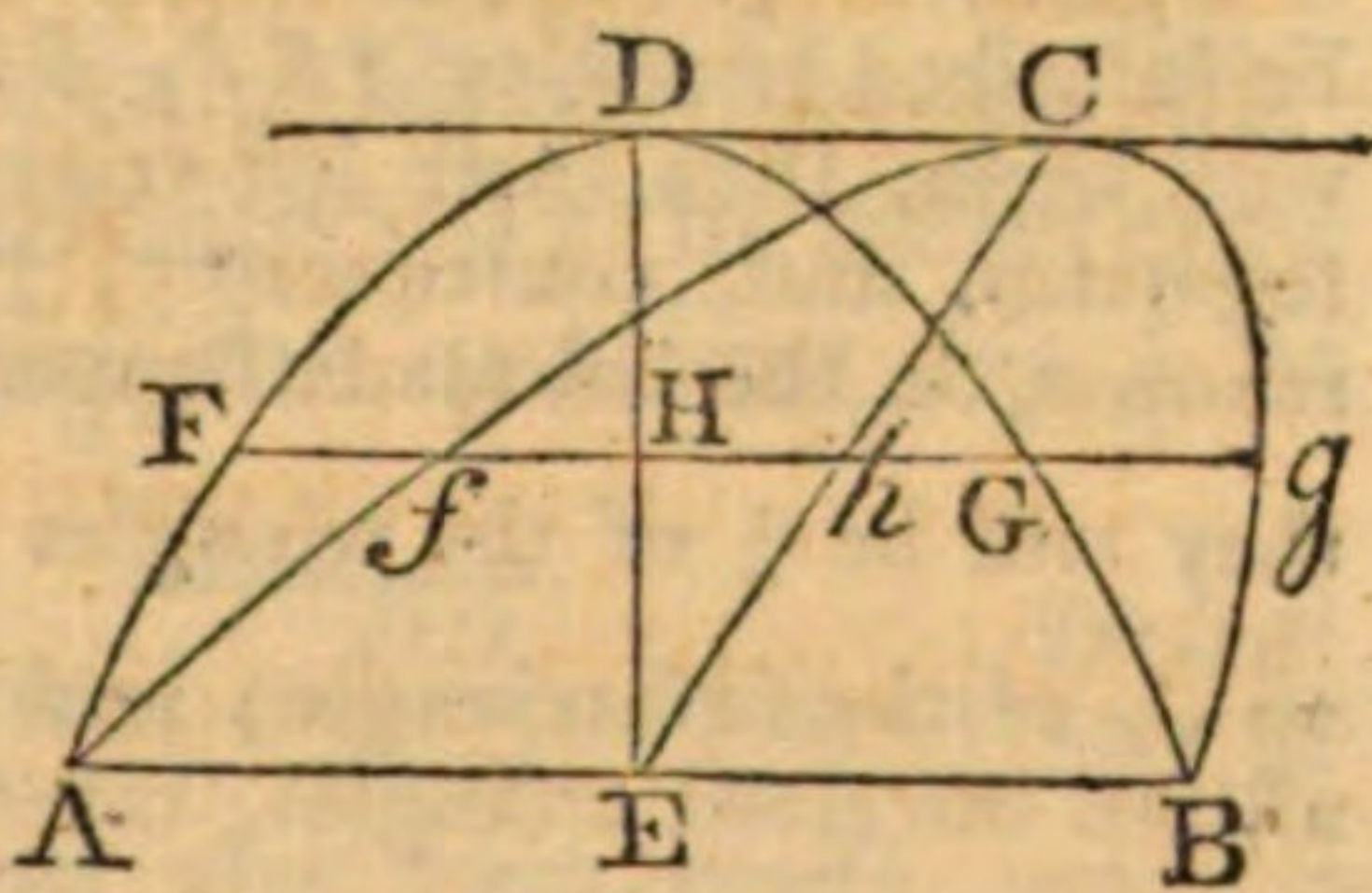
## COROLLARY III.

And hence, also, the segments  $FDG$ ,  $fCg$ , are equal to each other.——For they are of equal bases and altitudes.

## COROLLARY IV.

Moreover, the frustums  $AFOB$ ,  $AfgB$ , of equal ends and altitudes are equal to one another.

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## EXAMPLE I.

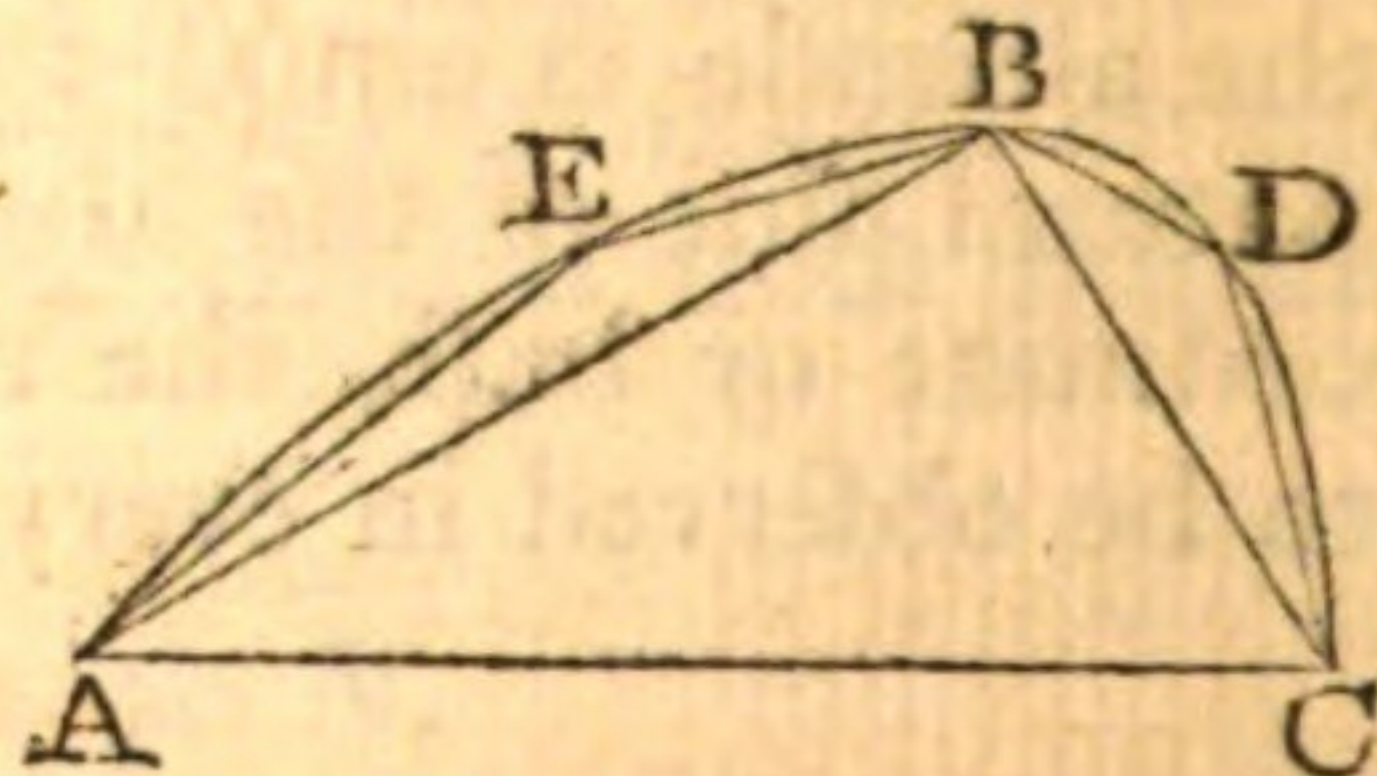
Required the area of a parabola, the abscissa being 2, and the ordinate, perpendicular to the abscissa, 6.

Here the altitude is 2, and the base or double ordinate is 12; therefore  $\frac{2 \times 2 \times 12}{3} = 16 =$  the area required.

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## COROLLARY V.

Let  $ABC$  be a triangle having the same base and altitude with the parabolic segment  $AEBDC$ ; then, because the triangle is half of the circumscribed parallelogram, it will be  $\frac{1}{4}$  of the parabolic segment, and consequently the segments  $AEB$ ,  $BDC$ , together, will be  $\frac{1}{4}$  of the whole portion  $AEBDC$ , or  $\frac{1}{3}$  of the triangle  $ABC$ .



Again, it will appear that, if in the two segments  $AEB$ ,  $BDC$ , be inscribed triangles  $AEB$ ,  $BDC$ , of the same bases and altitudes with them, then these last triangles will be  $\frac{1}{4}$  of their circumscribed segments, and, consequently,  $\frac{1}{4}$  of the triangle  $ABC$ ; if, in like manner, in the last made segments be inscribed the greatest triangles, they will be  $\frac{1}{4}$  of the triangles immediately preceding them, or  $\frac{1}{4 \cdot 4} = \frac{1}{16}$  of the first triangle; and so on continually: And consequently all the inscribed triangles, taken together, will be expressed by the series  $b \times : 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} \&c.$  where  $b$  denotes the first triangle; and which, when the series is infinitely continued, will denote the area of the parabolic segment.

## COROLLARY VI.

Hence the infinite series  $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \&c.$  will be  $= 1\frac{1}{3}$ , as we also know from other principles; and, consequently, the sum of any finite number of terms of the series  $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \&c.$  is less than  $\frac{1}{3}$ ; and the sum of the infinite number of them  $= \frac{1}{3}$ .



## EXAMPLE II.

If the base of a parabolic segment be 12, and its abscissa 2 make an angle of 30 degrees with it; what will be the area.

The sine of 30° being half of radius, the altitude will be  $2 \times \frac{1}{2} = 1$ ; and hence  $\frac{2}{3} \times 12 = 8 =$  the area.

## PROBLEM VI.

*To find the area of a frustum or zone of a parabola, included by two parallel right lines and the intercepted curves of the parabola.*

## RULE.

To one of the parallel ends add the quotient arising from the division of the square of the other by the sum of the said ends, multiply the sum by the altitude of the frustum or distance of the ends, and  $\frac{2}{3}$  of the product will be the area. Or divide the difference of the cubes of the diameters by the difference of their squares, and multiply the quotient by  $\frac{2}{3}$  of the altitude.

That is,  $D + \frac{dd}{D+d} \times \frac{2}{3}a$ , or  $d + \frac{DD}{D+d} \times \frac{2}{3}a$ , or  $\frac{D^3-d^3}{D^2-d^2} \times \frac{2}{3}a =$  the area,  $D, d$  being the two ends, and  $a$  the altitude.\*

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## \* DEMONSTRATION.

By the property of the parabola,  $DD-dd : a :: \left\{ \begin{array}{l} DD : \frac{aDD}{DD-dd} \\ dd : \frac{add}{DD-dd} \end{array} \right\}$   
 $=$  the altitudes of two compleat segments whose bases are the ends  $D, d$  of



## EXAMPLE I.

If the two parallel ends of the frustum of a parabola be 10 and 6, and the part of the abscissa, perpendicular to and connecting the middles of those ends, be 4; what will be the area.

Here the abscissa, being perpendicular to the ends, will be the altitude of the figure, and therefore

$$10 + \frac{6 \times 6}{10 + 6} \times \frac{4 \times 2}{3} = \frac{196}{6} = 32\frac{2}{3}, \text{ or } 6 + \frac{10 \times 10}{10 + 6} \times \frac{4 \times 2}{3} = \frac{196}{6} = 32\frac{2}{3} = \text{the area.}$$

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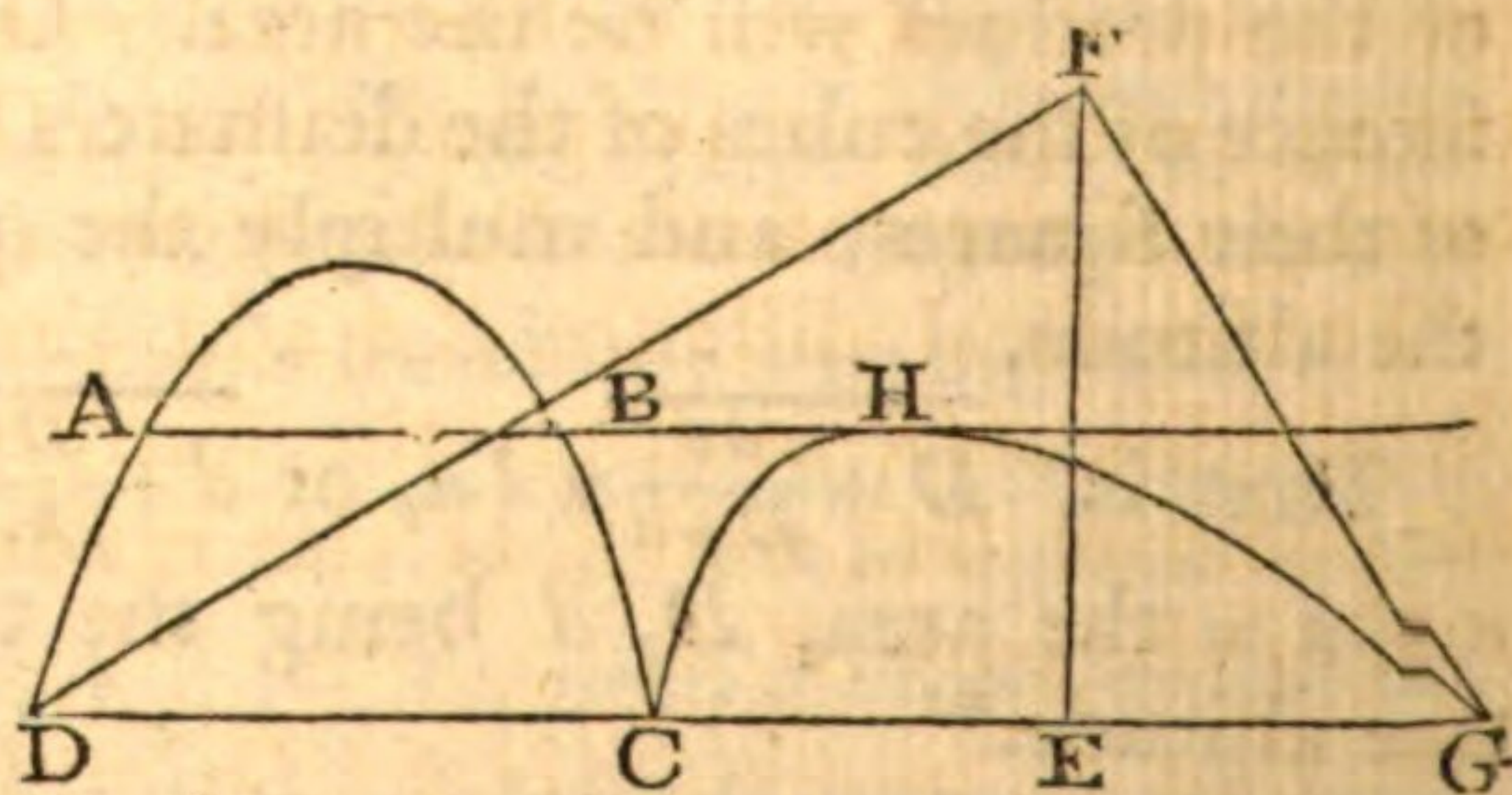
of the frustum; and, consequently, the frustum or their difference is  $\frac{2}{3}a \times \frac{D^3 \propto d^3}{DD \propto dd} = \frac{2}{3}a \times \frac{DD + Dd + dd}{D + d} = \frac{2}{3}a \times D + \frac{dd}{D + d} = \frac{2}{3}a \times d + \frac{DD}{D + d}$ . Q.E.D.

## COROLLARY.

Hence a parabolic frustum is equal to a parabola of the same altitude, and whose base is equal to one end of the frustum increased by a 3d proportional to the sum of the ends and the other.

Wherefore, in the one end, produced, of a parabolic frustum  $ABCD$ , take  $CE =$  the other end  $AB$ ; raise  $EF$  perpendicular to  $EC$  and  $= CD$ ; draw  $DF$ , and perpendicular to it  $FG$  meeting with  $CE$ , produced, in  $G$ : Then if upon the base  $CG$  be described a parabola  $CHG$  touching the line  $ABH$ , it will be equal to the frustum  $ABCD$ .

$$\text{For } CG = CE + EG = AB + EG = AB + \frac{FE^2}{ED} = AB + \frac{DC^2}{DC + AB}.$$





## EXAMPLE II.

If the part of the abscissa, connecting the middles of the two ends, make with them an angle of  $48^\circ 35\frac{5}{2}'$ ; required the area, the other dimensions being as in the first example.

The sine of  $48^\circ 35\frac{5}{2}'$  being  $\cdot 75$  or  $\frac{3}{4}$  very nearly, hence  $4 \times \frac{3}{4} = 3 =$  the altitude  $= \frac{3}{4}$  of that in the last example, and therefore the area here must be  $\frac{3}{4}$  of that above; that is,  $32\frac{2}{3} \times \frac{3}{4} = \frac{9^8}{4} = 24\frac{1}{2} =$  the area required.

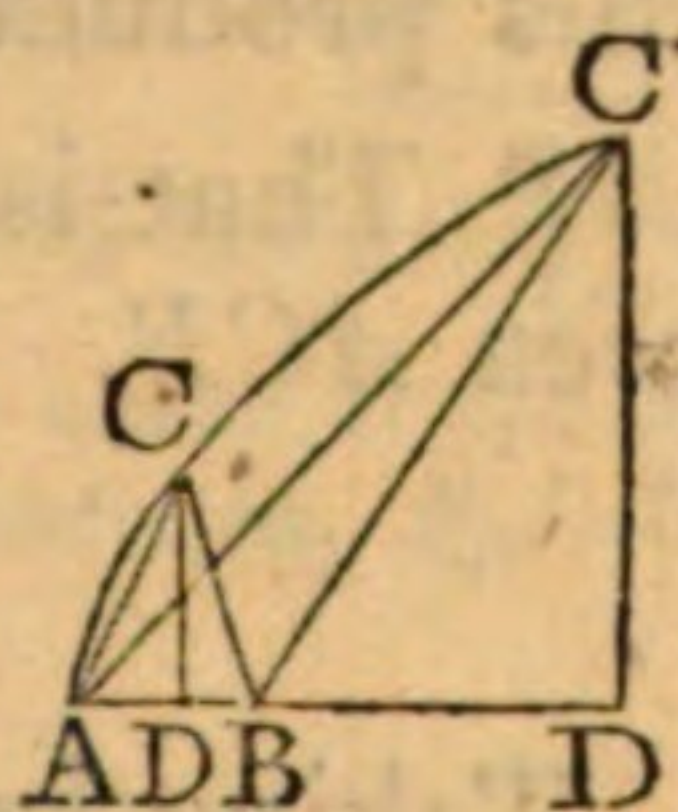
## PROBLEM VII.

*To find the area included by the focal distance, the line drawn from the focus to the curve, and the contained arc of the parabola.*

## RULE.

Upon the axe, or focal distance  $AB$ , produced if necessary, having demitted the ordinate or perpendicular  $CD$ , cutting off the abscissa  $DA$ ; then

To the focal distance  $BA$  add  $\frac{1}{3}$  of the abscissa  $AD$ ; multiply the sum by the ordinate  $DC$ , and  $\frac{1}{2}$  of the product will be the area of the part  $ACB$ .



\* That is,  $\overline{BA + \frac{1}{3}AD} \times \frac{1}{2}DC =$  the area.

Note, The focal distance  $AB$  is  $\frac{1}{4}$  of the parameter.

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## \* DEMONSTRATION.

Putting  $AD = x$ ,  $DC = y$ , and  $BA = a$ ;  $ACB = ACD \pm DCB$  will be  $= \frac{2}{3}xy \pm \frac{1}{2}y \times \mp x \pm a = \frac{2}{3}xy - \frac{1}{2}xy + \frac{1}{2}ay = \frac{1}{2}y \times \frac{1}{3}x + a$ .  
Q. E. D.



## EXAMPLE.

If the abscissa be 2, and the ordinate 6; required the area of  $ACB$ .

By case 1 prob. 2, we have  $2 : 6 :: 6 : 18 =$  the parameter, and hence  $\frac{1}{4} \cdot 18 = 4\frac{1}{2} = AB$ .

Then  $4\frac{1}{2} + \frac{2}{3} \times 3 = 5\frac{1}{6} \times 3 = 15\frac{1}{2} =$  the area required.

## RULE II.

subtract the focal distance, or distance between the focus and the beginning of the arc, from the distance between the focus and the end of the arc, and multiply the remainder by the said focal distance; then multiply the root of the product by the sum of the distance between the end of the arc and focus and double the focal distance, and  $\frac{1}{3}$  of this product will be the area.

\* That is,  $\frac{1}{3} \sqrt{AB \times CB - BA \times 2AB + BC} =$  the area  $ACB$ .

## EXAMPLE.

Taking here the same example as before, we have  $AB = 4\frac{1}{2}$ , and  $BC = DA + AB = 2 + 4\frac{1}{2} = 6\frac{1}{2}$ ; and hence  $\frac{1}{3} \sqrt{4\frac{1}{2} \times 2 \times 9 + 6\frac{1}{2}} = 15\frac{1}{2} =$  the area as before.

PRO-

## \* DEMONSTRATION.

By the nature of the parabola,  $z = CB$  is  $= BA + AD = a + x$ , and  $x = z - a$ , also  $\frac{1}{2}y = \sqrt{ax} = \sqrt{a \times z - a}$ ; which values of  $x$  and  $y$ , written for them in the last rule, give  $\frac{1}{3} \sqrt{a \times z - a \times 2a + z} =$  the area  $ACB$ . Q.E.D.



## PROBLEM VIII.

To find the curve surface of a paraboloid.

## RULE I.\*

To the square of the ordinate, or semi-diameter of the base, add 4 times that of the axe, and the root of

## \* DEMONSTRATION.

Calling the ordinate  $y$ , the abscissa  $x$ , the curve  $z$ , and the parameter  $p$ ; also  $a = 3.14159$  &c. the equation of the generating parabola will be  $px = yy$ ; hence  $p\dot{x} = 2y\dot{y}$ , and the fluxion of the surface

$$\dot{s} = 2ay\dot{z} = 2ay\sqrt{\dot{y}^2 + \dot{x}^2} = 2ay\sqrt{\dot{y}^2 + \frac{4y^2\dot{y}^2}{pp}} = \frac{4ay\dot{y}}{p}\sqrt{\frac{1}{4}pp + yy};$$

and, by taking the correct fluents, the surface  $s$  will be  $= \frac{4a}{3p} \times \frac{1}{4}pp + yy \Big|^{\frac{3}{2}}$

$$= \frac{2}{3}ay \times \frac{yy + 4xx \Big|^{\frac{3}{2}} - y^3}{4xx} = \frac{2}{3}ay \times \frac{yy + 4xx \Big|^{\frac{3}{2}} - y^3}{yy + 4xx \Big|^{\frac{3}{2}} - y^3}$$

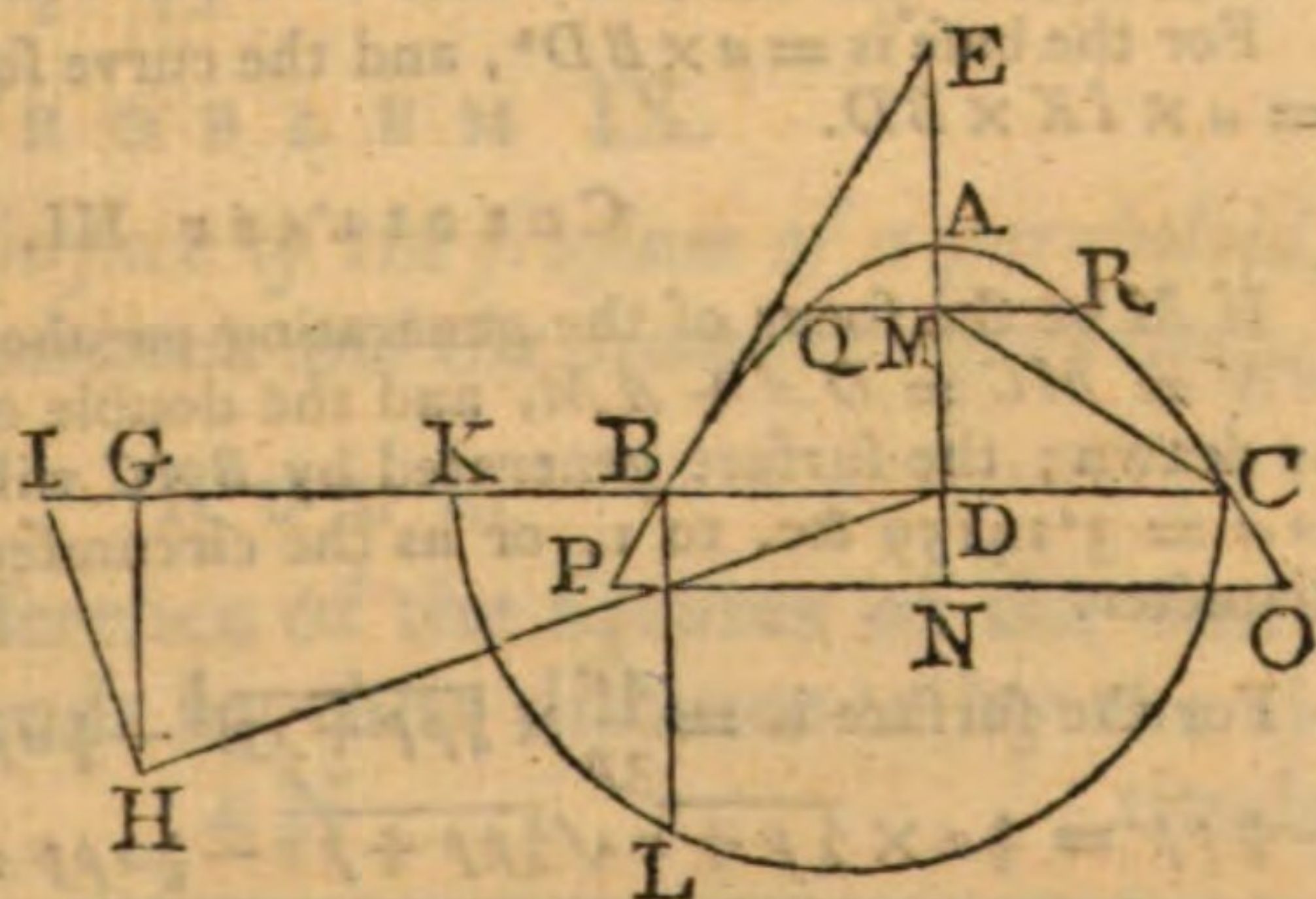
$$\times \frac{t^3 - y^3}{t^2 - y^2} = \frac{2}{3}ay \times \frac{tt + ty + yy}{t + y} = \frac{2}{3}ay \times t + \frac{yy}{t + y} = \frac{2}{3}ay \times y + \frac{tt}{t + y},$$

Putting  $t = \sqrt{yy + 4xx} = BE$  the tangent to the curve at the extremity of the ordinate and intercepted by the axe produced. Q.E.D.

## COROLLARY I.

In  $CB$ , produced, take  $BG =$  the tangent  $BE$ ; erect  $GH$  perpendicular to  $GB$  and equal to  $BD$ ; draw  $DH$ , and perpendicular thereto  $HI$  meeting  $BG$ , produced in  $I$ ; take  $BK = \frac{1}{3}$  of  $BI$ ; and upon  $KC$  describe a circle meeting in  $L$  with  $BL$  perpendicular to  $BC$ :

And the circle whose radius is  $BL$  will be equal to the curve surface of the paraboloid  $BAC$ .



For,



of the sum will be the tangent of the curve at the base and intercepted by the axe produced; let this tangent be called  $t$ , viz.  $t = \sqrt{yy + 4xx}$ ,  $x$  being the axe, and  $y$  the ordinate.

Then, by problem 6, find the area of the frustum of a parabola whose parallel ends are  $t$  and  $y$ , and its altitude  $= y$ .

And this area multiplied by 3.14159 &c. will be the curve surface of the paraboloid  $ABC$  required.

#### EXAMPLE.

Required the curve surface of a paraboloid whose axe is 20, and the diameter of its base 60.

Here

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$$\begin{aligned} \text{For, by the construction, } BL^2 &= KB \times BC = CB \times \frac{1}{3}BI = \\ \frac{2}{3}BD \times \overline{BG + GI} &= \frac{2}{3}BD \times BE + \frac{HG^2}{GD} = \frac{2}{3}BD \times BE + \frac{BD^2}{EB + BD} \\ &= \frac{2}{3}y \times t + \frac{yy}{t + y}. \end{aligned}$$

#### COROLLARY II.

The curve surface of the paraboloid is to the area of its circular base, as  $IK$  is to  $BD$ .

For the base is  $= a \times BD^2$ , and the curve surface  $= \frac{2}{3}IB \times BD \times a = a \times IK \times BD$ .

#### COROLLARY III.

If  $M$  be the focus of the generating parabola, and there be taken  $AN = MC = DA + AM$ , and the double ordinates  $PNO$ ,  $QMR$  be drawn; the surface generated by  $BAC$  will be to the area  $OPQR$ , as  $a = 3.14159$  &c. to 1, or as the circumference of a circle to its diameter.

$$\begin{aligned} \text{For the surface is } &= \frac{4a}{3p} \times \overline{\frac{1}{4}pp + yy}^{\frac{3}{2}} - \frac{1}{6}app = \frac{4a}{3p} \times \overline{\frac{1}{4}pp + px}^{\frac{3}{2}} \\ &- \frac{1}{6}pp = \frac{4}{3}a \times \frac{1}{4}p + x \sqrt{\frac{1}{4}pp + px} - \frac{1}{6}app = \frac{4}{3}a \times MC \sqrt{MC \times p} \\ &- \frac{1}{6}app = \frac{4}{3}a \times AN \times NO - \frac{4}{3}a \times AM \times MR = a \times PAO - QAR \\ &= a \times OPQR. \end{aligned}$$



Here  $\sqrt{30^2 + 40^2} = 50$  is the tangent. Then, by prob. 5,  $50 + \frac{30 \times 30}{50 + 30} \times \frac{2}{3} \times 30 \times 3.14159 \&c. = \frac{4900}{80} \times 20 \times 3.14159 \&c. = 1225 \times 3.14159 \&c. = 3848.451$  = the surface required.

## \* R U L E II.

Divide the difference between the cube of the tangent and the cube of the ordinate by 4 times the square of the axe, multiply the quotient by  $\frac{2}{3}$  of the ordinate, and the product multiplied by 3.14159 &c. will be the surface.

That is,  $\frac{t^3 - y^3}{4xx} \times \frac{2}{3}cy =$  the surface; where  $t = \sqrt{yy + 4xx}$  = the tangent,  $x$  = the abscissa or axe,  $y$  = the ordinate or semi-diameter of the base, and  $c = 3.14159 \&c.$

## E X A M P L E.

Taking the same example as in the last, in which  $x = 20$ ,  $y = 30$ , and therefore  $t = \sqrt{40^2 + 30^2} = 50$ ; we shall have  $\frac{50^3 - 30^3}{40 \times 40} \times 30 \times \frac{2}{3}c = \frac{980}{16} \times 20c = 1225 \times 3.14159 \&c. = 3848.451$  = the surface as before.

## P R O B L E M IX.

*To find the curve surface of the frustum of a paraboloid, having given its altitude and the diameters of its ends.*

## R U L E.

Divide the difference of the squares of the semi-diameters of the ends by their distance or by the altitude

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\* This rule appears in the investigation of the last.



titude of the frustum, or divide the difference of the squares of the whole diameters by 4 times the altitude, and the quotient will be the parameter of the axe of the generating parabola.

Find the two sums arising from the addition of the square of the parameter and the square of each diameter; multiply each sum into its root, and divide the difference of the products by the parameter; multiply the quotient by 3.14159 &c. and  $\frac{1}{6}$  of the product will be the surface of the frustum.

\* That is,  $\frac{\overline{pp + DD}^{\frac{3}{2}} - \overline{pp + dd}^{\frac{3}{2}}}{p} \times \frac{1}{6}c =$  the surface,  $D$  and  $d$  being the diameters of the ends,  $p =$  the parameter  $= \frac{DD - dd}{4a}$ ,  $a =$  the altitude, and  $c = 3.14159$  &c.

EX-

## \* DEMONSTRATION.

In the investigation of rule I of the last problem, it appears that  $\frac{4c}{3p} \times \overline{\frac{1}{4}pp + yy}^{\frac{3}{2}} - \frac{1}{6}cpp$  or  $\frac{c}{6p} \times \overline{pp + DD}^{\frac{3}{2}} - \frac{1}{6}cpp$  is the surface of the segment whose base diameter is  $D$ , and  $\frac{c}{6p} \times \overline{pp + dd}^{\frac{3}{2}} - \frac{1}{6}cpp =$  that whose diameter is  $d$ ; and, by taking the difference,  $\frac{1}{6}c \times \frac{\overline{pp + DD}^{\frac{3}{2}} - \overline{pp + dd}^{\frac{3}{2}}}{p}$  will express the surface of the frustum the diameters of whose ends are  $D, d$ .

And that the parameter is  $= \frac{DD - dd}{4a}$ , appears thus: Since  $pX = rY$ , and  $px = yy$ , therefore  $pX - px = rY - yy$ , and  $p = \frac{rY - yy}{X - x} = \frac{DD - dd}{4a}$ .



## EXAMPLE.

Required the curve surface of the frustum of a paraboloid, the altitude being  $25\frac{7}{8}$  and the diameters of its ends 48 and 15.

Here  $D = 48$ ,  $d = 15$ , and  $a = 25\frac{7}{8}$ .

Therefore  $\frac{DD-dd}{4a} = \frac{48^2 - 15^2}{4 \times 25\frac{7}{8}} = \frac{2079}{103\frac{1}{8}} = 20 = p$ .

Then  $\sqrt{pp+DD} = \sqrt{20^2 + 48^2} = \sqrt{2704} = 52$ .

And  $\sqrt{pp+dd} = \sqrt{20^2 + 15^2} = \sqrt{625} = 25$ .

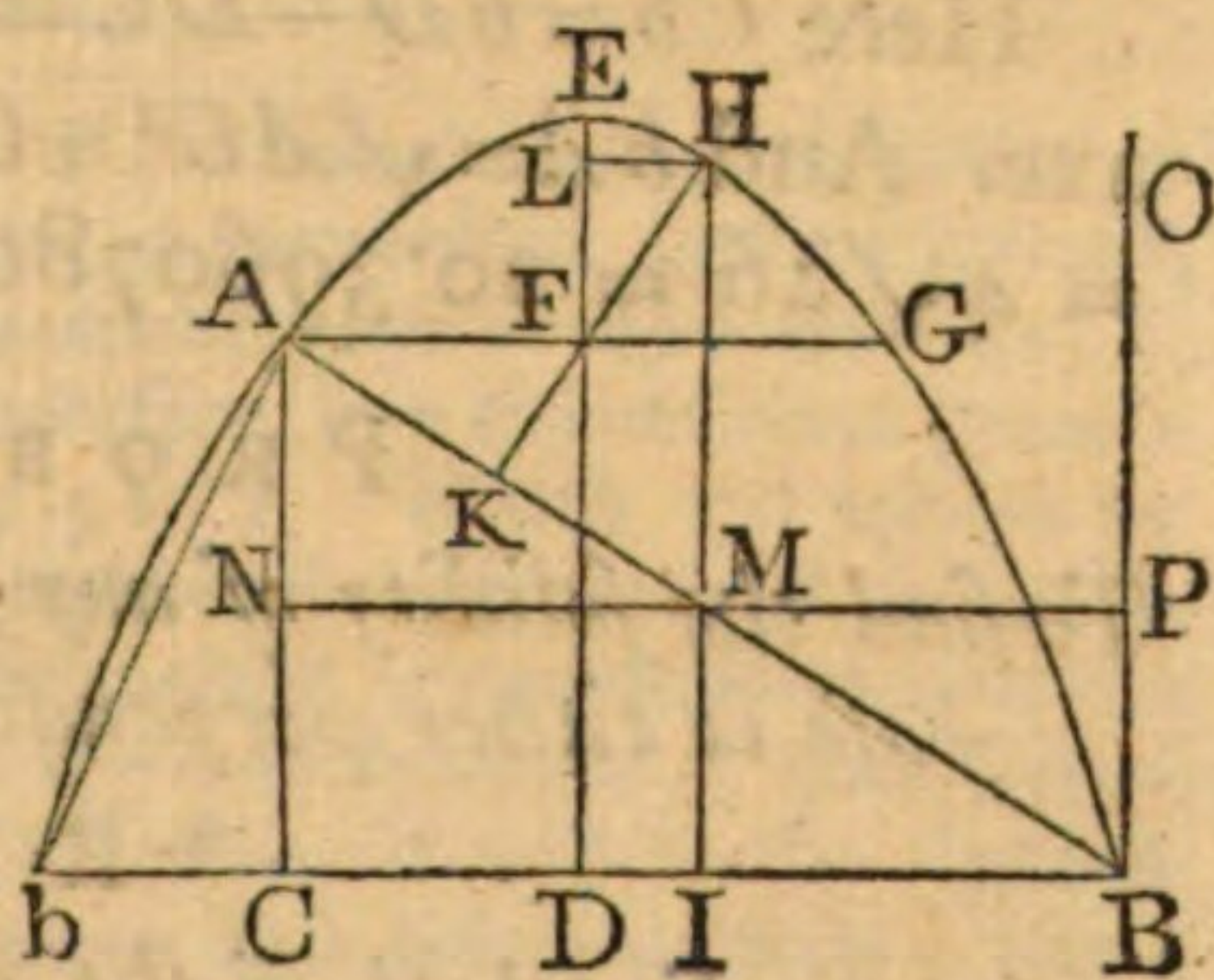
Hence  $\frac{2704 \times 52 - 625 \times 25}{20 \times 6} \times c = \frac{41661}{40} \times 3.14159 \&c.$   
 $= 4166.1 \times .785398 \&c. = 3272.0472885 =$  the surface required.

## PROBLEM X.

*If a paraboloid be cut by a plane, oblique to its axe, it is required to find the axes of the elliptic section.*

## RULE.

It is evident that  $AB$  is the transverse axe. And if  $Bb$  be a double ordinate to the axe and meeting the diameter  $AC$  in  $C$ , then  $BC$  will be the other or conjugate axe of the section. As is proved in corollary 7 to prop. 1 sect. 2.



## EXAMPLE.

If a paraboloid whose axe is  $45\frac{5}{7}$ , and its base or greatest double ordinate 32, be cut by a plane passing through the extremity of the base and the opposite side



side of the figure at the height of 20 above the base; what are the axes of the section.

Draw  $AF$  perpendicular to and meeting with the axe  $DE$  in  $F$ .—Then  $FE = ED - DF = 45\frac{5}{7} - 20 = 25\frac{5}{7}$ ; and, by the nature of the parabola,  $DE : EF :: DB^2 : FA^2 = \frac{DB^2 \times FE}{ED} = \frac{16^2 \times 25\frac{5}{7}}{45\frac{5}{7}} = \frac{16^2 \times 5\frac{1}{7}}{9\frac{1}{7}} = \frac{16^2 \times 36}{64} = 4 \times 36 = 144$ ; and hence  $FA = DC = \sqrt{144} = 12$ : wherefore  $CB = BD + DC = 16 + 12 = 28 =$  the conjugate axe. And  $AB = \sqrt{BC^2 + CA^2} = \sqrt{28^2 + 20^2} = 4\sqrt{7^2 + 5^2} = 4\sqrt{74} = 34.4093010682 =$  the transverse axe required.

#### EXAMPLE II.

Let the same paraboloid be cut through the extremity of the base by a plane cutting the figure again on the same side of the axe, and at the same distance of 20 above the base; to find the axes of the section.

Here  $Cb = bD - DC = 16 - 12 = 4 =$  the conjugate axe. And  $bA = \sqrt{AC^2 + Cb^2} = \sqrt{20^2 + 4^2} = 4\sqrt{5^2 + 1^2} = 4\sqrt{26} = 20.39607804 =$  the transverse.

#### PROBLEM XI.

*To find the solidity of any segment of a paraboloid, whose base is either perpendicular or oblique to the axe.*

##### \* RULE I.

*For both right and oblique segments.*

Multiply the base by half the altitude, and the product will be the content.

NOTE.

##### \* DEMONSTRATION.

Let  $b$  denote the base,  $a$  the altitude,  $x =$  any variable abscissa, or part of the diameter drawn through the middle of the base, and  $s =$



## NOTE.

When the base is perpendicular to the axe, it is a circle, and the altitude is equal to the whole length of the axe contained between the base and the vertex.—But if the base be oblique to the axe, it will be an ellipse, and the altitude will be the perpendicular

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cular

$s$  = the sine of the angle formed by this diameter and its ordinate, to the radius 1.

Then as  $a : sx :: b : \frac{bsx}{a}$  = the section corresponding to the abscissa  $x$ , by prop. 1 sect. 2. Wherefore  $\frac{sx}{a} \times bsx$  = the fluxion of the solid; and, by taking the fluent, the solid itself will be  $\frac{bsxxx}{2a}$ ; which, when  $sx = a$ , becomes  $\frac{1}{2}ab$  for the whole solid whose base is  $b$  and altitude  $a$ . Q.E.D.

## COROLLARY I.

A paraboloid is equal to half a prism of the same base and altitude; or a prism, a semi-spheroid or semi-sphere, a paraboloid, and a pyramid, all of equal bases and altitudes, are to one another as the numbers 1,  $\frac{2}{3}$ ,  $\frac{1}{2}$ , and  $\frac{1}{3}$ , or as 6, 4, 3, 2; and are, therefore, in a discontinued geometric proportion whose ratio is that of 3 to 2. Which will appear by comparing the above value of the paraboloid with those of the other solids.—When the base of the paraboloid is perpendicular to the axe, or when the paraboloid is right, the semi-spheroid or semi-sphere will, also, be right, and the prism and pyramid will be the right or common cylinder and cone, the common base being a circle; but if the paraboloid be oblique by having its base oblique to its axe, the common base of all the solids will be an ellipse, the semi-spheroid will, also, be oblique by having its base oblique to its axe, and the prism and pyramid may be either right or oblique.

## COROLLARY II.

If upon the diameter  $NP = BC$  = the conjugate axe of the section  $AB$  be described a circle, the oblique paraboloid or segment  $AHB$  will



cular demitted to the base from the vertex of the diameter drawn through the middle of the base, or it will be equal to the product arising from the multiplication of the part of the said diameter, intercepted by the vertex and the base, by the sine of the angle of inclination to the base.

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will be equal to the right paraboloid whose base is that circle, and its altitude  $MH$ . For, the diameter of the circle being the conjugate axe of the base of the oblique paraboloid, the circle will be to that base as the conjugate is to the transverse axe, that is, as  $CB$  to  $BA$ ; but  $KH$  is to  $HM$  in the same proportion of  $CB$  to  $BA$ ; consequently  $KH \times$  elliptic base  $AB$  will be  $= HM \times$  circular base  $NP$  or  $BC$ .

It is also evident, that all segments, right or oblique, having the same vertex  $H$ , and the transverse axes of whose bases pass through the same point  $M$  and terminate in  $AC$ ,  $OP$ , parallel to and equidistant from  $HM$ , will be equal to one another.

## COROLLARY III.

If  $B$ ,  $b$  denote the two bases or parallel ends of the frustum of a paraboloid, either right or oblique, and  $d$  the distance of the ends, or the altitude of the frustum; then  $\frac{1}{2}d \times \overline{B+b}$  will be the solidity of the frustum.

For, by this problem,  $\frac{1}{2}AB$  is the segment whose base is  $B$  and altitude  $A$ , and  $\frac{1}{2}ab$  is that whose base is  $b$  and altitude  $a$ ; wherefore the frustum, or the difference of the segments, is  $\frac{1}{2}AB - \frac{1}{2}ab$ : But,

by the nature of the paraboloid,  $B-b : d(A-a) :: B : A = \frac{Bd}{B-b}$ ,

and  $B-b : d :: b : a = \frac{bd}{B-b}$ ; which values of  $A$  and  $a$  being sub-

stituted for them, we obtain  $\frac{1}{2}AB - \frac{1}{2}ab = \frac{dB^2 - db^2}{2B - 2b} = \frac{1}{2}d \times \overline{B+b}$ .

## COROLLARY IV.

Let  $AGBb$  be a right frustum,  $HI$  the diameter to the double ordinate  $AB$ , and  $HK$  perpendicular to  $AB$ , that is,  $HK$  the altitude of the oblique segment  $AHB$ . Then the value of the oblique segment  $AHB$  is  $\frac{1}{2}n \times AB \times BC \times HK$ , putting  $n = .785398$  &c.

But



## EXAMPLE I.

Required the solidity of a right paraboloid whose  
axe is 30 and diameter of its base 40.

By the rule,  $40^2 \times 15 \times .785398 \&c. = 18849.5559215$   
= the solidity required.

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But, by the property of the figure,  $BD^2 - DC^2 : CA :: BD^2 - DI^2 : BD^2 - DI^2$   
 $\frac{BD^2 - DI^2}{BD^2 - DC^2} \times CA = IH$ ; hence  $MH = HI - IM = HI - \frac{1}{2}CA$   
 $= \frac{BD + DC}{BD - DC} \times \frac{1}{4}CA = \frac{BC \times CA}{8DI}$ ; and, by similar triangles,  $AB :$   
 $BC :: MH : HK = \frac{HM \times CB}{BA} = \frac{BC^2 \times CA}{AB \times 8DI}$ ; therefore the ob-  
lique segment  $AHB$ , or  $\frac{1}{2}n \times AB \times BC \times HK$ , will be  $= \frac{BC^3}{8DI} \times$   
 $\frac{1}{2}CA \times n = \frac{BC^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$ .

Or, by substituting the value of  $CA$ , viz.  $\frac{BD^2 - DC^2}{BD^2} \times DE$   
 $= \frac{BC \times 2DI}{BD^2} \times DE$ , instead of it, the same segment  $AHB$  will be  
 $= \frac{BC^4}{8BD^2} \times DE \times n = \frac{BC^4}{Bb^2} \times DE \times \frac{1}{2}n$ .

## COROLLARY V.

If from  $DE \times Bb^2 \times \frac{1}{2}n$ , the value of the right segment  $bEB$ , be  
taken the above value of the oblique segment  $AHB$ , there will remain  
 $\frac{Bb^4 - BC^4}{Bb^2} \times DE \times \frac{1}{2}n$ , or  $\frac{Bb^4 - BC^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$  for the value  
of the greater ungula  $bAB$ .

## COROLLARY VI.

And if from the said value of the oblique segment  $AHB$ , be taken  
that of the right segment  $AEG$ , viz.  $\frac{AG^4}{Bb^2} \times DE \times \frac{1}{2}n$  or  $\frac{AG^4}{Bb^2 - AG^2}$   
 $\times CA \times \frac{1}{2}n$ , there will remain  $\frac{BC^4 - AG^4}{Bb^2} \times DE \times \frac{1}{2}n$  or  $\frac{BC^4 - AG^4}{Bb^2 - AG^2}$   
 $\times CA \times \frac{1}{2}n$  for that of the less ungula  $AGB$ .



## EXAMPLE II.

If from the above right paraboloid be cut a part  $BAb$ , by a plane passing through the extremity of the base, and the opposite side at the height  $AC$  of  $22\frac{1}{2}$  above the base; it is required to find the content of the oblique segment  $AHB$ .

Here  $FE = ED - CA = 30 - 22\frac{1}{2} = 7\frac{1}{2}$ ; then, by the property of the parabola,  $\sqrt{DE} : \sqrt{EF} :: BD : AF = BD \sqrt{\frac{FE}{ED}} = 20 \sqrt{\frac{7\frac{1}{2}}{30}} = 20 \sqrt{\frac{1}{4}} = 20 \times \frac{1}{2} = 10$ ; hence  $BD + DC = BD + AF = 20 + 10 = 30 = CB$  the conjugate diameter of the base, by the last problem; and  $\sqrt{AC^2 + CB^2} = \sqrt{22\frac{1}{2}^2 + 30^2} = 7\frac{1}{2} \sqrt{3^2 + 4^2} = 7\frac{1}{2} \times 5 = 37\frac{1}{2} = BA$  the transverse. Now  $H$  being the vertex of the diameter  $HMI$  drawn through  $M$  the middle of  $AB$ , and  $HK$  perpendicular to  $AB$ , the triangles  $HKM$ ,  $MIB$ ,  $BAC$ , will be similar; but  $MB = \frac{1}{2}BA$ , and therefore  $MI = \frac{1}{2}CA = 11\frac{1}{4}$ ; but  $HL = DI = \frac{BD - DC}{2} = \frac{20 - 10}{2} = 5$ ; then, by the property of the parabola,  $BD^2 : BD^2 - LH^2 :: DE : HI = \frac{BD^2 - LH^2}{BD^2} \times DE = \frac{20^2 - 5^2}{20^2} \times 30 = \frac{4^2 - 1^2}{4^2} \times 30 = \frac{15^2}{8} = \frac{225}{8} = 28\frac{1}{8}$ ; hence  $HI - IM = 28\frac{1}{8} - 11\frac{1}{4} = 16\frac{7}{8} = MH$ ; and, by similar triangles,  $AB : BC :: MH : HK = \frac{MH \times CB}{BA} = \frac{16\frac{7}{8} \times 30}{37\frac{1}{2}} = 13\frac{1}{2} =$  the altitude.

Then  $CB \times BA \times \frac{1}{2}HK \times .785398 \&c. = 30 \times 37\frac{1}{2} \times 6\frac{7}{8} \times .785398 \&c. = 7593\frac{7}{8} \times .785398 \&c. = 5964\cdot1173033 =$  the segment  $AHB$  required.

RULE



## R U L E II.

*For the oblique segment.*

Divide the fourth power of  $BC$  by the square of  $Bb$ , multiply the quotient by  $\frac{1}{2}DE$ , and the product multiplied by  $\cdot 785398 \&c.$  will be the content of the oblique segment  $AHB$ .

That is,  $\frac{BC^4}{Bb^2} \times \frac{1}{2}DE \times n = AHB$ . By corollary 4.

## E X A M P L E.

Taking here the last example, we have  $\frac{BC^4}{Bb^2} \times \frac{1}{2}DE \times n = \frac{30^4}{40^2} \times 15 \times \cdot 785398 \&c. = \frac{30^2 \times 9 \times 15}{16} \times \cdot 785398 \&c. = 759375 \times \cdot 785398 \&c. = 5964 \cdot 1173033$  the same as before.

## P R O B L E M XII.

*To find the content of the frustum of a paraboloid, having its ends parallel to each other, and either right or oblique to the axe.*

## R U L E.

Multiply the sum of the two ends by their distance or altitude of the frustum, and half the product will be the content.—As is proved in corollary 3 to the last problem.

## N O T E.

The same things may be observed here as in the note to the last problem.

## E X A M P L E.

The length of a cask, composed of two equal frustums of a paraboloid, is 45 inches; required the

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content



content in ale and wine gallons, supposing the bung diameter to be 40 and the head diameter 20 inches.

Here  $40^2 + 20^2 \times .785398 \&c. \times 22\frac{1}{2} = 45000 \times .785398 \&c. = 35342.917352885 =$  the content in inches.

Then  $\frac{35342.91735}{282} = 125.329494$  ale gallons.

And  $\frac{35342.91735}{231} = 152.99964$  wine gallons.

### PROBLEM XIII.

*To find the solidity of the parabolic ungula bAB, or BAG, made by a plane passing through the opposite extremities of the ends of the frustum AGBb.*

### RULE.

Divide the difference of the 4th powers of the diameter of the base, and half the sum of the diameters of the ends of the frustum, by the difference of the squares of the said diameters; multiply the quotient by .785398 &c. and the product multiplied by half the altitude will produce the content.

That is,  $\frac{Bb^4 - BC^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n =$  the ungula bAB,

And  $\frac{BC^4 - AG^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n =$  the ungula BAG.—

As is proved in corol. 4 and 5 of prob. 10.

### EXAMPLE I.

If a vessel, in form of the frustum of a paraboloid, and open at the narrow end, be so placed, that the liquor in it may just cover the bottom, whose diameter is 40 inches, and rise to the height of  $22\frac{1}{2}$  inches



inches towards the top of it; required the quantity Of liquor contained in the vessel, supposing the diameter of the vessel at the upper edge of the liquor to be 20 inches.

Here  $Bb = 40$ ,  $CA = 22\frac{1}{2}$ , and  $AG = 20$ ; consequently  $BC = \frac{40^2 - 20^2}{2} = 30$ .

Hence  $\frac{40^4 - 30^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{.785398 \text{ \&c.}}{2} = \frac{17500}{96} \times 90 \times .785398 \text{ \&c.} = 12885.438618$  cubic inches; which being divided by 282 or by 231, will give 45.693044 ale, or 55.781119 wine gallons.

## EXAMPLE II.

If the vessel have its dimensions inverted, that is, if  $AGB$  be the part filled,  $AB$  being the surface of the liquor, the bottom diameter  $AG$  being 20, the diameter  $Bb$  at the top of the liquor 40, and the altitude  $AC$  22 $\frac{1}{2}$  inches; required the quantity of liquor.

Here  $\frac{BC^4 - AG^4}{Bb^2 - AG^2} \times AC \times \frac{1}{2}n = \frac{30^4 - 20^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{.785398 \text{ \&c.}}{2} = \frac{6500}{96} \times 90 \times .785398 \text{ \&c.} = 4786.020058$  cubic inches = 16.971702 ale gallons = 20.718701 wine gallons.

## PROBLEM XIV.

To find the solidity of the slice  $ABC$  cut off a paraboloid  $ADE$  by a plane  $FCG$  parallel to the axe  $DH$ .

## RULE.

Multiply the base of the slice by the square of the diameter of the base of the whole paraboloid, and divide







## EXAMPLE I.

If a cask, composed of two equal paraboloidal frustums, having its length = 40, its bung diameter 32, and its head diameter 24 inches, when laid with its axe parallel to the horizon, want so much of being full that there are  $3\frac{1}{5} = 3.2$  inches of the bung diameter left dry; it is required to find how many ale gallons will be necessary to fill the cask.

Here 3.2 is the versed sine of the base of the slice, and  $\frac{3.2}{32} = .1 =$  the tabular versed sine, to which, in the table, corresponds the area .04087528, and hence  $.04087528 \times 32^2 = 41.85628672 =$  the base of the slice.

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But

## COROLLARY I.

The slice  $ABC$  is equal to a paraboloid of the same altitude, and whose base is equal to the difference between  $\frac{2}{3}$  of the rectangular area  $H B G I$  and a fourth proportional to the areas  $B G^2$ ,  $H A^2$ ,  $F A G F$ .

## COROLLARY II.

From the demonstration it appears that the slice is also equal to

$$a \times B A G - \frac{y^3 \sqrt{rr - yy}}{3rr}.$$

## COROLLARY III.

When  $B$  coincides with  $H$ ,  $A$  will be  $= a$ , and  $r = y$ ; and therefore the rule becomes  $a \times H A K =$  half the paraboloid.

## COROLLARY IV.

From  $a \times H A K =$  half the paraboloid, subtract  $a \times B A G - \frac{ay^3 \sqrt{rr - yy}}{3rr} =$  the slice  $ABC$ , and there will remain  $a \times H B G K + \frac{ay^3 \sqrt{rr - yy}}{3rr} = a \times H B G K + \frac{1}{3} yy \sqrt{DL \times LH}$  for the value of the complementary slice  $D H B C$ ;  $CL$  being drawn parallel to  $HA$ .

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But, by the property of the circle,  $2\sqrt{32 - 3 \cdot 2 \times 3 \cdot 2} = 2\sqrt{28 \cdot 8 \times 3 \cdot 2} = 19 \cdot 2 =$  the chord of the base of the slice, and  $16 - 3 \cdot 2 = 12 \cdot 8 =$  its distance from the center or middle of the bung diameter. Also, by the nature of the parabola,  $16^2 - 12^2 : 20 :: 9 \cdot 6^2 : \frac{9 \cdot 6^2 \times 20}{28 \times 4} = \frac{24^2}{35} = 16\frac{1}{3}\frac{6}{5} =$  the altitude of the slice, or half the length of the empty part.

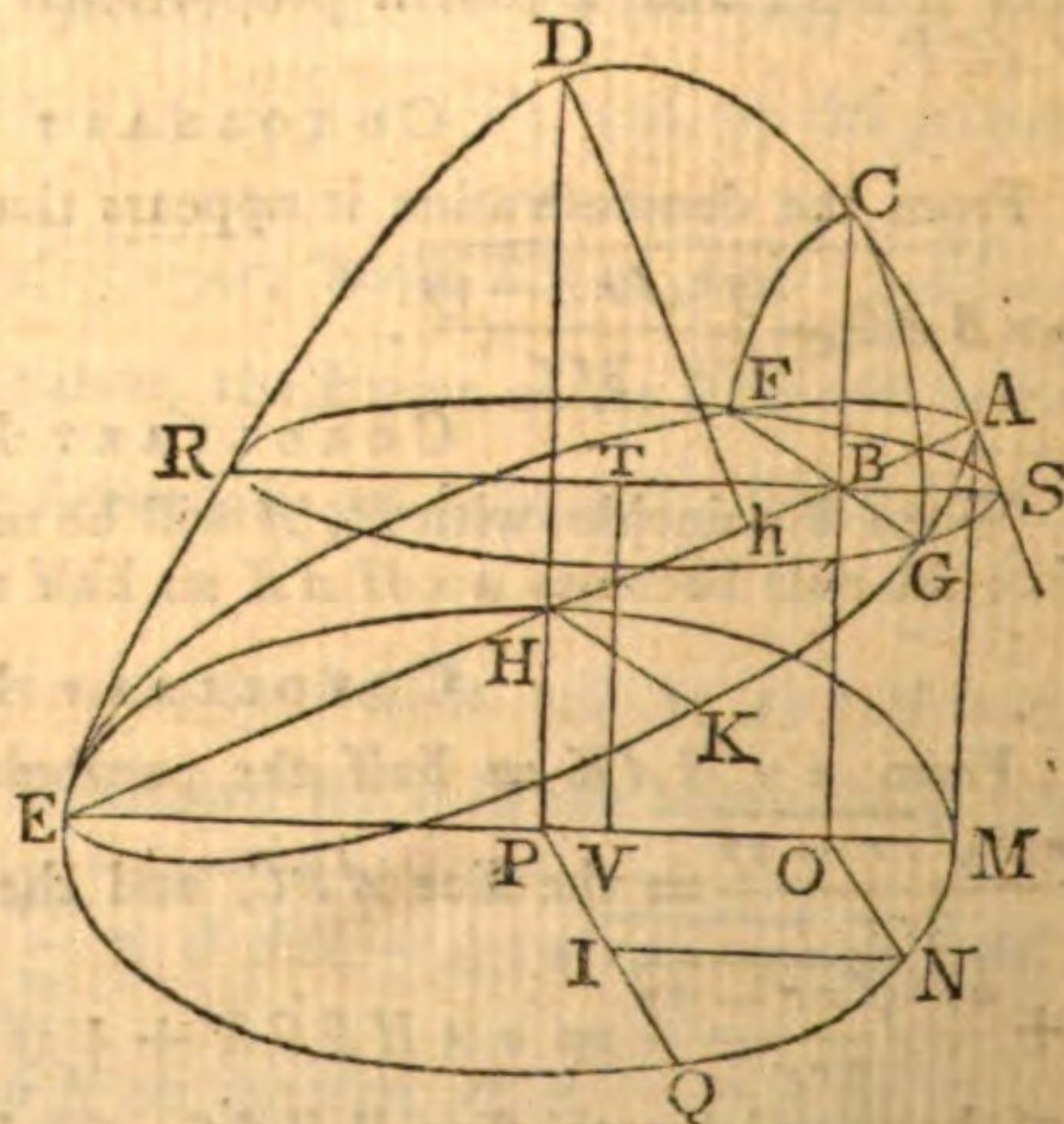
Then,  $\frac{41 \cdot 85628672 \times 32^3}{19 \cdot 2 \times 19 \cdot 2} - \frac{19 \cdot 2 \times 12 \cdot 8}{3} \times \frac{24^2}{35} =$   
 $\frac{41 \cdot 85628672 \times 5^3}{3^3} - 6 \cdot 4 \times 12 \cdot 8 \times \frac{24^2}{35} =$   
 $41 \cdot 85628672 \times 5 - 4 \times 19 \cdot 2^2 \times \frac{8^2}{7} = 565 \cdot 2611072$   
 cubic inches =  $2 \cdot 004472$  ale gallons = the content of the empty part required.

Ex-

## COROLLARY V.

After the same manner may be found the slice parallel to the axe of an oblique segment  $ADE$ .

Thus, since  $\frac{2}{3}FG \times CB \times BA \times s. \angle ABC$  is the fluxion of the slice  $ACB$ , by putting  $a = DH$  the diameter of the double ordinate  $AE$ ,  $t = HA$  the semi-transverse axe of the section  $AFEGA$ ,  $c = HK = \frac{1}{2}EM =$  the semi-conjugate,  $x = AB$ , and  $y = BG$ ; we shall have  $cc :$   
 $a :: yy : \frac{ayy}{cc} = BC$ ; but  
 $cc : tt :: yy : tt - t - x^2$ ,



hence



## EXAMPLE II.

If the same cask be placed in the same manner, and  $6\frac{2}{5}$  inches of the bung diameter be dry; required how many ale gallons the cask wants of being full.

Here the dry inches being more than  $16 - 12 = 4$  the difference of the bung and head semi-diameters, shews that  $6\frac{2}{5} - 4 = 2\frac{2}{5}$  inches of the head diameter also

hence  $t - x = \frac{t\sqrt{cc-yy}}{c}$ , and  $\dot{x} = \dot{B}A = \frac{ty\dot{y}}{c\sqrt{cc-yy}}$ ; also s.

$L ABC$  is  $= \frac{c}{t}$ ; consequently the above fluxion will become  $= \frac{4}{3}y$

$\times \frac{ay\dot{y}}{cc} \times \frac{ty\dot{y}}{c\sqrt{cc-yy}} \times \frac{c}{t} = \frac{4ay^4\dot{y}}{3cc\sqrt{cc-yy}}$ ; the fluent of which viz.  $\frac{ac}{t}$

$\times$  elliptic area  $BAG - \frac{ay^3\sqrt{cc-yy}}{3cc}$  will be the value of the slice  $BCA$ .

Or, if  $Dh$  be drawn perpendicular to  $AE$ , since, by similar triangles, it is  $t : c :: a : \frac{ac}{t} = Dh$ , the same slice  $BCA$  will be ex-

pressed by  $Dh \times BAG - \frac{ay^3\sqrt{cc-yy}}{3cc} = Dh \times BAG -$

$\frac{DH \times FG^3 \sqrt{EM^2 - FG^2}}{12EM^2}$ .

And if  $CB$  be produced to meet  $EM$  in  $O$ , and  $ON$  be perpendicular to  $EM$  and meet the circle  $ENM$  in  $N$ ; then will the circle  $ENM$  be to the ellipse  $EGA$ , as well as the segment  $NMO$  to the segment  $GAB$ , as  $ME$  to  $EA$ , as  $c$  to  $t$ ; and therefore the value

of the slice  $BCA$  will also be  $a \times MNO - \frac{ay^3\sqrt{cc-yy}}{3cc} = DH \times$

$MNO - \frac{DH \times FG^3 \sqrt{EM^2 - FG^2}}{12EM^2}$ . Where the circular area

$MNO$  is the projection of the elliptic area  $ABG$ , by lines parallel to the diameter  $DH$  upon a plane perpendicular to the same.

More



also is dry; and therefore the empty part of the cask will be double the difference of two slices the versed fines of whose bases are  $6\frac{2}{5}$  and  $2\frac{2}{5}$ , the whole diameters being 32 and 24.

Now the tabular versed fines will be  $\frac{6.4}{32} = .2$ , and  $\frac{2.4}{24} = .1$ , whose tabular areas are .11182380 and .04087528; hence  $32^2 \times .1118238 = 114.5075712 =$  the base of the greater slice, and  $24^2 \times .04087528 = 23.54416128 =$  that of the less.

Again,

---

Moreover, since  $yy : cc :: BC = A : a = HD = \frac{Acc}{yy}$ , by substituting this value of  $a$  instead of it, the solidity of the same slice will become  $\frac{Acc^3}{yy} \times AGB - \frac{1}{3} Ay \sqrt{c^2 - y^2} = BC \times \frac{PM^2}{NO^2} \times MNO - \frac{1}{3} NO \times OP$ .

COROLLARY VI.

Hence the slice  $FGAC$  is equal to a paraboloid whose altitude is  $BC$ , and its base equal to the difference between  $\frac{2}{3}$  of the rectangular area  $PINO$  and the fourth proportional to the areas  $NO^2$ ,  $MP^2$ ,  $2MNO$ .

COROLLARY VII.

When  $B$  coincides with  $H$ , the rule becomes barely  $DH \times MOP = Db \times HKA$  for the value of  $DAH = DEH =$  half the oblique segment.

COROLLARY VIII.

By taking the slice from the semi-segment, we obtain  $DH \times PONQ + DH \times \frac{y^3 \sqrt{cc - yy}}{3cc} = DH \times PONQ + DH \times \frac{NO^3 \times OP}{3PM^2}$  for the value of the complemental slice  $DCBH$ .

COROLLARY IX.

And by adding this last to the semi segment, there results  $DH \times ENO + DH \times \frac{NO^3 \times OP}{3PM^2} = \frac{BC \times PM^2}{ON^2} \times ENO + BC \times \frac{1}{3} NO \times OP$  for the value of the slice  $EDCBE$  greater than the semi-segment.

C o-



Again,  $16 - 6.4 = 9.6 =$  the distance of the chord of the base of the greater slice from the middle of the bung diameter, and  $12 - 2.4 = 9.6 =$  the distance of that of the less from the middle of the head diameter; and hence the former chord will be  $= 2\sqrt{16^2 - 9.6^2} = 2\sqrt{25.6 \times 6.4} = 25.6$ , and the latter  $= 2\sqrt{12^2 - 9.6^2} = 2\sqrt{21.6 \times 2.4} = 14.4$ .

Moreover, by the nature of the parabola, as

$$16^2 - 12^2 \text{ or } 28 \times 4 : 20 :: \left\{ \begin{array}{l} 12 \cdot 8^2 : \frac{20 \times 3 \cdot 2^2}{7} = \frac{32^2}{35} = 29\frac{9}{35} \\ 7 \cdot 2^2 : \frac{20 \times 1 \cdot 8^2}{7} = \frac{18^2}{35} = 9\frac{9}{35} \end{array} \right\}$$

$=$  the altitudes of the slices.

Consequently,  $\frac{114.5075712 \times 32^2}{25.6^2} - \frac{25.6 \times 9.6}{3} \times \frac{32^2}{35} =$   
 $\frac{114.5075712 \times 5^2}{4^2} - 25.6 \times 3.2 \times \frac{32^2}{35} =$   
 $114.5075712 \times 5 - .4 \times 25.6^2 \times \frac{8^2}{7} = 2837.886683 =$   
 the double of the greater slice.

4 R

And

## COROLLARY X.

If  $RG SFR$  be a circle perpendicular to  $BC$ , and if from its center  $T$  be demitted the perpendicular  $TV$ ; then, by taking the oblique slice  $ABC$  from the right slice  $SBC$ , we shall obtain  $BC \times \frac{RS^2 \times BSG - EM^2 \times OMN + FG^3 \times \frac{1}{8}PV}{FG^2}$  for the value of the ungula or cuneus  $AFGSA$  contained between the two sections  $RFSGR$ ,  $EFAGE$ . And the same will hold though both the sections be oblique.

## COROLLARY XI.

And, in the same manner,  $BC \times \frac{EM^2 \times OEN - RS^2 \times BRG + FG^3 \times \frac{1}{8}PV}{FG^2}$  will be the value of the opposite ungula or cuneus  $EGFRE$ .



And  $\frac{23'54416128 \times 24^2}{14'4^2} - \frac{14'4 \times 9'6}{3} \times \frac{18^2}{35} =$   
 $\frac{23'54416128 \times 5^2}{3^2} - \frac{14'4 \times 9'6}{3} \times \frac{18^2}{35} =$   
 $23'54416128 \times 5 - 4 \times 14'4^2 \times \frac{6^2}{7} = 178'852147 =$   
 the double of the less.

Wherefore  $2837'886683 - 178'852147 =$   
 $2659'034536$  inches =  $9'4292$  ale gallons = the  
 quantity required.

### PROBLEM XV.

*To find the solidity of a parabolic spindle.*

#### \* RULE.

Multiply the area of the greatest circle, or middle section, by the length, and  $\frac{8}{15}$  of the product will be the content.

That

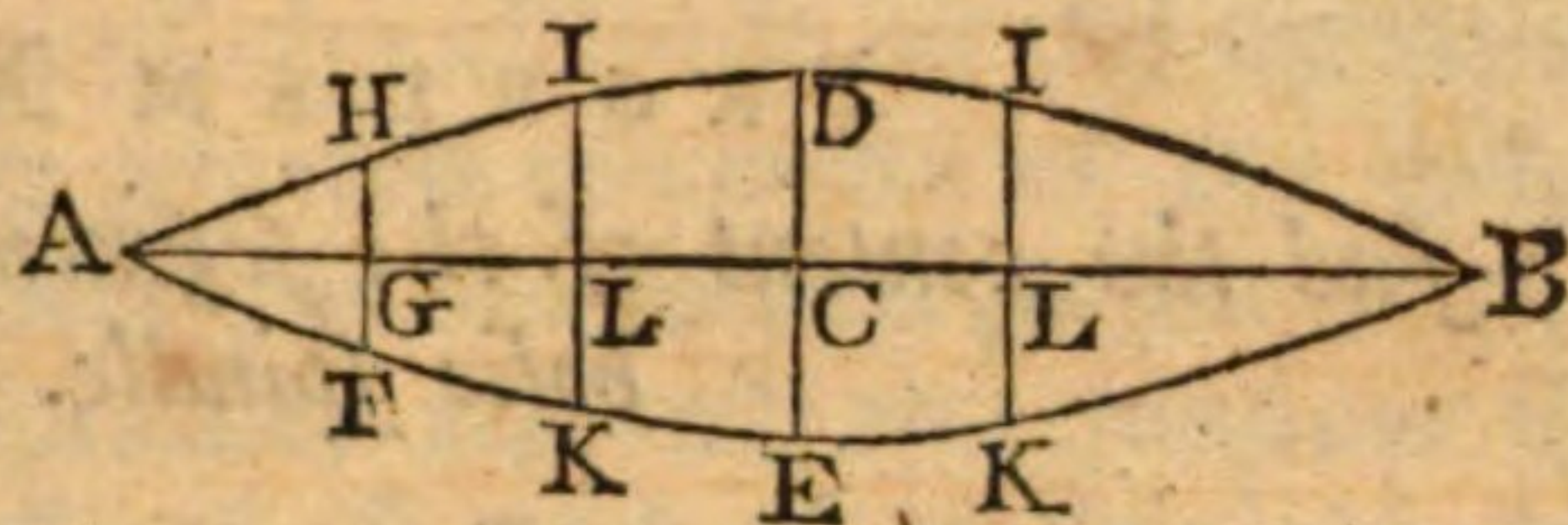
#### \* DEMONSTRATION.

Putting  $a = DC$ ,  $b = CA$ ,  $c = 3'14159$  &c.  $x = AG$ , and  $y = GH$ ; we shall have, by the property of the parabola, as  $AG^2 : CD :: AG \times GB : GH = a \times \frac{2bx - xx}{bb} = \frac{2bx - xx}{p} = y$ , putting  $p = \frac{bb}{a}$  = the parameter of  $DC$ ; and hence the fluxion of the solid  $cyyx$  will be  $\frac{cx}{pp} \times \overline{2bx - xx}^2$ ; the fluent of which  $cx^3 \times \frac{4bb - bx + \frac{1}{3}xx}{pp}$  will be a general expression for the segment  $AFH$ ; which, when  $x = b$ , gives  $\frac{8cb^5}{15pp} = \frac{8}{15}caab$  for the value of the semi-spindle  $ADE$ . Q.E.D.

Co-



That is,  $\frac{8}{15} n$   
 $\times DE^2 \times AB =$   
 the whole solid  
 $ADBEA$ ;  $AB$   
 being the length



or axe,  $DE$  the greatest diameter or double the  
 abscissa of the generating parabola  $ADB$ , and  $n =$   
 $\cdot 785398 \text{ \&c.}$

## E X A M P L E.

Required the solidity of a parabolic spindle whose  
 length is 80 and greatest diameter 32.

By the rule,  $\cdot 785398 \text{ \&c.} \times 32^2 \times 80 \times \frac{8}{15} =$   
 $\cdot 785398 \text{ \&c.} \times 43690\frac{2}{3} = 34314\cdot 5693576 =$  the con-  
 tent required.

P R O-

## COROLLARY I.

A parabolic femi-spindle is to a cylinder of the same base and alti-  
 tude as 8 to 15, and to a paraboloid of the same base and altitude as  
 8 to  $7\frac{1}{2}$  or as 16 to 15.

## COROLLARY II.

It appears from the demonstration that the segment  $AHF$  is ex-  
 pressed by  $cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{3}xx}{pp} = c \times AG^3 \times DC^2 \times$   
 $\frac{\frac{4}{3}AC^2 - CA \times AG + \frac{1}{3}GA^2}{AC^4}$ .

## COROLLARY III.

If from the value of the femi-spindle  $ADE$  be taken that of the  
 segment  $AFH$ , there will remain  $\frac{8cb^5}{15pp} - cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{3}xx}{pp}$   
 $= \frac{cz}{pp} \left( \text{or } \frac{ca^2z}{b^4} \right) \times \overline{b^4 - \frac{2}{3}b^2z^2 + \frac{1}{3}z^4}$  for the value of the frustum  
 $EFHD$ , putting  $z$  instead of  $b - x = CG$ .—And if instead of  $z$ ,  
 in



## PROBLEM XVI.

To find the content of the segment AFH of a parabolic spindle.

## RULE.

To  $\frac{4}{3}$  of the square of the length of the semi-spindle, add  $\frac{1}{5}$  of the square of the altitude of the segment, from the sum take the product of the said length and altitude, and call the difference  $P$ .

Mul-

in the two last terms of this expression, be substituted its value  $b\sqrt{\frac{a-y}{a}}$ , the value of the said frustum will be denoted by  $c \times \frac{8aa + 4ay + 3yy}{15} = c \times GC \times \frac{8DE^2 + 4DE \times FH + 3FH^2}{60}$ .

## COROLLARY IV.

If to, or from, the frustum  $DEFH = cA \times \frac{8D^2 + 4Dd + 3d^2}{60}$ ,  
be added, or subtracted, the frustum  $DIKE = ca \times \frac{8D^2 + 4Dd + 3d^2}{60}$ ,  
there will result  $c \times \frac{8D^2 \times A \pm a + dA \times 4D + 3d \pm da \times 4D + 3d}{60}$

for the value of the frustum  $HIKF$ , neither of whose ends pass through the center of the spindle, the upper or under signs being used according as the said center is within or without the frustum, and wherein  $D$  represents the diameter through the center of the spindle,  $d$  the diameter of the less end of the frustum, and  $\delta$  that of the greater; also  $A$  the distance of  $d$  from the center of the spindle, and  $a$  that of  $\delta$  from the same. Or in a cask of this form, standing upon one end,  $D$  will be the bung diameter,  $d$  the head diameter,  $2A$  the length of the cask,  $A \pm a = w =$  the wet inches of  $2A$ , and  $\delta$  the diameter at the surface of the liquor, the value of which diameter is as determined in the following corollary.

## COROLLARY V.

Since, by the property of the parabola,  $A^2 : a^2 :: D - d : D - \delta$ ,  
the value of  $\delta$  will be  $\frac{DA^2 - a^2 \times D - d}{AA}$ .



Multiply the cube of the altitude of the segment by the square of the greatest diameter of the spindle, divide the product by the fourth power of the length of the semi-spindle, and call the quotient  $\mathcal{Q}$ .

Then the product of  $P$  and  $\mathcal{Q}$  multiplied by  $\cdot 785398$  &c. will give the content of the segment.

That is,  $\frac{\frac{4}{3}AC^2 - GA \times AG + \frac{1}{5}GA^2}{AC^4} \times AG^3 \times DE^2 \times \cdot 785398$  &c. = the segment  $AFH$ , by corollary 2 of the last problem.

#### EXAMPLE.

If the diameter of the base of the segment of a parabolic spindle be 24, and its altitude 20; what will be the content, supposing the length of the whole spindle to be 80.

Here  $AC = 40$ ,  $AG = 20$ , and  $GH = 12$ ; then  $GC = 40 - 20 = 20$ , and, by the parabola,  $AC^2 - CG : GH :: AC^2 : CD = \frac{AC^2 \times GH}{AC^2 - CG^2} = \frac{40^2 \times 12}{60 \times 20} = 4 \times 4 = 16$ ; the double of which gives  $DE = 32$ .

Then, by the rule,  $\frac{4 \times 40^2}{3} - 40 \times 20 + \frac{20^2}{5} \times \frac{20^3 \times 32^2}{40^4}$   
 $\times \cdot 785398$  &c. =  $\frac{4}{3} - \frac{1}{3} + \frac{1}{20} \times 5 \times 32^2 \times \cdot 785398$  &c.  
 $= \frac{80 - 30 + 3}{3} \times 16^2 \times \cdot 785398$  &c. =  $4522\frac{2}{3} \times \cdot 785398$   
 &c. =  $3552\cdot 094093659$  = the content required.

#### PROBLEM XVII.

*To find the content of the frustum of a parabolic spindle, one of the ends of the frustum passing through the center of the spindle.*

#### RULE.

Add into one sum 8 times the square of the diameter of the greater end, 3 times the square of that  
 4 S of



of the less, and 4 times the product of the diameters; multiply the sum by the length, and the product multiplied by  $\frac{1}{15}$  of .785398 &c. will be the content.

That is,  $\frac{3DE^2 + 4DE \times FH + 3FH^2}{15} \times CG \times .785398 \text{ \&c.}$   
 = the frustum  $DEFH$ , by corollary 3 to prob. 15.

## E X A M P L E.

Required the content of a cask in form of the middle zone of a parabolic spindle, the bung diameter being 32 inches, the head diameter 24, and length 40 inches.

The cask is evidently two equal frustums, whose greater diameters are 32 and least 24, and their lengths each 20; wherefore  $\frac{8 \times 32^2 + 4 \times 32 \times 24 + 3 \times 24^2}{15}$   
 $\times 40 \times .785398 \text{ \&c.} = 8^3 \times \frac{8 \times 4^2 + 3 \times 4^2 + 3 \times 3^2}{3} \times$   
 $.785398 \text{ \&c.} = \frac{11 \times 16 + 27}{3} \times 8^3 \times .785398 \text{ \&c.} = 34645\frac{1}{3}$   
 $\times .785398 \text{ \&c.} = 27210.3811703 \text{ inches} = 96.49071337$   
 ale gallons = the content required.

## P R O B L E M XVIII.

*To find the content of the frustum of a parabolic spindle, neither of whose ends pass through the center of the spindle.*

## R U L E.

Multiply the diameter of each end by its distance from the diameter in the middle of the spindle, and multiply each product by the sum of 3 times the said diameter of the end and 4 times the said middle diameter;



diameter; to or from the product belonging to the less diameter add or subtract that belonging to the greater, according as the center of the spindle is within or without the frustum; to the sum or difference add 8 times the product arising from the multiplication of the length of the frustum by the square of the diameter in the middle of the spindle; and this sum multiplied by  $\frac{1}{15}$  of  $\cdot 785398$  &c. will produce the content of the frustum.

That is,

$$\frac{8DE^2 \times GL + FH \times GC \times \overline{4DE + 3FH} \pm IK \times LC \times \overline{4DE + 3IK}}{15}$$

$\times \cdot 785398$  &c. = the frustum  $FHIK$ , using the upper or under sign according as  $GL$  is greater or less than  $GC$ ; as is proved in corollary 4 to prob. 15.

*Note*, The value of the greater diameter  $IK$  is  $\frac{DE \times GC^2 - LC^2 \times \overline{DE - FH}}{GC^2}$ , as in corollary 5 to the same problem.

#### EXAMPLE I.

If the liquor in the cask, in the example to the last problem, when standing upon its end with its axe perpendicular to the horizon, rise to the height of 30 inches; what quantity of liquor will there be.

Here, by the note,  $IK = \frac{32 \times 20^2 - 10^2 \times \overline{32 - 24}}{20 \times 20} = 30$ .

Then, by the rule,

$$\frac{8 \times 32^2 \times 30 + 24 \times 20 \times \overline{4 \times 32 + 3 \times 24} + 30 \times 10 \times \overline{4 \times 32 + 3 \times 30}}{15} \times$$

$\cdot 785398$  &c. =  $8 \times 16^2 + 4 \times 128 + 72 + 5 \times 64 + 45$   
 $\times 8 \times \cdot 785398$  &c. =  $27144 \times \cdot 785398$  &c. =  
 $21318 \cdot 8477473$  cubic inches =  $75 \cdot 59875088$  ale  
gallons = the quantity required.

Ex-



## EXAMPLE II.

If in the same cask, placed as before, the liquor rise but to the height of 10 inches; how much will be in it.

Here all the dimensions are the same quantities as in the last example, excepting  $GL$  which here is only 10 instead of 30, and the cask is less than half full.

Wherefore, by the rule,

$$\frac{8 \times 32^2 \times 10 + 24 \times 20 \times 4 \times 32 + 3 \times 24 - 30 \times 10 \times 4 \times 32 + 3 \times 30}{15} \times 8 \times$$

$$\times .785398 \&c. = \frac{8 \times 16^2 + 12 \times 128 + 72 - 15 \times 64 + 45}{3} \times 8 \times$$

$$.785398 \&c. = 7501\frac{1}{3} \times .785398 \&c. = 5891.533423$$

cubic inches = 20.89196249 ale gallons = the quantity required.

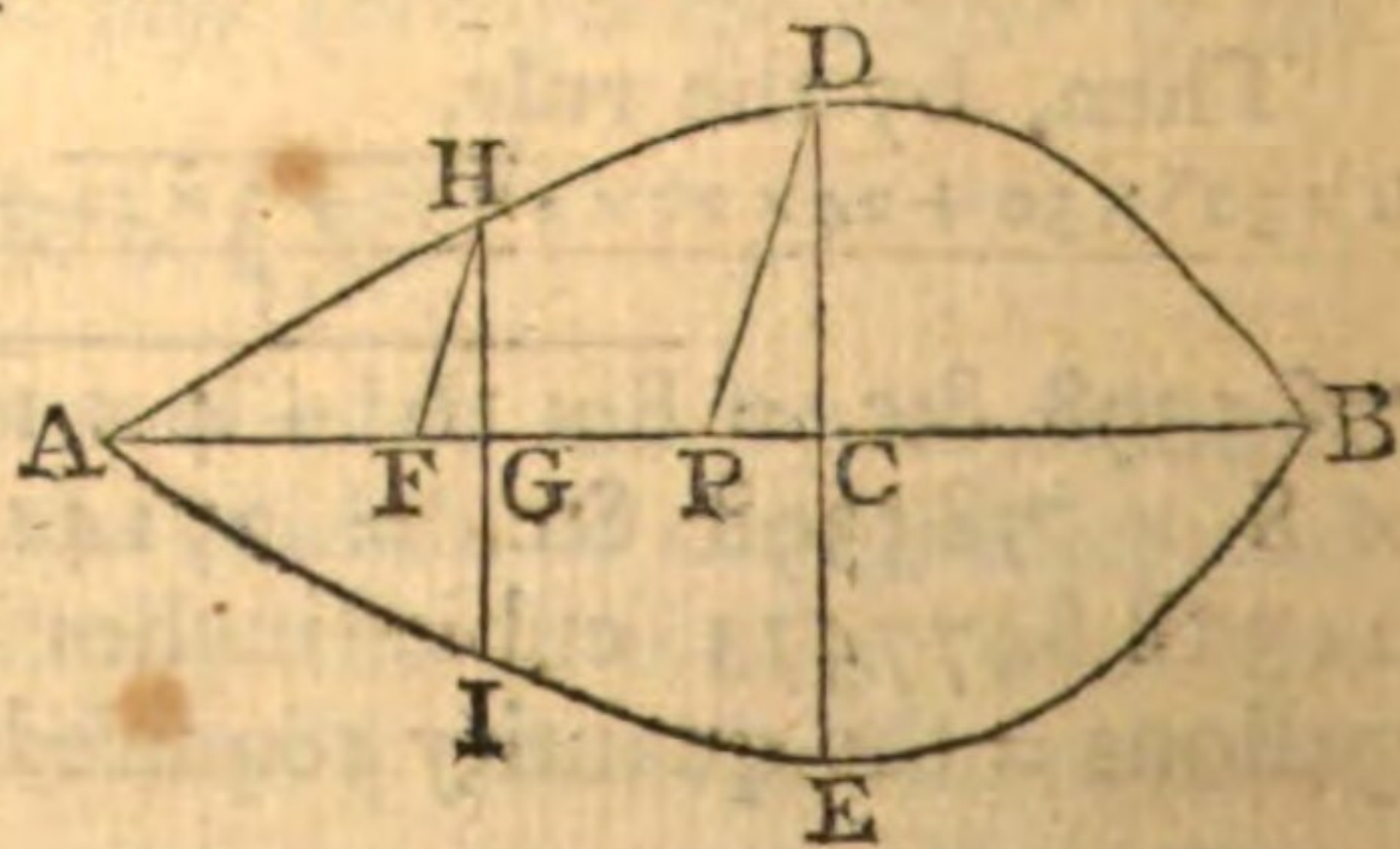
## PROBLEM XIX.

To find the content of the universal parabolic spindle, or solid generated by the revolution of a parabolic segment round its base, being an ordinate to any diameter.

## RULE.

Multiply the area of the greatest section by the length, and  $\frac{8}{15}$  of the product will be the content; as in the common spindle.

That is,  $DE^2 \times AB \times \frac{8}{15} n =$  the solid  $ADB EA$ , generated from the revolution of the parabolic segment  $ADB$  about its base  $AB$  which is



a double



a double ordinate to the diameter  $DP$ , and  $DCE$  being perpendicular to  $AB$ .\*

## EXAMPLE.

If an oblique parabola, whose base is inclined to its diameter in an angle of 30 degrees, be turned about its base; required the value of the solid generated by its revolution; the base of the parabola,

4 T

or

## \* DEMONSTRATION.

Let  $GH$  be parallel to  $CD$ , and  $HF$  parallel to  $DP$ ; and put  $a = DP$ ,  $\frac{1}{2}b = PA = \frac{1}{2}AB$ ,  $\frac{1}{2}c = CD = \frac{1}{2}DE$ ,  $z = AF$ , and  $p = 4^n = 3.14159$  &c. Then, by the property of the parabola,  $AP^2 : AF \times FB :: PD : FH ::$  (by sim.  $\Delta$ s.)  $CD : GH = \frac{CD \times BF \times FA}{AP^2} = 2c \times \frac{bz - zz}{bb} :: CP = \sqrt{PD^2 - DC^2} = d :$   
 $GF = \frac{GH \times PC}{CD} = 4d \times \frac{bz - zz}{bb} : \text{hence } AG = GF + FA =$   
 $z \times \frac{bb + 4bd - 4dz}{bb}$ , and the fluxion  $p \times HG^2 \times GA$  of the solid will be  $p \dot{z} \times \frac{bb + 4bd - 8dz}{bb} \times \frac{4cc}{b^4} \times \overline{bz - zz}^2 = 4pc^2 z^2 \dot{z} \times$   
 $\frac{b^3 \times b + 4d - 2b^2 z \times b + 8d + bz^2 \times b + 20d - 8dz^2}{b^6}$ ; the fluent of which, viz.  $4pc^2 z^3 \times$   
 $\frac{\frac{1}{3}b^3 \times b + 4d - \frac{1}{2}b^2 z \times b + 8d + \frac{1}{3}bz^2 \times b + 20d - \frac{4}{3}dz^2}{b^6}$  will be a general expression for the value of the segment; which, when  $z$  becomes  $= b$ , will be  $\frac{2}{15} pccb = \frac{8}{15} nbcc$  for the content of the whole solid  $ADBEA$ . Q.E.D.

## COROLLARY.

If any parabolas  $ADB$ ,  $ACB$ , [fig. at corol. 1 to prob. 5] having the same base  $AB$  and between the same parallels  $AB$ ,  $CD$ , be turned round their common base; they will generate equal spindles.

For the lengths and greatest diameters will be the same in all of them.







$$ac \times \frac{\overline{b \pm d}^3 \times \overline{3b \mp d - b\sqrt{\frac{a-x}{a} \pm d}}^3 \times \overline{3b\sqrt{\frac{a-x}{a} \mp d}}}{6bb} \text{ for the}$$

value of the solid required, in which the upper signs respect the solid  $dkfc$  generated by  $cf$ , and the under the solid  $DKFC$  generated by  $CF$ .

## COROLLARY I.

When  $E$  arrives at  $A$ , then  $y=0$ , and the expression becomes  $ac \times \overline{b \pm d}^3 \times \frac{3b \mp d}{6bb} = c \times HG \times CB^3 \times \frac{cB + 2CH}{6CH} = \left( \text{by substituting for } HG \text{ its value } \frac{CH^2 \times BA}{CB \times Bc} \right) c \times CB^2 \times BA \times \frac{Bc + 2CH}{6Bc}$  for the solid  $CAD$ .

And if  $CM$  be an ordinate to the diameter  $AB$ , we shall have as  $cB : BC :: BA : AM = \frac{AB \times BC}{Bc}$ ; hence  $BA \times \frac{Bc + 2CH}{6Bc}$  will be  $= \frac{2BA + AM}{6}$ , and consequently the above value of the solid  $CAD$  becomes  $c \times CB^2 \times \frac{2BA + AM}{6}$

## COROLLARY II.

When  $B$  coincides with  $H$ ,  $d$  will be  $= 0$ , and then the rule becomes  $ac \times \frac{b^4 - y^4}{2bb} = c \times \frac{1}{2} HI \times \overline{bb + yy}$  for the frustum  $CFfc$  generated about  $HI$ , as in problem 12. Or  $c \times CH^2 \times \frac{1}{2} HG$  for the whole paraboloid, as in prob. 11.

*The INVESTIGATION otherwise.*

Supposing  $FL$  to be an ordinate to the diameter  $AB$  whose parameter is  $p$ , and putting  $m$  and  $n$  for the  
the



the sine and cosine of the angle  $L$  to the radius 1; we shall have  $FL = \sqrt{p \times LA} = \sqrt{px}$ , where  $x$  is now  $= LA$ ; then  $FE = m\sqrt{px}$ , and  $EL = n\sqrt{px}$ ; hence  $EA = AL + LE = x + n\sqrt{px}$ . Then the fluxion of the solid  $c \times FE^2 \times EA = cm^2 px \times x + \frac{1}{3}nx\sqrt{\frac{p}{x}}$ , whose fluent gives  $cm^2 px \times \frac{1}{2}x + \frac{1}{3}n\sqrt{px} = cyy \times \frac{3x + 2n\sqrt{px}}{6} = c \times FE^2 \times \frac{2EA + AL}{6}$  for the value of the solid  $AFK$  generated by the revolution of  $AF$ . And  $c \times CB^2 \times \frac{2BA + AM}{6}$  for the solid  $CAD$  as above.

## S C H O L I U M.

It is evident that the surface of this solid might be found by the method by which we determined that of an elliptic spindle; and the surface of the parabolic spindle also might easily be determined if there appeared an occasion for it, but not by the same method.

## S E C T I O N V.

*Of hyperbolic Lines, Areas, Surfaces, and Solidities.*

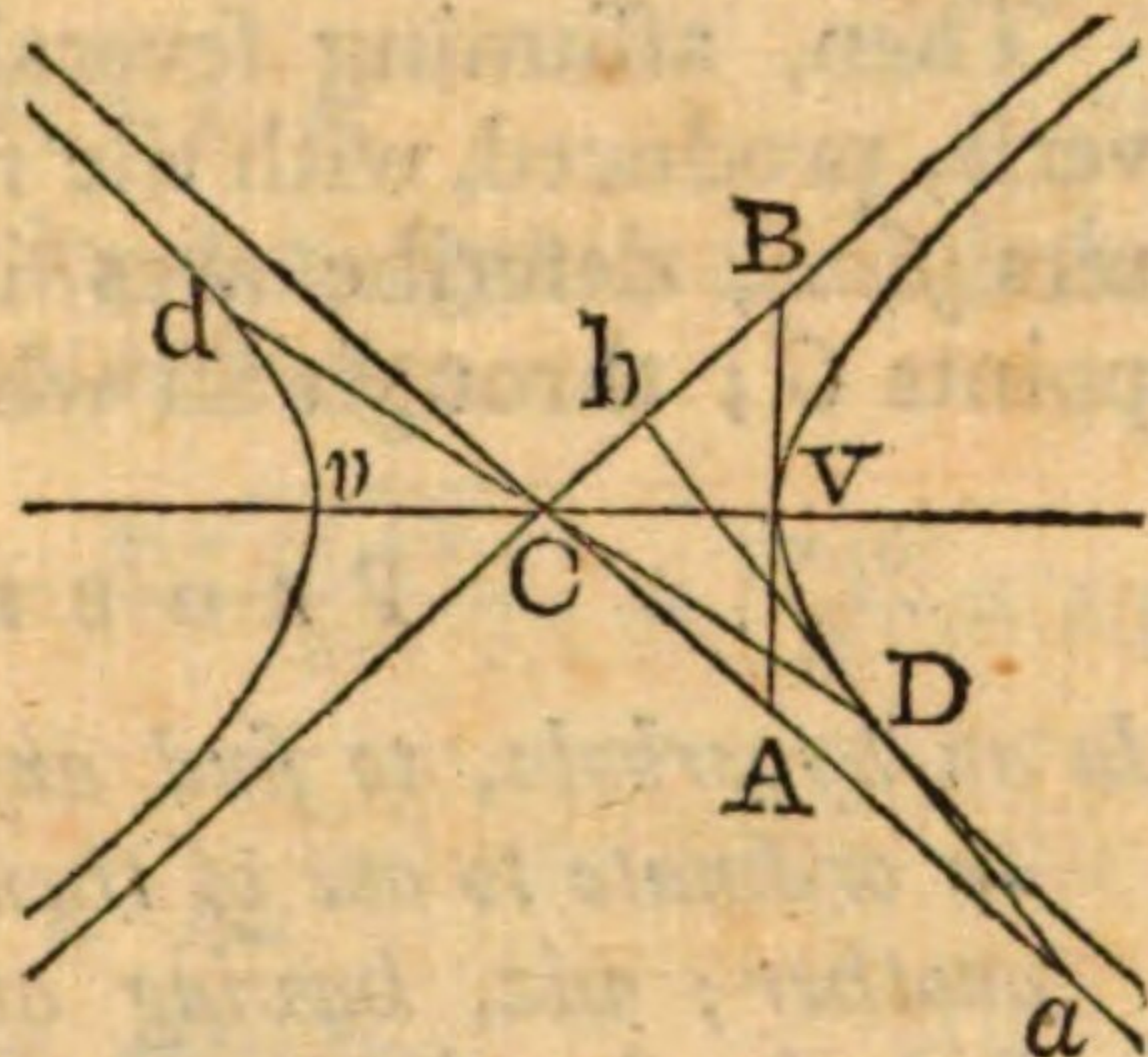
## D E F I N I T I O N S.

**T**O what hath been said of the hyperbola among the conic sections in general, may be added the following observations.

If  $AVB$  touch the hyperbola in the vertex  $V$ , and be equal to the conjugate axe, viz.  $AV$  and  $VB$  each equal to the half of the conjugate axe, and if from the



the center  $C$  through the extremities  $A, B$ , right lines  $CA, CB$ , be drawn, those lines will be what are called asymptotes to the hyperbola; that is, they are lines which continually approach to the curve.



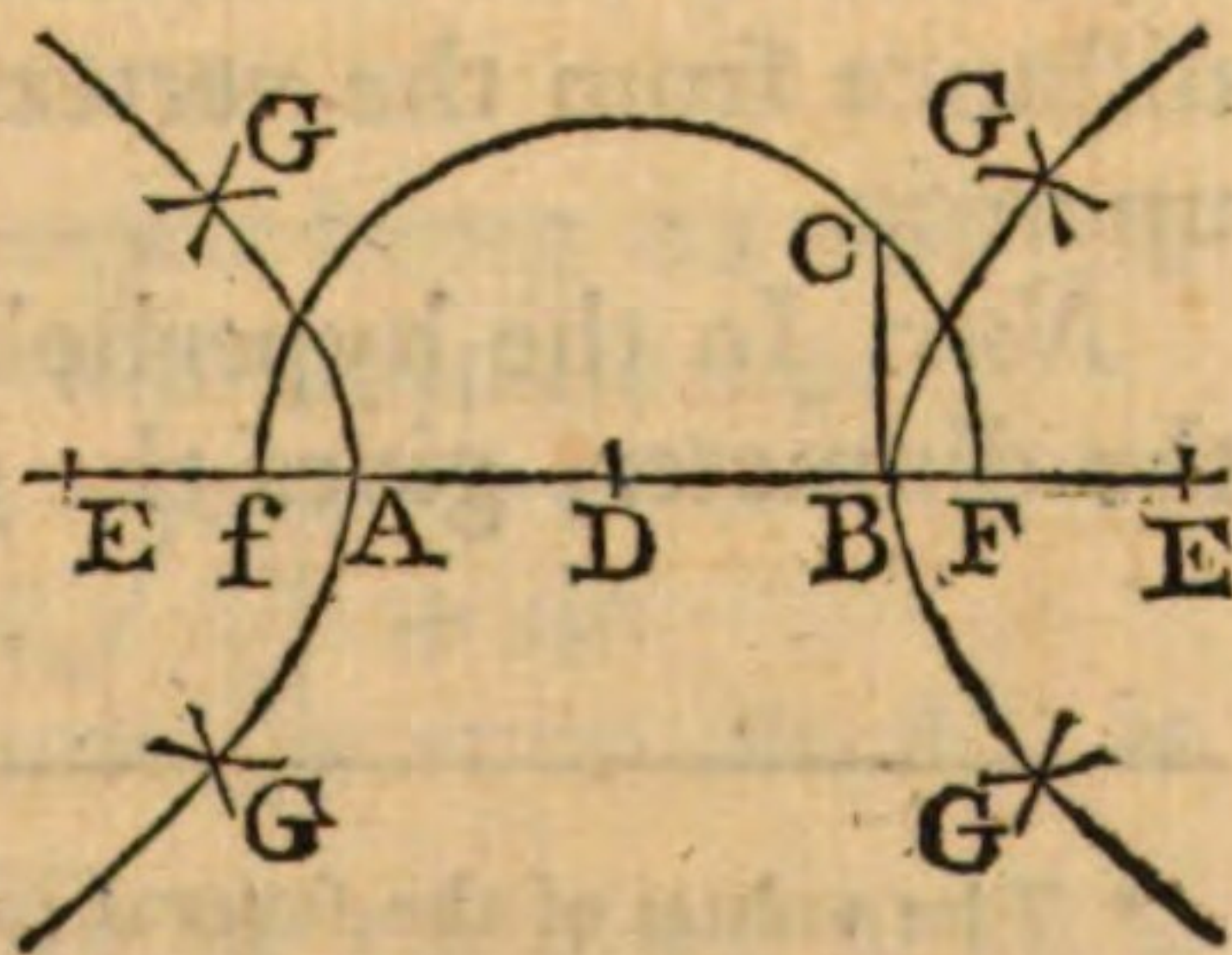
And like as the conjugate axe is a tangent to the curve at the vertex of the transverse and bounded by the asymptotes, so if to the extremity  $D$  of any diameter  $Dd$  be drawn a tangent  $aDb$  and terminated by the asymptotes, then  $ab$  will be the conjugate to  $Dd$ .

When the symptotes form a right angle at the center, the hyperbola is said to be right-angled; or, also equilateral, because that then its axes are equal to each other.

### PROBLEM I.

*To construct an hyperbola, having given the transverse and conjugate axes  $AB$  and  $2BC$ .*

The semi-conjugate  $BC$  being erected perpendicular to  $AB$ , and  $D$  being the middle of  $AB$  or the center of the hyperbola; with the center  $D$  and radius  $DC$ , describe an arc meeting with  $AB$ , both ways produced, in  $F$  and  $f$  the two focufes.



4 U

Then,



Then, assuming several points  $E$  in the transverse, produced, with the radiuses  $AE$ ,  $BE$ , and centers  $f$ ,  $F$ , describe arcs intersecting in the several points  $G$ ; through all which draw the curve.

### PROBLEM II.

*In an hyperbola, to find any two conjugate diameters, an ordinate to one of them, and its abscissa, one from another; viz. having any three of them given, to find the fourth.*

#### CASE I.

*To find the ordinate.*

#### \* RULE.

As any diameter:

Is to its conjugate::

So is the mean proportional between the two abscissas:

To the ordinate.

That is, as  $d : c :: \sqrt{d+x} \times x : \frac{c}{d} \sqrt{d+x} \times x = y$ ; where  $d$  denotes the diameter,  $c$  its conjugate,  $y$  an ordinate to the diameter, and  $x$  its abscissa, or its distance from the vertex of the diameter, measured upon it.

*Note.* In the hyperbola, the less abscissa added to the diameter, gives the greater abscissa.

Ex-

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\* The values of the several quantities in the four cases of this problem are easily found from the general property of the curve, viz.  $dd : cc :: xx \over d+x : yy$ . The demonstration of which together with that of the preceding problem belongs to conics.



## EXAMPLE.

If the diameter be 24, its conjugate 21, and the less abscissa 8; what is the ordinate.

By the rule,  $\frac{c\sqrt{d+x \times x}}{d} = \frac{21\sqrt{24+8 \times 8}}{24} = \frac{7 \times 16}{8} = 14$   
= the ordinate required.

## CASE II.

*To find the abscissas.*

## RULE.

As the conjugate:

Is to the diameter ::

So is the root of the sum of the squares of the ordinate and semi-conjugate:

To the distance between the bottom of the ordinate and the center.

Then, to and from this distance add and subtract the semi-diameter, and the sum and difference will give the two abscissas.

That is,  $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = x$  the greater or less abscissa according as the upper or under sign is used.

## EXAMPLE.

The diameter and its conjugate being 24 and 21, required the two abscissas to the ordinate 14.

Here  $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = \frac{24\sqrt{\left(\frac{21}{2}\right)^2 + 14^2}}{21} \pm 12 =$   
 $\frac{4\sqrt{21^2 + 28^2}}{7} \pm 12 = 4\sqrt{3^2 + 4^2} \pm 12 = 20 \pm 12 =$   
32 and 8 the two abscissas required.

CASE



## C A S E III.

*To find the conjugate.*

## R U L E.

As the mean proportional between the abscissas:  
 Is to the ordinate ::  
 So is the diameter:  
 To its conjugate.

That is,  $\sqrt{d+x \times x} : y :: d : \frac{dy}{\sqrt{d+x \times x}} = c.$

## E X A M P L E.

What is the conjugate to the diameter 24, the less abscissa being 8, and its ordinate 14.

Here  $\frac{dy}{\sqrt{d+x \times x}} = \frac{24 \times 14}{\sqrt{32 \times 8}} = \frac{24 \times 14}{16} = 21 =$  the conjugate required.

## C A S E IV.

*To find the diameter.*

## R U L E.

To or from the root of the sum of the squares of the ordinate and semi-conjugate, add or subtract the semi-conjugate, according as the less or greater abscissa is used: Then, as the square of the ordinate is to the product of the abscissa and conjugate, so will the sum or difference, above found, be to the diameter.

That is,  $cx \times \frac{\sqrt{\frac{1}{4}cc + yy} \pm \frac{1}{2}c}{yy} = d.$

Ex-



## EXAMPLE.

The less abscissa being 8, and its ordinate 14, required the diameter, supposing the conjugate to be 21.

$$\begin{aligned} \text{Here } cx \times \frac{\sqrt{\frac{1}{4}cc + yy} + \frac{1}{2}c}{yy} &= 21 \times 8 \times \frac{\sqrt{\frac{21^2}{2} + 14^2} + \frac{21}{2}}{14 \times 14} \\ &= 3 \times \frac{\sqrt{21^2 + 28^2} + 21}{7} = 3 \times \sqrt{3^2 + 4^2} + 3 = 3 \times 5 + 3 = \\ &24 = \text{the diameter required.} \end{aligned}$$

## PROBLEM III.

Having given any two abscissas  $X, x$ , and their ordinates  $Y, y$ , to find the diameter  $d$  to which they belong.

## \* RULE.

Multiply each abscissa by the square of the ordinate belonging to the other; multiply, also, the square of each abscissa by the square of the other's ordinate; then divide the difference of the latter products by the difference of the former, and the quotient will be the diameter to which the ordinates belong.

$$\text{That is, } \frac{XXyy \oslash xxYY}{xYY \oslash Xyy} = d.$$

4 X

E x-

## \* DEMONSTRATION.

By the nature of the hyperbola,  $dX + XX : dx + xx :: YY : yy$ ,  
or  $dXyy + XXyy = d xYY + xxYY$ ; hence  $d = \frac{XXyy - xxYY}{xYY - Xyy}$ .  
Q. E. D.



## EXAMPLE.

If two abscissas be 1 and 8, and their two ordinates  $4\frac{3}{8}$  and 14; what is the diameter to which they belong.

$$\text{Here } \frac{XXyy - xxyy}{xyy - Xyy} = \frac{8^2 \times 8 \times 4\frac{3}{8} \times 4\frac{3}{8} - 1 \times 1 \times 14 \times 14}{1 \times 14 \times 14 - 8 \times 4\frac{3}{8} \times 4\frac{3}{8}} =$$

$$\frac{35 \times 35 - 14 \times 14}{14 \times 14 - 35 \times 4\frac{3}{8}} = \frac{5 \times 5 - 2 \times 2}{2 \times 2 - 5 \times \frac{5}{8}} = \frac{21 \times 8}{7} = 24 = \text{the diameter required.}$$

## PROBLEM IV.

Given any three equidistant ordinates  $a, b, c$ , with their common distance or common difference  $d$  of their abscissas, to find the diameter  $D$  to which they belong, together with its conjugate  $C$ .

## \* RULE.

From the sum of the squares of the extreme ordinates take the double of that of the middle one, and call the difference  $A$ .

From

## \* DEMONSTRATION.

If  $x$  be the abscissa to the least ordinate  $a$ , by the nature of the hyperbola we shall have these three equations,

$$aa = \frac{CC}{DD} \times \overline{D+x} \times x = \frac{CC}{DD} \times \overline{Dx+xx},$$

$$bb = \frac{CC}{DD} \times \overline{D+x+d} \times \overline{x+d} = \frac{CC}{DD} \times \overline{Dx+xx+2dx+Dd+dd},$$

$$cc = \frac{CC}{DD} \times \overline{D+x+2d} \times \overline{x+2d} = \frac{CC}{DD} \times \overline{Dx+xx+4dx+2Dd+4dd}.$$

And by taking the double of the second from the sum of the first and third, we obtain this equation,  $aa - 2bb + cc = \frac{CC}{DD} \times 2dd$ ; and

$$\text{hence } \frac{CC}{DD} = \frac{aa - 2bb + cc}{2dd}.$$

Again,



From the square of the difference between  $A$  and the double of the said square of the middle ordinate take 4 times the square of the product of the extreme ordinates, and call the square root of the remainder  $B$ .

Then, as  $A$  is to  $B$ , so will the given common distance be to the diameter to which the ordinates belong.

$$\text{That is, } \frac{d\sqrt{-aa+4bb-cc}^2-4aacc}{aa-2bb+cc} = D.$$

And as the common distance of the ordinates is to the root of  $\frac{1}{2}A$ , so will the diameter  $D$  be to its conjugate  $C$ .

$$\text{That is, } C = \frac{D\sqrt{\frac{1}{2}A}}{d}.$$

E x-

Again, taking the first equation from the second, leaves  $bb-aa$   
 $= \frac{CC}{DD} \times 2dx + Dd + dd$ ; hence  $x = \frac{DD}{CC} \times \frac{bb-aa}{2d} - \frac{D+d}{2} =$   
 $d \times \frac{bb-aa}{aa-2bb+cc} - \frac{D+d}{2} = d \times \frac{2bb-\frac{3}{2}aa-\frac{1}{2}cc}{aa-2bb+cc} - \frac{1}{2}D.$

Consequently  $aa = \frac{CC}{DD} \times \overline{D+x} \times x = \frac{aa-2bb+cc}{2dd} \times$   
 $d \times \frac{2bb-\frac{3}{2}aa-\frac{1}{2}cc}{aa-2bb+cc} - \frac{1}{2}D^2$ ; and hence  $\frac{d\sqrt{-aa+4bb-cc}^2-4aacc}{aa-2bb+cc}$   
 $= \frac{dB}{A} = \text{the diameter } D.$

Also from the equation  $\frac{CC}{DD} = \frac{aa-2bb+cc}{2dd}$ , is deduced  
 $\frac{D}{d} \sqrt{\frac{aa-2bb+cc}{2}} = \frac{D\sqrt{\frac{1}{2}A}}{d} = \text{the conjugate } C. \quad Q.E.D.$

## COROLLARY.

Hence the sum of the squares of the extreme ordinates is greater than double the square of the middle one.



## EXAMPLE.

Required the diameter and its conjugate to which belong the three ordinates  $4\frac{3}{8}$ , 14, and  $2\frac{1}{8}\sqrt{65}$ , their common distance being 7.

$$\text{Here } \frac{d\sqrt{4bb - aa - cc| - 4aacc}}{aa - 2bb + cc} =$$

$$7\sqrt{28^2 - \frac{35^2}{8^2} - \frac{65 \times 21^2}{8^2} - \frac{4 \times 65 \times 35^2 \times 21^2}{8^4}} =$$

$$\frac{\frac{35^2}{8^2} - 2 \times 14^2 + \frac{65 \times 21^2}{8^2}}{5^2 - 2 \times 16^2 + 65 \times 3^2} = \frac{7 \times 336}{98} = 24 = \text{the diameter } D.$$

$$\text{And } \frac{24}{7}\sqrt{\frac{35^2}{2 \times 8^2} - 14^2 + \frac{65 \times 21^2}{2 \times 8^2}} = 8\sqrt{\frac{5^2}{2} - 16^2 + \frac{65 \times 3^2}{2}}$$

$$= \frac{3}{2}\sqrt{50 - 32^2 + 9 \times 130} = \frac{3}{2}\sqrt{196} = \frac{3}{2} \times 14 = 21 = \text{the conjugate } C.$$

## PROBLEM V.

*To find the length of any arc of an hyperbola, the arc beginning at the vertex.*

## \* RULE I.

Put  $a$  = the semi-transverse axe,  $c$  = the semi-conjugate,  $q = \frac{cc + aa}{c^4}$ ,  $y$  = an ordinate to the axe drawn

## \* DEMONSTRATION.

Putting  $x$  = the abscissa of the ordinate  $y$ , we shall obtain, from the nature of the curve,  $x = \frac{a\sqrt{cc + yy}}{c} - a$ , and therefore  $\dot{x} =$

$\frac{a\dot{y}y}{c}$



drawn from the end of the arc required,  $A$  = the hyperbolic logarithm of  $\frac{y + \sqrt{cc + yy}}{c} = 2.302585093$  x common logarithm of  $\frac{y + \sqrt{cc + yy}}{c}$ ,  $B = \frac{y\sqrt{cc + yy} - ccA}{2}$ ,  $C = \frac{y^3\sqrt{cc + yy} - 3ccB}{4}$ ,  $D = \frac{y^5\sqrt{cc + yy} - 5ccC}{6}$ ,  $E = \frac{y^7\sqrt{cc + yy} - 7ccD}{8}$ , &c.

Then will the length of the arc be expressed by  $c$  drawn into the series  $A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E$  &c.

## EXAMPLE.

Required the length of the curve corresponding to the ordinate 10, the transverse and conjugate axes being 80 and 60.

Here  $a = 40$ ,  $c = 30$ , and  $y = 10$ ; hence  $q = \frac{cc + aa}{c^4} = 2500$

$$\frac{ay\dot{y}}{c\sqrt{cc + yy}}; \text{ hence } z = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{y} \sqrt{\frac{cc + \frac{cc + aa}{c^4}yy}{cc + yy}} = \frac{cy}{\sqrt{cc + yy}} \sqrt{1 + \frac{cc + aa}{c^4}yy} = \frac{cy}{\sqrt{cc + yy}} \sqrt{1 + qyy} = \frac{cy}{\sqrt{cc + yy}} \times \left( 1 + \frac{q}{2}y^2 - \frac{q^2}{2.4}y^4 + \frac{3q^3}{2.4.6}y^6 - \frac{3.5q^4}{2.4.6.8}y^8 \right) \&c.$$

But the fluent of  $\frac{\dot{y}}{\sqrt{cc + yy}}$  is the hyp. log. of  $\frac{y + \sqrt{cc + yy}}{c} = A$ , that of  $\frac{y^2\dot{y}}{\sqrt{cc + yy}}$  is  $\frac{y\sqrt{cc + yy} - ccA}{2} = B$ , that of  $\frac{y^4\dot{y}}{\sqrt{cc + yy}}$  is  $\frac{y^3\sqrt{cc + yy} - 3ccB}{4} = C$ , that of  $\frac{y^6\dot{y}}{\sqrt{cc + yy}}$  is  $\frac{y^5\sqrt{cc + yy} - 5ccC}{6} = D$ , &c.

And therefore  $z$  is  $= c \times \left( A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E \right)$  &c. Q.E.D.



$= \frac{2500}{810000} = \frac{1}{324}$ ,  $\sqrt{cc + yy} = 10\sqrt{10} = 31.62277662$ ,  
 and  $\frac{y + \sqrt{cc + yy}}{c} = 1.387425887$ , whose hyp. log. is  
 $.3274501 = A$ ; also  $\frac{y\sqrt{cc + yy} - ccA}{2} = 10.76133 = B$ ,  
 $\frac{y^3\sqrt{cc + yy} - 3ccB}{4} = 641.796405 = C$ ,  $\frac{y^5\sqrt{cc + yy} - 5ccC}{6} =$   
 $45698.97\frac{1}{3} = D$ , and  $\frac{y^7\sqrt{cc + yy} - 7ccD}{8} = 3540529.3125$   
 $= E$ .

Then +  $A = .327450$   $- \frac{q^2}{2.4} C = .000764$   
 $+ \frac{q}{2} B = .016607$   $- \frac{3.5q^4}{2.4.6.8} E = .000012$   
 $+ \frac{3q^3}{2.4.6} D = .000084$   $- .000776$   
 $+ .344141$   
 $- .000776$   
 $.343365 =$  the sum of the series,  
 which multip. by  $c = 30$   
 produces  $10.30095 =$  the length of the arc  
 required.

## \* RULE II.

Using the same symbols as in the last rule, the arc  
 will be expressed by the series  $y \times : 1 + \frac{a^2}{6c^4}y^2 - \frac{a^4 + 4a^2c^2}{40c^8}y^4$   
 $+ \frac{a^6 + 4a^4c^2 + 8a^2c^4}{112c^{12}}y^6 - \frac{5a^8 + 24a^6c^2 + 48a^4c^4 + 64a^2c^6}{1152c^{16}}y^8 \&c.$   
 OR

## \* DEMONSTRATION.

For, by the demonst. of the last rule,  $z$  is  $= y \sqrt{\frac{cc + \frac{aa + cc}{cc}yy}{cc + yy}}$   
 $=$  (by extracting the root of the numerator and denominator, and  
 then dividing the one by the other)  $y \times : 1 + \frac{a^2}{2c^4}y^2 - \frac{a^4 + 4a^2c^2}{2.4c^8}y^4$   
 $+ \frac{a^6 + 4a^4c^2 + 8a^2c^4}{4.4c^{12}}y^6 - \frac{5a^8 + 24a^6c^2 + 48a^4c^4 + 64a^2c^6}{4.4.8c^{16}}y^8 \&c.$   
 And, by taking the fluents, we obtain the series as in the rule.



or  $y \times : 1 + \frac{a^2 y^2}{6c^4} A - \frac{a^2 + 4c^2}{c^4} \cdot \frac{3y^2}{20} B + \frac{a^4 + 4a^2 c^2 + 8c^4}{a^2 + 4c^2} \cdot \frac{5y^2}{14c^4} C$   
 $- \frac{5a^6 + 24a^4 c^2 + 48a^2 c^4 + 64c^6}{a^4 + 4a^2 c^2 + 8c^4} \cdot \frac{7y^2}{72c^4} D \&c.$  putting  $A, B,$   
 $C, \&c.$  for the first, second, third,  $\&c.$  term.

## EXAMPLE.

Taking the same example as to the last rule, in which  $a = 40$ ,  $c = 30$ , and  $y = 10$ ; we shall obtain

$$A = 1 \qquad C = .003170$$

$$B = 0.032922 \qquad E = \frac{59}{\phantom{000000}}$$

$$D = \frac{398}{\phantom{000000}} \qquad - .003229$$

$$+ 1.033320$$

$$- .003229$$

the sum = 1.030091 which multiplied

by  $y = 10$

produces 10.30091 = the arc, nearly the same as before.

## RULE III.\*

As the sum of 15 times the parameter of the axe and a fourth proportional to the axe, the abscissa, and

## \* DEMONSTRATION.

First  $x = \frac{a\sqrt{c^2 + y^2}}{c} - a$  (using the same symbols as in the last rules) =  $\frac{ay^2}{2c^2} - \frac{ay^4}{2.4c^4} + \frac{3ay^6}{2.4.6c^6} - \frac{3.5ay^8}{2.4.6.8c^8} \&c.$

Then proceed as in rule 5 for the elliptic arc.

$$\text{Thus } y \times \frac{A + \overline{B + 1} \cdot x}{A + Bx} = y \times \frac{A + \overline{B + 1} \times : \frac{ay^2}{2c^2} - \frac{ay^4}{8c^4} \&c.}{A + B \times : \frac{ay^2}{2c^2} - \frac{ay^4}{8c^4} \&c.} =$$

$$y \times : 1 + \frac{ay^2}{2Ac^2} - \frac{Aa + 2Ba^2}{8A^2c^4} y^4 \&c. \text{ which put } = y \times : 1 + \frac{a^2 y^2}{6c^4}$$

$$- \frac{a^4 + 4a^2 c^2}{40c^8} y^4 \&c. \text{ the value of the arc in the last rule. Then, by}$$

equating



and the sum of 9 times the axe and 21 times the parameter; is to the sum of 15 times the parameter and a fourth proportional to the axe, the abscissa, and the sum of 19 times the axe and 21 times the parameter; so is the ordinate, to the length of the curve, very nearly.

$$\text{That is, } \frac{15p + \frac{19t + 21p}{t}x}{15p + \frac{9t + 21p}{t}x} \times y = \text{the length of the}$$

curve nearly; where  $t$  = the transverse axe,  $p$  = its parameter or a third proportional to the transverse and conjugate,  $x$  = the abscissa, and  $y$  = the ordinate corresponding to the arc.

Ex-

equating the corresponding terms, we have  $\frac{a}{2Ac^2} = \frac{a^2}{6c^4}$ , or  $A = \frac{3c^2}{a} = \frac{3}{2}p$ ; and  $\frac{Aa + 2Ba^2}{8A^2c^4} = \frac{a^4 + 4a^2c^2}{40c^8}$ , hence  $B = \frac{9a^2 + 21c^2}{10aa}$   
 $= \frac{9a + \frac{21cc}{a}}{10a} = \frac{9t + 21p}{10t}$ , putting  $p$  for the parameter and  $t$  for the whole transverse axe.

$$\text{Consequently } y \times \frac{A + \overline{B + 1} \cdot x}{A + Bx} \text{ is } = y \times \frac{\frac{3}{2}p + \frac{19t + 21p}{10t}x}{\frac{3}{2}p + \frac{9t + 21p}{10t}x} =$$

$$y \times \frac{15p + \frac{19t + 21p}{t}x}{15p + \frac{9t + 21p}{t}x}. \quad \text{Q.E.D.}$$

#### COROLLARY.

A right line may be found nearly equal to an hyperbolic arc by proceeding in the same manner as in the corollary to rule 5 for the arc of the ellipse, taking here  $PG = \frac{1}{2}AP + \frac{19AK + 21AP}{10AK} \times AD$ .







ducts of the semi-transverse multiplied by the ordinate and the semi-conjugate multiplied by the said distance of the ordinate from the center, by the product of the semi-axes; and the remainder will be the area cut off by the double ordinate.\*

That

\* DEMONSTRATION.

The fluxion of the area  $ABD$  is  $= y\dot{v} =$  (by the nature of the curve)  $\frac{cv\sqrt{vv-aa}}{a}$ , the fluent of which is  $= \frac{cv\sqrt{vv-aa}}{2a} - \frac{ac}{2} \times$   
 h. l. of  $\frac{v+\sqrt{vv-aa}}{a} =$  the area  $ABD$ , the double of which will  
 be  $\frac{cv\sqrt{vv-aa}}{a} - ac \times$  h. l. of  $\frac{v+\sqrt{vv-aa}}{a} = vy - ac \times$  h. l.  
 of  $\frac{v}{a} + \frac{y}{c} = vy - ac \times$  h. l. of  $\frac{ay+cv}{ac} =$  the whole segment  
 $DAF.$   $\mathcal{Q}, E, D.$

COROLLARY I.

The same seg.  $DAF$  or  $\frac{cv\sqrt{vv-aa}}{a} - ac \times$  h. l. of  $\frac{v+\sqrt{vv-aa}}{a}$   
 is, also,  $= \frac{ay\sqrt{cc+yy}}{c} - ac \times$  h. l. of  $\frac{y+\sqrt{cc+yy}}{c} = ac \times$   
 $\frac{q\sqrt{1+qq}}{c} -$  h. l. of  $\frac{q+\sqrt{1+qq}}{c} =$   
 $ac \times \mathcal{Q}\sqrt{\mathcal{Q}\mathcal{Q}-1} -$  h. l. of  $\mathcal{Q} + \sqrt{\mathcal{Q}\mathcal{Q}-1}$ , putting  $q = \frac{y}{c}$ ,  
 and  $\mathcal{Q} = \frac{v}{a}.$

COROLLARY II.

If from the triangle  $CBD = \frac{1}{2}vy$ , be taken the semi-segment  
 $ABD = \frac{1}{2}vy - \frac{1}{2}ac \times$  h. l. of  $\frac{ay+cv}{ac}$ , there will remain  $\frac{1}{2}ac \times$   
 h. l. of  $\frac{ay+cv}{ac}$  for the value of the hyperbolic sector  $CAD$ . When  
 $y$  or  $v$  are supposed infinite; then the sector becomes the space  $CAG$   
 between



That is,  $CB \times BD - CA \times AE \times \text{hyp. log. of } \frac{CA \times BD + AE \times CB}{CA \times AE} = vy - ac \times \text{hyp. log. of } \frac{ay + cv}{ac} =$   
 the area  $ADF$ , putting  $a =$  the semi-transverse axe  
 $CA$ ,  $c = AE$  the semi-conjugate,  $y = BD$  the ordi-  
 nate, and  $v = BC$  its distance from the center.

*Note*

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between the asymptote and the curve infinitely produced, and the  
 expression for it becomes infinite.

COROLLARY III.

When the hyperbola is equilateral, then  $a$  is  $= c$ , and the area  
 $DAF = vy - aa \times \text{h. l. of } \frac{v+y}{a}$ .

COROLLARY IV.

When  $A$  and  $C$  coincide, the hyperbola becomes a triangle, and  
 because  $CA$  and  $AE$  are then each  $= 0$ , the rule will become barely  
 $CB \times BD$  or  $AB \times BD$  for the area.

COROLLARY V.

Similar segments of any hyperbolas are to each other as the rect-  
 angles of their axes. (Where by similar segments are meant those  
 whose ordinates or abscissas are as their like axes or diameters.) For, by  
 cor. I, the seg. being expressed by  $ac \times q \sqrt{1+qq} - \text{h. l. of } q + \sqrt{1+qq}$   
 or  $ac \times \mathcal{Q} \sqrt{\mathcal{Q}\mathcal{Q}-1} - \text{h. l. of } \mathcal{Q} + \sqrt{\mathcal{Q}\mathcal{Q}-1}$ , and  $q = \frac{y}{c}$ , or

$\mathcal{Q} = \frac{v}{a}$ , being the same in all similar segments, those segments will  
 be as  $ac$ .—And hence, similar segments of equilateral hyperbolas  
 are as the squares of their axes, or of their like diameters.

Hence, also, hyperbolas of the same transverse axe and abscissa,  
 are to one another as their conjugate axes; but if their bases or or-  
 dinates and conjugate axes be the same, they will be as their trans-  
 verse axes. For when  $a$  and  $v$  are the same, the quantity  $\frac{cv\sqrt{vv-aa}}{a}$

$- ac \times \text{h. l. of } \frac{v+\sqrt{vv-aa}}{a}$  will be as  $c$ ; and if  $c$  and  $y$  be the

same, the quantity  $\frac{ay\sqrt{cc+yy}}{c} - ac \times \text{h. l. of } \frac{y+\sqrt{cc+yy}}{c}$  will be as  $a$ .

And



*Note 1.* The same rules will serve for any other two conjugate diameters and their abscissa and ordinate; multiplying the result by the sine of the angle formed by the abscissa and ordinate, to the radius 1.

*Note 2.* The hyperbolic logarithm is equal to the common logarithm multiplied by 2.302585093.

Ex-

And, consequently, having the quadrature of any one hyperbola, from it we may find that of any other hyperbola; viz. knowing the the area answering to every abscissa in any one hyperbola, we can find the area answering to any abscissa in any other hyperbola. For we can find a similar abscissa in the squared hyperbola, with the corresponding area. Then, as the rectangle of the axes of the squared hyperbola is to the rectangle of those of the hyperbola proposed, so is the area of the former to that of the latter.—Thus, let  $t, c$  be the transverse and conjugate axes of any proposed hyperbola, and  $x$  its abscissa; also  $T, C$  the axes of the squared hyperbola: Then,  $t : T :: x : \frac{Tx}{t} =$  the similar abscissa in the squared hyperbola, whose

corresponding area call  $A$ : Then, as  $TC : A :: tc : \frac{tcA}{TC} =$  the area of the hyperbola proposed.—Now, it will be most convenient to have the squared hyperbola an equilateral one whose axes are each an unit. For then we need only divide the abscissa of the hyperbola proposed by its transverse axe, and multiply the product of its axes by the area answering to this quotient in the squared or equilateral hyperbola. And this is similar to the finding of the segments of circles and ellipses from those of a circle whose segments are already calculated and ranged in tables.

#### COROLLARY VI.

If  $a$  and  $c$  be any other two conjugate diameters, and  $s$  the sine of the angle formed by  $a$  and its ordinates, to the radius 1; it is extremely evident that  $\frac{scv\sqrt{vv-aa}}{a} = sac \times \text{h. l. of } \frac{v + \sqrt{vv-aa}}{a}$  will denote the oblique segment cut off by a double ordinate to the diameter  $a$ . And if, in this expression, for  $sa$  and  $sv$  be substituted respec-



## EXAMPLE.

Required the area of an hyperbola whose base is 24 and altitude 10, the transverse axe being 30.

Here  $a = 15$ ,  $y = 12$ , and  $v = 25$ ; hence  $c =$   

$$\frac{ay}{\sqrt{vv-aa}} = \frac{15 \times 12}{\sqrt{40 \times 10}} = \frac{180}{20} = 9.$$

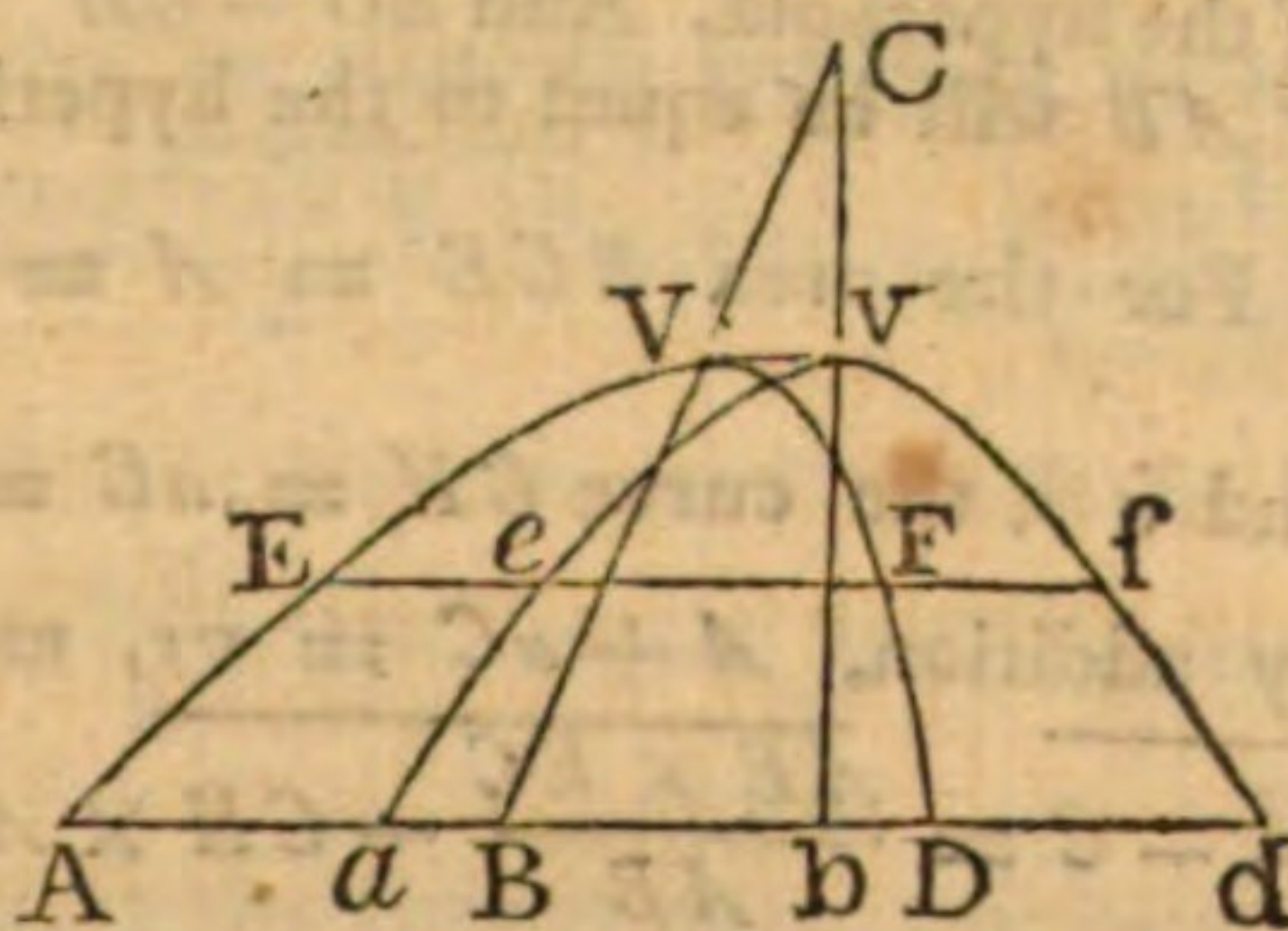
5 A

Then

respectively  $A$  and  $V$ , the said oblique hyperbola will be expressed by  $\frac{cV\sqrt{VV-AA}}{A} - Ac \times \text{h. l. of } \frac{V+\sqrt{VV-AA}}{A}$  which also expresses a right hyperbola whose transverse and conjugate axes are  $A$ ,  $c$ , and the distance of its base from the center  $V$ .

And hence is evident the following construction which *SCHOOTEN* first demonstrated, in a very operose manner, in his *Exercitationes Mathematicæ Lib. 4.*

Viz. Let  $AD$  be a double ordinate to the semi-diameter  $CV$  of the oblique hyperbola  $AVD$ . Draw  $Vv$  parallel and  $Cvb$  perpendicular to  $AD$ ; and with the semi-transverse  $Cv$  and semi-conjugate = the semi-conjugate of  $CV$  describe the right hyperbola  $avd$ . And the right hyperbola  $avd$  will be =  $AVD$  the oblique one proposed; also  $AD = ad$ , and every other corresponding ordinates  $EF$ ,  $ef$ , equal to each other.



## COROLLARY VII.

Hence it appears that all hyperbolas having the same center and equal bases, and between the same parallels  $Ad$ ,  $Vv$ , infinitely produced, are equal to each other; as are, also, their corresponding sections  $EF$ ,  $ef$ , parallel to the bases; and likewise the intercepted frustums  $AEFD$ ,  $aefd$ .

Co-

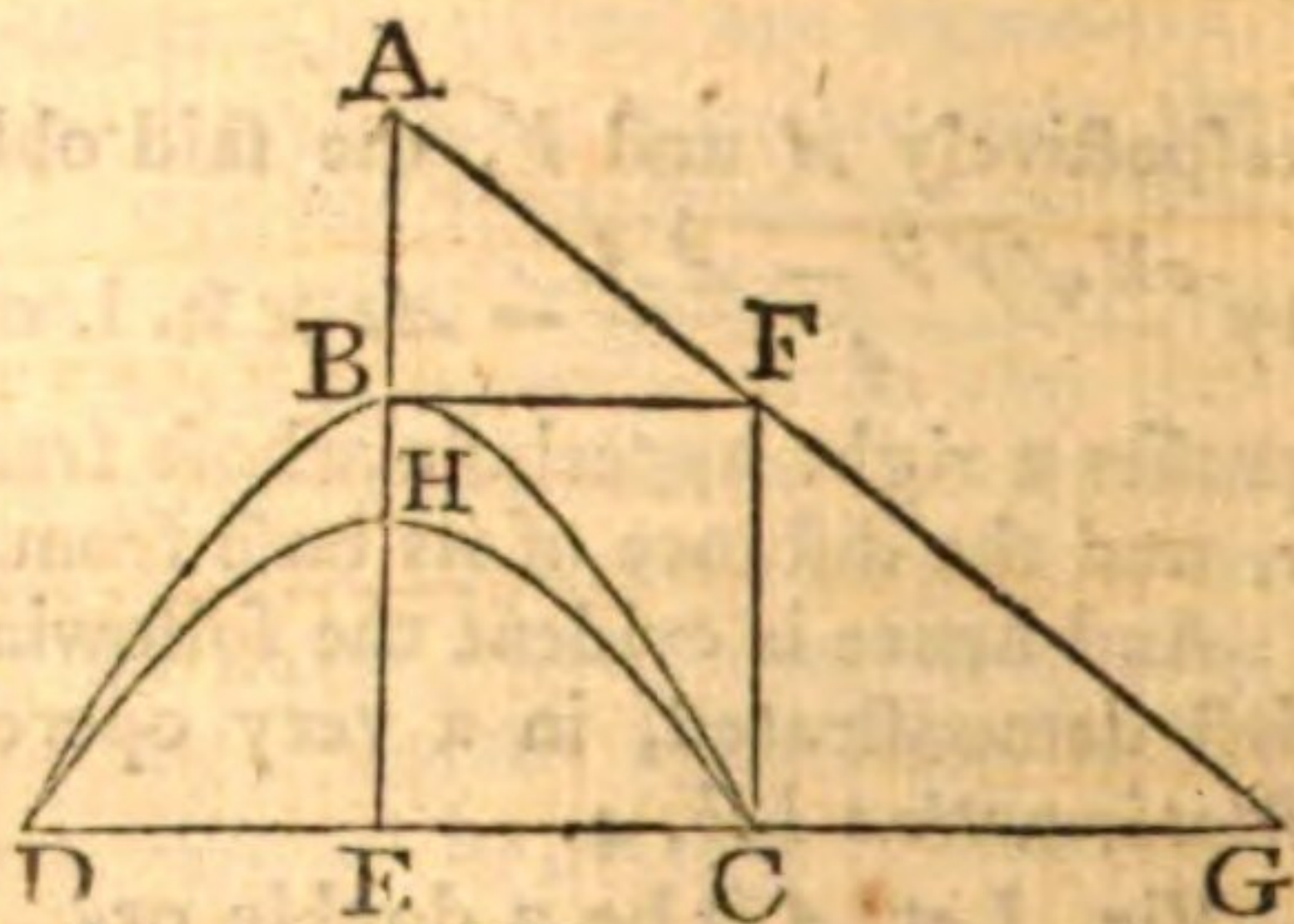


Then  $\frac{ay + cv}{ac} = \frac{15 \times 12 + 9 \times 25}{15 \times 9} = \frac{4+5}{3} = 3$ , whose hyperbolic logarithm is  $1.098612288\frac{2}{3}$ , which being multiplied by  $ac = 15 \times 9$ , produces  $148.31265897$ ; and this being taken from  $vy = 25 \times 12 = 300$ , leaves  $151.68734103 =$  the area required.

RULE

COROLLARY VIII.

Let  $AB$  be the semi-transverse axis of the hyperbola  $CBD$ ; draw  $CF$  parallel and  $FB$  perpendicular to  $BA$ , and draw  $AFG$  meeting  $DC$ , produced, in  $G$ ; then upon the base  $CD$  of the hyperbola describe a parabola  $CHD$  having its parameter equal to the conjugate axis of the hyperbola. And  $EG - CH \times AB$  will be equal to the hyperbolic area  $BCE$ .



For the area  $BCE = A = \frac{1}{2}vy - \frac{1}{2}ac \times \text{h. l. of } \frac{ay + cv}{ac}$ ,  
and  $a \times$  the curve  $CH = aC = \frac{1}{2}vy + \frac{1}{2}ac \times \text{h. l. of } \frac{ay + cv}{ac}$ ,  
by addition,  $A + aC = vy$ , and therefore  $A = vy - aC = a \times$   
 $\frac{vy}{a} - C = \frac{AE \times EC}{AB} - CH \times AB = EG - CH \times AB$ .

So that the quadrature of the hyperbola depends upon the rectification of the parabola.

COROLLARY IX.

By subtracting the segment  $vy - ac \times \text{h. l. of } \frac{ay + cv}{ac}$  from the segment  $VY - ac \times \text{h. l. of } \frac{aY + cV}{ac}$ , there will result  $VY - vy - ac \times \text{h. l. of } \frac{aY + cV}{ay + cv}$  for the frustum included by two parallel double ordinates  $2Y, 2y$ , their distances from the center being  $V, v$ .



## \* RULE II.

Putting  $x$  = the abscissa  $AB$ , and the rest of the letters as in the first rule, also  $z = \frac{x}{2a+x}$ . Then

$2xy \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3 - \frac{1}{7.9.11}z^4 \&c.$  or  
 $2xy \times : \frac{1}{3} - \frac{1}{5}Az - \frac{1}{7}Bz - \frac{3}{9}Cz - \frac{5}{11}Dz \&c.$  will be  
 the area of the semi-segment  $ABD$ . (Fig. to rule I)  
 putting  $A, B, C, \&c.$  for the first, second, third, &c.  
 term.

Ex-

## \* DEMONSTRATION.

The fluxion of the area is  $= y\dot{x} = \frac{cx\sqrt{2ax+xx}}{a}$ . Then instead  
 of  $\frac{x}{2a+x}$  write  $z$ , in order that the fluent quantity  $z$  may be less  
 than 1, and by that means the series be made to converge by its  
 powers, and the said fluxion  $y\dot{x} = \frac{cx\sqrt{2ax+xx}}{a}$  will be  $= \frac{4acz^{\frac{3}{2}}}{1-z}$   
 $= 2acz^{\frac{3}{2}} \times : 1.2 + 2.3z + 3.4z^2 + 4.5z^3 \&c.$  and the fluent is  
 $4acz^{\frac{3}{2}} \times : \frac{1.2}{3} + \frac{2.3}{5}z + \frac{3.4}{7}z^2 + \frac{4.5}{9}z^3 \&c. =$  the semi-segment  
 $ABD$ .

Now although this series converge by the powers of  $z$ , yet the co-  
 efficients actually diverge; in order, then, to make the coefficients to  
 converge, let the series be multiplied by some convenient quantity less  
 than 1, as  $1-z$ , and the quantity  $4acz^{\frac{3}{2}}$  divided by the same, and the  
 result will be  $\frac{8acz^{\frac{3}{2}}}{1-z} \times : \frac{1.1}{1.3} + \frac{2.2}{3.5}z + \frac{3.3}{5.7}z^2 + \frac{4.4}{7.9}z^3 \&c.$  for the  
 said area  $ABD$ . And, to make the series converge still quicker, let  
 the same operation be performed upon this series, and there will re-  
 sult  $\frac{8acz^{\frac{3}{2}}}{1-z} \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3 - \frac{1}{7.9.11}z^4 \&c.$   
 for the value of the said area.

But



## EXAMPLE.

Taking here the last example, in which  $x = 10$ ,  
 $y = 12$ , and  $2a = 30$ ;  $z$  will be  $= \frac{x}{2a+x} = \frac{10}{30+10}$   
 $= \frac{10}{40} = \frac{1}{4}$ .

Then

---

But  $\frac{8acz^{\frac{3}{2}}}{1-z|^2} = \frac{2cx\sqrt{2ax+xx}}{a} = 2xy$ ; wherefore  $2xy \times :$   
 $\frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3 \&c. = \text{the area } ABD. \text{ Q.E.D.}$

## COROLLARY I.

An hyperbola is always less than a parabola of the same base and altitude; for  $2xy \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 \&c.$  is always less than  $2xy \times \frac{1}{3} = \frac{2}{3}xy = \text{the parabola}$ ; but as the altitude is diminished, the hyperbola approaches nearer and nearer to the measure of the parabola, till, at last, they vanish in a ratio of equality.

## COROLLARY II.

An hyperbola, is, evidently, for any finite altitude, always greater than a triangle of the same base and altitude; but as the altitude is increased, the triangle approaches nearer and nearer to the value of the hyperbola, till, at length, when the altitude is infinite, they become accurately equal to each other. Which I demonstrate, by the method of increments, after the notation of the ingenious *Mr Emerson*, thus:

When the altitude  $x$  is infinite, then  $z = \frac{x}{2a+x} = \frac{x}{x} = 1$ , and the general series expressing the value of the area will become  $2xy \times : \frac{1}{3} - \frac{1}{1.3.5} - \frac{1}{3.5.7} - \frac{1}{5.7.9} \&c.$ —Now to find the sum  $s$  of any number  $n$  of the negative terms; since it is evident that the  $n$  term is  $\frac{1}{2n-1.2n+1.2n+3}$ , or, putting  $v = 2n-1$ , then  $v = 2n = 2$ , and



Then  $A = \frac{1}{3} = .33 \text{ \&c.}$

|                          |             |
|--------------------------|-------------|
| $B = \frac{1}{5} Az =$   | $.01666667$ |
| $C = \frac{1}{7} Bz =$   | $51524$     |
| $D = \frac{3}{9} Cz =$   | $4960$      |
| $E = \frac{5}{11} Dz =$  | $5639$      |
| $F = \frac{7}{13} Ez =$  | $76$        |
| $G = \frac{9}{15} Fz =$  | $11$        |
| $H = \frac{11}{17} Gz =$ | $2$         |

the sum of these negative terms is  $.017318039$  which being taken from the first term  $\frac{1}{3}$ , leaves  $.316015294$ ; and this multiplied by  $2xy = 240$ , produces  $75.84367056 =$  the semi-segment, the double of which is  $151.68734112 =$  the area required.

5 B

RULE

the  $n$  term  $= \frac{1}{v.v+2.v+3} = \frac{1}{vvv}$ ; the  $n+1$  term or  $s$  will be

$\frac{1}{vvv}$ ; and the integral  $s = A - \frac{1}{2vvv} = A - \frac{1}{4vv}$ : But when  $n$  is

$= 1$ , then  $s$  ought to be  $\frac{1}{vvv}$  = (in this case)  $A - \frac{1}{4vv}$ , hence  $A =$

$\frac{1}{4vv} + \frac{1}{vvv} = \frac{v+4}{4vvv} = \frac{1+4}{4.1.3.5} = \frac{5}{60} = \frac{1}{12}$ ; and consequently

the correct integral is  $s = \frac{1}{12} - \frac{1}{4vv} = \frac{1}{12} - \frac{1}{4.2n+1.2n+3}$ ,

which when  $n$  is infinite becomes accurately  $= \frac{1}{12}$ , that is, the sum

of the infinite series  $\frac{1}{1.3.5} - \frac{1}{3.5.7} \text{ \&c.}$  is  $= \frac{1}{12}$ ; which being taken

from  $\frac{1}{3}$  leaves  $\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$ , and hence  $2xy \times \frac{1}{4} = \frac{1}{2}xy =$  the area of the hyperbola when  $x$  is infinite, and is therefore equal to a triangle of the same base and altitude.

## COROLLARY III.

Hence the space included between the arc of an hyperbola, infinitely produced, and its chord, is of a finite magnitude, although it be



## R U L E III.

Take  $BH = \frac{3AB^2}{8AC} =$  to three-fourths of a third proportional to the transverse axe  $2AC$ , and the abscissa  $AB$ ; upon the diameter  $AI =$  the parameter of the axe, describe a circle meeting with  $BD$  in  $K$ , and with  $HL$ , parallel to  $BD$ , in  $L$ ; and draw  $AK$ ,  $AL$ . [See fig. to rule 1.]

Then will the segment  $FAD$  be nearly equal to  $\frac{4AL + AD}{15} \times 4AB$ , or  $= \frac{4\sqrt{tx + \frac{1}{4}xx} + \sqrt{tx}}{15} \times \frac{4cx}{t}$ , putting

be of an infinite length. For this space is the difference between the hyperbola and triangle, which, as above, is nothing, that is, nothing in comparison with an infinite space.

## COROLLARY IV.

By cor. 3 rule 2 for the segment of a circle, the area of a circle whose diameter is 1, is  $n = 2 \times \frac{1}{3} + \frac{1}{1.3.5} - \frac{1}{3.5.7} + \&c.$  and, by cor. 2 to this problem,  $\frac{1}{2} = 2 \times \frac{1}{3} - \frac{1}{1.3.5} - \frac{1}{3.5.7} - \&c.$  from hence are deduced the following equations:

$$\text{I. } n = \frac{5}{6} - 4 \times \frac{1}{3.5.7} + \frac{1}{7.9.11} + \frac{1}{11.13.15} + \frac{1}{15.17.19} \&c.$$

$$\text{II. } n = \frac{1}{2} + 4 \times \frac{1}{1.3.5} + \frac{1}{5.7.9} + \frac{1}{9.11.13} + \frac{1}{13.15.17} \&c.$$

$$\text{III. } n = \frac{2}{3} + 12 \times \frac{1}{1.3.5.7} + \frac{1}{5.7.9.11} + \frac{1}{9.11.13.15} \&c.$$

$$\text{IV. } \frac{1}{96} = \frac{1}{1.3.5.7} + \frac{2}{5.7.9.11} + \frac{3}{9.11.13.15} \&c.$$



ting  $t$  = the transverse  $2 AC$ ,  $c$  = the conjugate, and  $x = AB$  the abscissa.\*

EXAMPLE.

Taking the same example, in which  $t = 30$ ,  $c = 18$ , and  $x = 10$ ; we shall have  $\frac{\sqrt{tx} + 4\sqrt{tx + \frac{3}{4}x^2}}{15} \times \frac{4cx}{t} = \frac{\sqrt{300} + 4\sqrt{375}}{450} \times 720 = \frac{10\sqrt{3} + 20\sqrt{15}}{5} \times 8 = \sqrt{3} + 2\sqrt{15} \times 16 = 151.64828 = \text{the area nearly.}$

RULE

\* DEMONSTRATION.

By proceeding as in rule 2 for the circular segment, we obtain  $\frac{cx\sqrt{tx}}{t} \times \frac{2}{3} + \frac{x}{5t} - \frac{x^2}{28t^2}$  for the true value of the semi-segment  $ADB$ . Then, after the method used in demonstrating rule 6 for the cir. segment, suppose  $ADB$  to be  $= \frac{cmx\sqrt{tx + \frac{3}{4}xx}}{t} + \frac{cnx\sqrt{tx}}{t} = \frac{cx\sqrt{tx}}{t} \times \frac{m}{m+n} + \frac{3mx}{8t}$  &c. hence, comparing the like terms,  $m+n = \frac{2}{3}$ , and  $\frac{3m}{8} = \frac{1}{5}$ ; and therefore  $m = \frac{8}{15}$ , and  $n = \frac{2}{3} - \frac{8}{15} = \frac{2}{15}$ . Which values being substituted for them, we have  $ADB =$

$$\begin{aligned} \frac{2cx}{t} \times \frac{4\sqrt{tx + \frac{3}{4}xx} + \sqrt{tx}}{15} &= 2x \times \frac{4\sqrt{\frac{c^2x}{t} + \frac{3c^2x^2}{4tt}} + \sqrt{\frac{c^2x}{t}}}{15} = \\ &= 2AB \times \frac{4\sqrt{AB + \frac{3AB^2}{8AC}} \times AI + \sqrt{BA \times AI}}{15} = 2AB \times \\ \frac{4\sqrt{HA \times AI} + \sqrt{BA \times AI}}{15} &= 2AB \times \frac{4AL + AD}{15}. \quad Q.E.D. \end{aligned}$$

And this rule was first given by Sir I. Newton, but without demonstration.



## R U L E IV.

If  $BH$  be taken  $= \frac{5xx}{7t} = \frac{5AB^2}{14AC} =$  to five-sevenths of the said third proportional to the axe  $2AC$ , and the abscissa  $AB$ ; and the lines drawn as in the last rule.

Then will  $\frac{21AL + 4AK}{75} \times 4AB$ , or  $\frac{21\sqrt{tx} + \frac{5}{7}xx + 4\sqrt{tx}}{75} \times \frac{4cx}{t}$  express the area nearer than by the last rule.\*

Ex-

## \* DEMONSTRATION.

Taking  $BH = \frac{Axx}{t}$ , let, as before, the area be expressed by

$$\begin{aligned} AB \times m \times AL + n \times AK &= AB \times m \sqrt{HA \times AI} + n \sqrt{BA \times AI} \\ &= mx \sqrt{x + \frac{Axx}{t}} \times \frac{cc}{t} + nx \sqrt{x \times \frac{cc}{t}} = m \sqrt{tx + Axx} + n \sqrt{tx} \\ \times \frac{cx}{t} &= \frac{cx \sqrt{tx}}{t} \times : m + n + \frac{mA}{2t} - \frac{mA^2x^2}{8tt} \&c. \text{ which being com-} \\ \text{pared with the true series } \frac{cx \sqrt{tx}}{t} \times : \frac{2}{3} + \frac{x}{5t} - \frac{x^2}{28t^2} \&c. \text{ we obtain} \\ \text{these three equations, } m + n &= \frac{2}{3}, \frac{mA}{2} = \frac{1}{5}, \text{ and } \frac{mA^2}{8} = \frac{1}{28}; \\ \text{whence } A &= \frac{5}{7}, m = \frac{14}{25}, \text{ and } n = \frac{8}{75}. \text{ Consequently } BH = \frac{Axx}{t} \\ &= \frac{5xx}{7t}, \text{ and the semi-segment } = 2AB \times \frac{21AL + 4AK}{75} = \frac{2cx}{t} \times \\ &\frac{21\sqrt{tx} + \frac{5}{7}xx + 4\sqrt{tx}}{75}. \quad Q.E.D. \end{aligned}$$

Which likewise is another rule given by *Sir I. Newton*, without demonstration.



## EXAMPLE.

Taking, still, the same example, we have

$$\frac{21\sqrt{tx} + \frac{5}{7}xx + 4\sqrt{tx}}{75} \times \frac{4cx}{t} = \frac{21\sqrt{371\frac{1}{7}} + 4\sqrt{300}}{75} \times \frac{720}{30} =$$

$$\frac{30\sqrt{182} + 40\sqrt{3}}{75} \times 24 = \frac{3\sqrt{182} + 4\sqrt{3}}{5} \times 16 = 151.68133$$

= the area nearly.

## PROBLEM VII.

*To find the area of the frustum of an hyperbola, included between two parallel double ordinates.*

## RULE.

Multiply the semi-transverse by the one ordinate, and its distance from the center by the semi-conjugate, and take the sum of the products; do the same by the other ordinate and its distance from the center, taking the sum of the products also; divide the greater sum by the less, and multiply the hyperbolic logarithm of the quotient by the product of the semi-axes; then subtract this last product from the difference of the products arising from the multiplication of each ordinate by its distance from the center, and the remainder will be the area required.

That is,  $VY - vy - ac \times \text{hyp. log. of } \frac{aY + cV}{ay + cv} =$  the area included by  $2Y$  and  $2y$ ,  $V$  and  $v$  being their respective distances from the center, and  $a$  and  $c$  the semi-axes. As is proved in cor. 8 to rule 1 to the last problem.

Or, if there be calculated the two segments whose bases are  $2Y$  and  $2y$ , their difference will be the area sought.



## EXAMPLE.

To find the area of the frustum whose altitude is 5; its less end being, as in the example to the last problem, 24, its distance from the center of the hyperbola 25, the transverse axe 30, and the conjugate 18.

Here  $a = 15$ ,  $c = 9$ ,  $v = 25$ ,  $y = 12$ , and  $V = 30$ ; hence  $\gamma = \frac{c\sqrt{VV - aa}}{a} = \frac{9\sqrt{30^2 - 15^2}}{15} = 9\sqrt{2^2 - 1^2} = 9\sqrt{3} = 15.58845727$ .

Then  $\frac{a\gamma + cV}{ay + cv} = \frac{15 \times 9\sqrt{3} + 9 \times 30}{15 \times 12 + 9 \times 25} = \frac{9\sqrt{3} + 3 \times 6}{12 + 3 \times 5} = \frac{\sqrt{3} + 2}{3} = 1.244016936$ , whose hyperbolic logarithm is .2183456, which multiplied by  $ac = 15 \times 9$  will produce 29.476656.

And  $V\gamma - vy = 30 \times 9\sqrt{3} - 25 \times 12 = 30 \times 9\sqrt{3} - 10 = 30 \times 5.58845727 = 167.653718$ .

Wherefore  $167.653718 - 29.476656 = 138.177062$  = the area required.

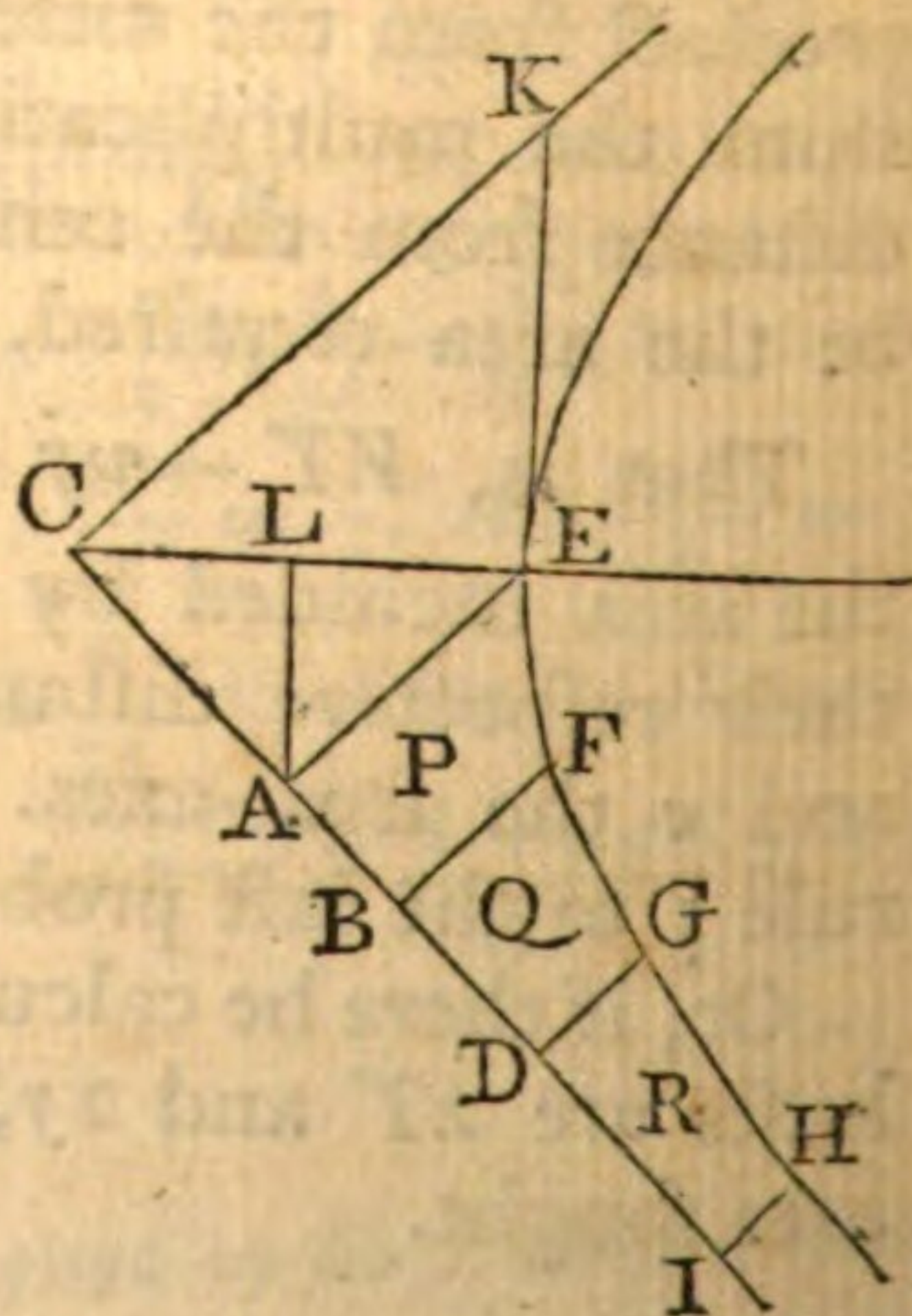
## PROBLEM VIII.

To find the area included between two ordinates BF, DG, to an asymptote CD of an hyperbola, drawn parallel to the other asymptote.

## RULE.

From the vertex E of the curve draw EA parallel to BF.

Then





Then the square of  $CA$  or of  $AE$  multiplied by the hyperbolic logarithm of the quotient arising from the division of  $CD$  by  $CB$ , or from the division of  $BF$  by  $DG$ , will produce the area of  $BFGD$ .\*

Ex-

## \* DEMONSTRATION.

Put  $CA = a$ ,  $AB = x$ , and  $BF = y$ ; then, by the property of the curve,  $aa = CB \times BF = \overline{a+x} \times y$ , and hence  $y = \frac{aa}{a+x}$ , and  $y \dot{x} = \frac{aa\dot{x}}{a+x}$ ; whose fluent is  $aa \times \text{h. l. of } \frac{a+x}{a} = CA^2 \times \text{h. l. of } \frac{CB}{CA} = \text{the area } ABFE$ .

In like manner  $ADGE$  is  $= CA^2 \times \text{h. l. of } \frac{CD}{CA}$ .

And, taking the difference, we have  $BDGF = CA^2 \times \text{h. l. of } \frac{CD}{CA} - \text{h. l. of } \frac{CB}{CA} = CA^2 \times \text{h. l. of } \frac{CD}{CB}$ ; or by writing  $\frac{CA^2}{BF}$  for  $CB$ , and  $\frac{CA^2}{DG}$  for  $CD$ , the same area will be  $= CA^2 \times \text{h. l. of } \frac{BF}{DG}$ . Q.E.D.

## COROLLARY I.

If  $CB, CD, CI$ , &c. be in geometrical progression; then  $\frac{CD}{CB}$  will be  $= \frac{CI}{CD}$ , &c. and consequently the spaces  $P, Q, R$ , &c. equal to one another; or the spaces  $ABFE, ADGE, AIKE$ , &c. a series of arithmeticals, and are the logarithms of the geometricals  $\frac{CB}{CA}, \frac{CD}{CA}, \frac{CI}{CA}$ , &c. to the modulus  $CA$ ; or the said spaces will form a scale of hyperbolic logarithms whose absolute numbers are  $CB, CD, CI$ , &c. when  $CA$  is 1.

Co-



## EXAMPLE.

Required the area  $BDGF$  included by two ordinates  $BF$ ,  $DG$ , whose lengths are 8 and 6; the semi-transverse  $CE$  being 15, and the semi-conjugate  $EK = 9$ .

Draw  $AL$  perpendicular to  $CE$ ; then because  $EA$  is  $= AC$ ,  $CL$  will be  $= LE$ ; and in the similar triangles  $ALC$ ,  $CEK$ , because  $CL$  is  $= \frac{1}{2}CE$ ,  $CA$  will be  $= \frac{1}{2}CK$ , and so  $CA^2 = \frac{1}{4}CK^2 = \frac{CE^2 + EK^2}{4} = \frac{15^2 + 9^2}{4} = 9 \times \frac{5^2 + 3^2}{4} = 9 \times \frac{17}{2} = \frac{153}{2}$ .

But

## COROLLARY II.

The space  $ABFE$  is also equal to  $a^2 \times : \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \&c.$

For the fluxion of the area  $y\dot{x} = \frac{a\dot{a}x}{a+x}$  is  $= a\dot{a}x \times : \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} \&c.$  whose fluent is  $a^2 \times : \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \&c. = CA^2 \times : \frac{AB}{CA} - \frac{AB^2}{2CA^2} + \frac{AB^3}{3CA^3} \&c.$

After the same manner  $ADGE$  is  $= CA^2 \times : \frac{AD}{CA} - \frac{AD^2}{2CA^2} + \frac{AD^3}{3CA^3} \&c.$

And, consequently, by taking the difference, we obtain  $BDGF = CA^2 \times : \frac{AD - AB}{CA} - \frac{AD^2 - AB^2}{2CA^2} + \frac{AD^3 - AB^3}{3CA^3} \&c.$

## COROLLARY III.

Hence the hyperbolic logarithm of  $\frac{a+x}{a}$  is  $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \&c.$



But the hyperbolic logarithm of  $\frac{BF}{DG} = \frac{8}{6} = \frac{4}{3}$  is  
 .287682.

Wherefore  $.287682 \times \frac{153}{2} = 22.007673 =$  the  
 area required.

PROBLEM IX.

*To find the curve surface of an hyperboloid.*

RULE.

Let  $a$  and  $c$  be the femi-axes of the generating hyperbola, and  $v$  the distance of its base from the center. Also let  $A = \frac{aa}{\sqrt{aa+cc}}$  be the femi-transverse of another hyperbola whose femi-conjugate is  $c$  the same with that of the former. Then find, by problem 7, the area of the frustum of this latter hyperbola whose two ends are distant from the center by  $v$  and  $a$ ; multiply this area by 3.14159 &c. the circumference of a circle whose diameter is 1, and the product will be the surface required.

That is,  $p \times vY - ay - Ac \times \text{hyp. log. of } \frac{AY + cv}{Ay + ac} =$   
 the surface required;  $p$  being = 3.14159 &c. and  
 $Y, y$ , the ordinates, of the latter hyperbola, whose  
 distances from the center are  $v, a$ .\*

5 D

Ex-

\* DEMONSTRATION.

Put, here,  $w =$  the ordinate to the abscissa  $v$ , and  $z =$  the curve;  
 then  $a : c :: \sqrt{vv - aa} : \frac{c\sqrt{vv - aa}}{a} = w$ , and  $\dot{w} = \frac{cvv}{a\sqrt{vv - aa}}$ ;  
 hence



## EXAMPLE.

Required the curve surface of an hyperboloid whose altitude is 10, the transverse and conjugate axes of the generating hyperbola being 30 and 18.

$$\text{Here } a = 15, c = 9, A = \frac{aa}{\sqrt{aa+cc}} = \frac{15^2}{\sqrt{15^2+9^2}} = \frac{75}{\sqrt{34}},$$

$$v = 25, Y = \frac{c\sqrt{vv-AA}}{A} = \frac{9\sqrt{25^2-\frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = 15, \text{ and } y$$

$$= \frac{c\sqrt{aa-AA}}{A} = \frac{9\sqrt{15^2-\frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = \frac{27}{5} = 5\frac{2}{5}.$$

Hence

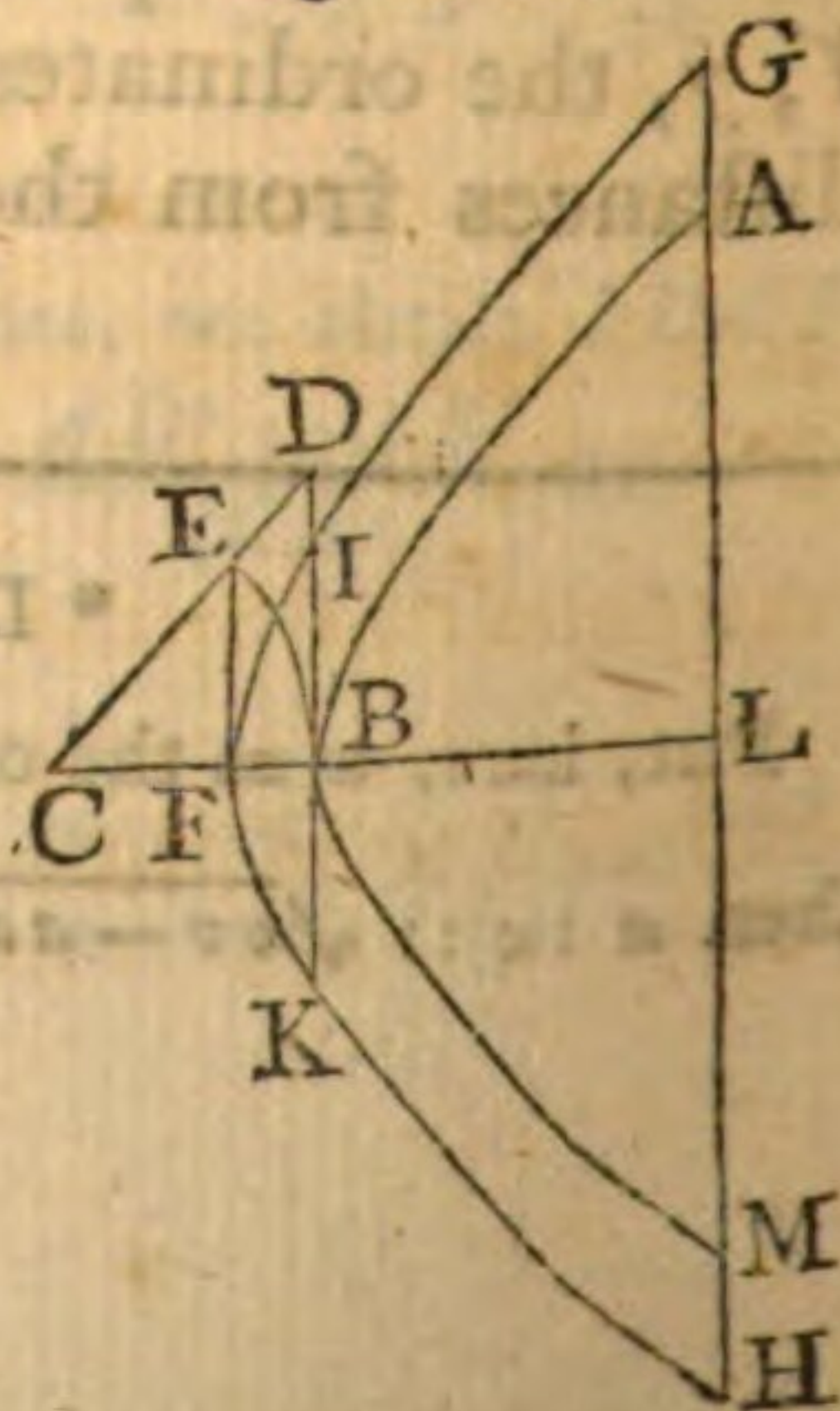
$$\begin{aligned} \text{hence the fluxion of the surface will be } 2pwz &= 2pw\sqrt{\dot{v}\dot{v}+\dot{w}\dot{w}} \\ &= \frac{2pc\sqrt{vv-aa}}{a} \sqrt{\dot{v}\dot{v} + \frac{ccvv\dot{v}\dot{v}}{aa \times vv-aa}} = \frac{2pcv\sqrt{aa \times \dot{v}\dot{v}-aa+ccv\dot{v}}}{aa} \\ &= \frac{2pcv\sqrt{aa+cc}}{aa} \sqrt{vv-\frac{aaaa}{aa+cc}} = \frac{2pcv\sqrt{vv-AA}}{A}; \text{ which} \end{aligned}$$

is, evidently, equal to  $p$  drawn into the fluxion of an hyperbolic segment, the semi-axes being  $A$ ,  $c$ , and abscissa  $v$ . And by taking the correct fluent we obtain the expression in the rule above given, which is the difference of two segments whose bases are distant from the center by  $v$  and  $a$ ; or the frustum whose two ends are distant from the center by the same quantities  $v$  and  $a$ . Q. E. D.

## COROLLARY I.

Hence if  $ABM$  be the generating hyperbola,  $BC$  its semi-transverse,  $BD$  perpendicular to  $BC$  and equal to the semi-conjugate. — Draw  $CD$ ; make  $CE = CB$ , and on  $CB$  let fall the perpendicular  $EF$ ; with the semi-transverse  $CF$  and semi-conjugate of  $ABM$  describe the hyperbola  $GKH$ .

Then





Hence  $vY - ay = 25 \times 15 - 15 \times 5\frac{2}{5} = 19\frac{3}{5} \times 15 = 294$ ;  $Ac = \frac{75 \times 9}{\sqrt{34}} = \frac{75 \times 9\sqrt{34}}{34} = 115.7615451$ ;

$$\text{and } \frac{AY + cv}{Ay + ca} = \frac{\frac{75 \times 15}{\sqrt{34}} + 9 \times 25}{\frac{75 \times 27}{5\sqrt{34}} + 9 \times 15} = \frac{5}{3} \times \frac{5 + \sqrt{34}}{3 + \sqrt{34}} = \frac{19 + 2\sqrt{34}}{15}$$

$= 2.0441269$ ; whose hyperbolic logarithm is  $.7149706$ ; which multiplied by  $Ac = 115.7615451$  produceth  $82.7661049$ , which being taken from  $vY - ay = 294$  leaves  $211.2338951$ , which multiplied by  $3.14159$  &c. produces  $663.610853 =$  the curve surface required.

PRO-

Then, as the diameter of a circle is to its circumference, so is the frustum  $G I K H$ , included by the parallels  $B I$ ,  $A M$ , produced, to the surface generated by  $A B$ .

For, by similar triangles,  $DC : CB :: EC : CF = A = \frac{BC^2}{CD} = \frac{BC^2}{\sqrt{CB^2 + BD^2}} = \frac{aa}{\sqrt{aa + cc}}$ ; and all the rest is evident.

#### COROLLARY II.

Putting  $Z = \frac{FL}{2CF + FL}$ , and  $z = \frac{FB}{2CF + FB}$ ; and proceeding as in rule 2 for the hyperbolic segment, we shall obtain the surface generated by  $BA$  equal to  $3.14159$  &c. drawn into the difference of

the serieses  $2FL \times GH \times : \frac{1}{3} - \frac{1}{1.3.5}Z - \frac{1}{3.5.7}Z^2 - \frac{1}{5.7.9}Z^3$  &c.

and  $2FB \times IK \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3$  &c.



## PROBLEM X.

*To find the solidity of an hyperboloid.*

## RULE I.\*

From the sum of the height of the solid and the transverse axe of the generating hyperbola subtract  $\frac{1}{3}$  of the said height, and divide the remainder by 2 times the said sum; then multiply continually together

## \* DEMONSTRATION.

Let  $t$  be = the transverse, and  $c$  = the conjugate axe of the generating hyperbola,  $x$  = its abscissa or the altitude of the solid,  $y$  = the ordinate or radius of the base, and  $p = 3.14159$  &c.

Then  $yy = cc \times \frac{tx + xx}{tt}$ , and the fluxion of the solid or  $pyy\dot{x}$  is  $= pcc\dot{x} \times \frac{t+x}{tt}$ , whose fluent is  $pccxx \times \frac{\frac{1}{2}t + \frac{1}{3}x}{tt} = pxyy \times \frac{t + \frac{2}{3}x}{2t + 2x} = pxyy \times \frac{t + x - \frac{1}{3}x}{2t + 2x} =$  the measure of the solid  $= \frac{1}{2}$  base  $\times$  altitude  $\times \frac{t + x - \frac{1}{3}x}{t + x}$ . Q.E.D.

## COROLLARY I.

An hyperboloid is to a paraboloid of the same base and altitude, as  $t + \frac{2}{3}x$  is to  $t + x$ ; and therefore the former is always less than the latter by the quantity  $\frac{1}{6}pxyy \times \frac{x}{t+x}$ : which difference, when  $x$  is infinitely little or nothing, is nothing; and the hyperboloid approaches nearer and nearer to the paraboloid as the common altitude is diminished, till at last they vanish in a ratio of equality; also when the altitude is very little the hyperboloid is equal to the paraboloid very nearly: but when  $x$  is infinite, the same difference  $\frac{1}{6}pxyy \times \frac{x}{t+x}$  becomes barely  $\frac{1}{6}pxyy$ , and the value of the infinitely long hyperboloid is  $\frac{1}{2}pxyy - \frac{1}{6}pxyy = \frac{1}{3}pxyy =$  a cone of the same base and altitude, to which measure the hyperboloid continually approaches, and when the altitude is very great it is equal to the cone very nearly.

C o-



together, the quotient, the altitude of the solid, and the area of the base, and the product will be the solidity required.

That is, base  $\times$  altitude  $\times \frac{t + a - \frac{1}{3}a}{2t + 2a} = parr \times \frac{t + a - \frac{1}{3}a}{2t + 2a} =$  the solidity, putting  $a$  for the altitude,  $r$  the radius of the base,  $t$  the transverse axe, and  $p = 3.14159$  &c.

5 E

Ex-

## COROLLARY II.

When  $x$  is  $= nt$ , the general expression becomes  $\frac{1}{2}$  base  $\times$  altitude  $\times \frac{1 + \frac{2}{3}n}{1 + n}$ , where  $n$  may be any number integral or fractional. When  $n$  is infinitely little, this expression becomes  $\frac{1}{2}$  base  $\times$  altitude or  $=$  the paraboloid of the same base and altitude, as before; if  $n$  be infinitely great, it will become  $\frac{1}{2}$  base  $\times$  altitude  $\times \frac{2}{3}$ , that is,  $\frac{2}{3}$  of the paraboloid, or  $=$  the cone of the same base and altitude, as above; when  $n$  is  $= 1$ , or  $t = x$ , the expression becomes  $\frac{1}{2}$  base  $\times$  altitude  $\times \frac{5}{6}$ , or  $\frac{5}{6}$  of the paraboloid. Moreover, when  $n$  is between 1 and 0, the hyperboloid is between  $\frac{5}{6}$  and  $\frac{6}{6}$  of the paraboloid; and when  $n$  is between 1 and infinite, the hyperboloid is between  $\frac{5}{6}$  and  $\frac{4}{6}$  of the paraboloid of the same base and altitude.

## COROLLARY III.

If the generating hyperbola be equilateral, since  $t$  is then  $= c$ ,  $yy$  will be  $= tx^2 + xx$ , and hence  $t = \frac{yy - xx}{x}$ ; which being substituted instead of it, the general expression for the hyperboloid will become  $px \times \frac{yy - \frac{1}{3}xx}{2}$ , which is the same expression as that for the spheric segment, differing only in the sign of the latter term, it being  $-\frac{1}{3}xx$  here and  $+\frac{1}{3}xx$  there.

## COROLLARY IV.

When  $t$  is  $= 0$ , the rule becomes  $\frac{1}{3}$  base  $\times$  altitude, as it ought; for when  $t$  is  $= 0$ , the hyperboloid becomes a cone.



## EXAMPLE.

Required the content of an hyperboloid whose altitude is 10, the radius of its base 12, and the ordinate to the middle of the abscissa or height  $3\sqrt{7}$ .

Here, by prob. 3, we shall have  $\frac{10^2 \times 3^2 \times 7 \propto 5^2 \times 12^2}{5 \times 12^2 \propto 10 \times 3^2 \times 7}$   
 $= \frac{70 - 40}{8 - 7} = \frac{30}{1} = 30 = \text{the transverse.}$

Then  $parr \times \frac{t + a - \frac{1}{2}a}{2t + 2a} = p \times 10 \times 144 \times \frac{40 - \frac{10}{2}}{80} =$   
 $p \times 1440 \times \frac{3\frac{1}{2}}{8} = p \times 1440 \times \frac{11}{24} = 3.14159 \&c. \times 660$   
 $= 2073.451151369 = \text{the content required.}$

## RULE II.

To the square of the radius of the base add the square of the diameter in the middle between the base and top, multiply the sum by the altitude, the product by 3.14159 &c. and  $\frac{1}{6}$  of the last product will be the content of the segment.

That is,  $\frac{rr + dd}{6} \times ap = \text{the content of the segment, putting } r \text{ and } d \text{ for the radius of the base and diameter in the middle, } a \text{ for the altitude, and } p = 3.14159 \&c. \text{—This is proved in corollary 1 to rule 2 for the next problem.}$

## EXAMPLE.

Taking here the last example, in which  $r = 12$ ,  $d = 6\sqrt{7}$ , and  $a = 10$ ;

We shall have  $\frac{12^2 + 36 \times 7}{6} \times 10p = 11 \times 6 \times 10p = 660p = \text{the content the same as before.}$

PRO-



## PROBLEM XI.

*To find the content of the frustum of an hyperboloid.*

## \* RULE I.

From the sum of the squares of the semi-diameters of the two ends subtract  $\frac{1}{3}$  of a fourth proportional to the square of the transverse, the square of the conjugate, and the square of the altitude of the frustum; multiply the remainder by the said altitude, and the product by 3.14159 &c. and the half of the last product will be the content of the frustum.

That is,  $DD + dd - \frac{aacc}{3tt} \times \frac{1}{2}pa =$  the content, putting  $D$  and  $d$  for the semi-diameters of the ends,  $a =$  the altitude,  $t =$  the transverse,  $c =$  the conjugate, and  $p = 3.14159$  &c.

Ex-

## \* DEMONSTRATION.

Using here  $y, Y$  for the ordinates or semi-diameters of the ends,  $x$  for the altitude; and putting  $A$  for the distance of the less ordinate  $y$  from the vertex of the whole solid; since  $YY$  is  $= \frac{t+A+x \times A+x}{tt}$   $\times cc$ , we shall have the fluxion of the solid  $\dot{s} = pYY\dot{x} = pcc\dot{x} \times \frac{At + AA + 2Ax + tx + xx}{tt}$ , and the fluents give  $s = pccx \times \frac{At + AA + Ax + \frac{1}{2}tx + \frac{1}{3}xx}{tt}$ ; and this, by substituting  $\frac{yy}{cc}$  for  $\frac{At + AA}{tt}$ , and  $\frac{YY}{cc}$  for  $\frac{At + AA + 2Ax + tx + xx}{tt}$  becomes  $YY + yy - \frac{ccxx}{3tt} \times \frac{1}{2}px$ . Q.E.D.

## COROLLARY I.

When the generating hyperbola is equilateral,  $t$  is  $= c$ , and the rule becomes  $YY + yy - \frac{1}{3}xx \times \frac{1}{2}px$ , which is the same with the rule for the frustum of the sphere.

Co-



## EXAMPLE.

If a cask in the form of two frustums of an hyperboloid have its bung diameter 32 inches, its head diameter 24, and the diameter in the middle between the bung and head  $\frac{8}{5}\sqrt{310}$ ; required the content in ale gallons, the length being 40 inches.

By problem 4 we shall have

$$\begin{aligned} & \frac{10\sqrt{-12^2 + \frac{64}{25} \times 310 - 16^2} - 4 \times 12^2 \times 16^2}{12^2 - \frac{32}{25} \times 310 + 16^2} = \\ & \frac{10\sqrt{\frac{64 \times 310}{25} - 20^2} - 2^2 \times 12^2 \times 16^2}{20^2 - \frac{32 \times 310}{25}} = \frac{10\sqrt{248 - 125}^2 - 120^2}{125 - 124} \\ & = 30\sqrt{41^2 - 40^2} = 30\sqrt{81} = 270 = \text{the transverse axe } t. \\ & \text{And } \frac{270}{10}\sqrt{\frac{12^2}{2} - \frac{16 \times 310}{25} + \frac{16^2}{2}} = 27 \times 4\sqrt{\frac{3^2}{2} - \frac{62}{5} + \frac{4^2}{2}} \\ & = 108\sqrt{\frac{5^2}{2} - \frac{62}{5}} = 108\sqrt{\frac{125 - 124}{10}} = 108\sqrt{\frac{1}{10}} = \text{the} \\ & \text{conjugate axe } c. \end{aligned}$$

Then

## COROLLARY II.

When the axe is = 0, or the hyperboloid becomes a cone; since then  $T:y :: A+x:A$ , or  $T-y:y :: x:A = \frac{xy}{T-y}$ , and  $t:c :: A:y$ , we shall have  $\frac{cc}{tt} = \frac{yy}{AA} = \frac{T-y|^2}{xx}$ , and therefore  $\frac{ccxx}{3tt} = \frac{T-y|^2}{3}$ ; which being substituted for it in the rule  $TT + yy - \frac{ccxx}{3tt} \times \frac{1}{2}px$ , produces  $TT + Ty + yy \times \frac{1}{2}px$  for the frustum of the cone, as it ought.



Then  $DD + dd - \frac{aacc}{3tt} \times \frac{1}{2}ap \times 2 = 16^2 + 12^2 - \frac{20^2 \times 108^2}{30 \times 270^2}$   
 $\times 20p = 20^2 - \frac{20^2 \times 2^2}{30 \times 5^2} \times 20p = \frac{30 \times 25 - 4}{3} \times 4^2 \times 2p =$   
 $\frac{746 \times 32}{3} \times p = 7957\frac{1}{3} \times 3.14159 \&c. = 24998.69994216$   
 inches = 88.64787213 ale gallons.

## R U L E II.

Add together the squares of the greatest and least  
 femi-diameters and the square of the whole diameter  
 5 F in

## \* DEMONSTRATION.

Put  $x$  = the abscissa whose ordinate is  $d$ , the axes being  $t$  and  $c$ ; and, from the nature of the hyperbola, we shall have these three equations:

$$\begin{cases} tt\delta\delta = cc \times \overline{t+x} \times x & = cc \times \overline{tx+xx}, \\ ttdd = cc \times \overline{t+x-\frac{1}{2}a} \times \overline{x-\frac{1}{2}a} & = cc \times \overline{tx+xx-ax-\frac{1}{2}at+\frac{1}{4}aa}, \\ ttDD = cc \times \overline{t+x+\frac{1}{2}a} \times \overline{x+\frac{1}{2}a} & = cc \times \overline{tx+xx+ax+\frac{1}{2}at+\frac{1}{4}aa}, \end{cases}$$

from the sum of the two latter of which subtract the double of the former, and there will result  $tt \times DD - 2\delta\delta + dd = \frac{1}{2}aacc$ ; and hence  $\frac{aacc}{3tt}$  will be  $= \frac{2DD - 4\delta\delta + 2dd}{3}$ .

Which being substituted instead of it in the last rule will give  $\frac{DD + 4\delta\delta + dd}{6} \times ap$  for the content of the frustum required.

Q.E.D.

## COROLLARY I.

If  $d$  the least diameter be supposed to become infinitely little or nothing, the above rule will become  $\frac{DD + 4\delta\delta}{6} \times ap$  for the content of a segment.

## COROLLARY II.

An hyperbolic frustum is equal to a cylinder of the same altitude and whose diameter is  $\sqrt{\frac{DD + 4\delta\delta + dd}{6}}$ . And the segment is equal to the cylinder whose diameter is  $\sqrt{\frac{DD + 4\delta\delta}{6}}$ , the altitude being the same.



in the middle, multiply the sum by the altitude, and the product by 3.14159 &c. and one-sixth of the last product will be the content.

That is,  $\frac{DD + 4\delta\delta + dd}{6} \times ap =$  the content, putting  $D$ ,  $\delta$ , and  $d$  for the greatest, middle, and least semi-diameters,  $a =$  the altitude, and  $p = 3.14159$  &c.

## E X A M P L E.

Taking here the same Example as before, we

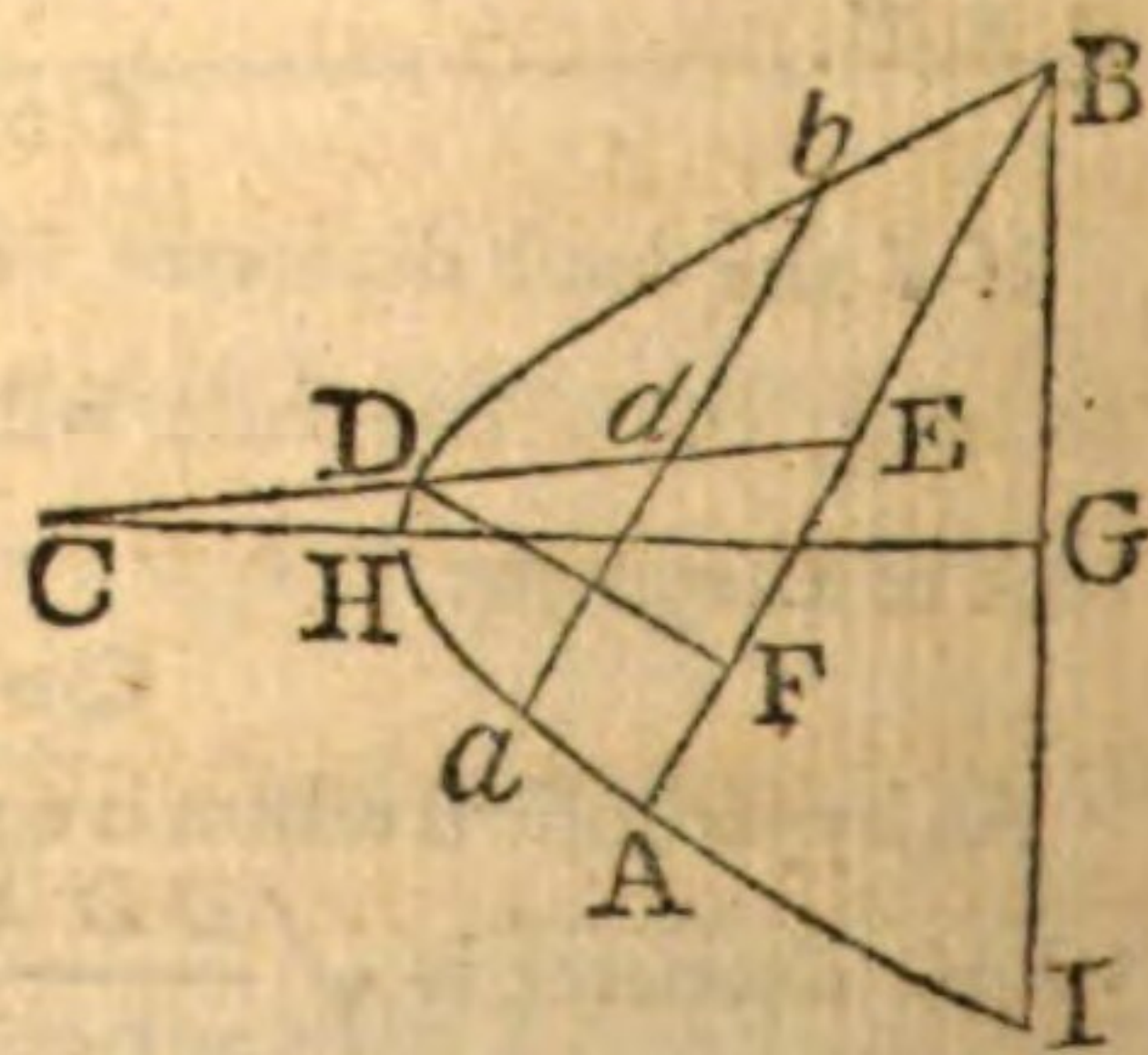
shall have  $\frac{16^2 + \frac{64 \times 310}{25} + 12^2}{6} \times 40p = \frac{20^2 + \frac{64 \times 62}{5}}{6} \times 40p = \frac{5 \times 20^2 + 64 \times 62}{3} \times 4p = 7957\frac{1}{3}p =$  the content the same as before.

## P R O B L E M XII.

*To find the content of the segment cut off an hyperboloid by a plane oblique to the axe of the solid.*

## R U L E.

From the sum of  $2CD$  and  $DE$  subtract  $\frac{1}{3}$  of  $DE$ , and divide the remainder by the double of the said sum; multiply the quotient by the product of the base and perpendicular altitude, and the last product will be the content.



That



That is,  $\frac{2CD + DE - \frac{1}{3}DE}{4CD + 2DE} \times DF \times \text{base} = \text{the content of } ADB.$ —Where  $2CD$  is the diameter to the double ordinate  $AB$  or transverse diameter of the the base,  $DE$  the continuation of the diameter to the middle of  $AB$ , and  $DF$  perpendicular to  $AB$ .\*

PRO-

## \* DEMONSTRATION.

Put  $2CD = a$ ,  $DE = b$ ,  $EB = c$ ,  $Dd = x$ ,  $db = y$ ,  $B =$  the elliptic base whose transverse axe is  $AB$ , and  $s =$  the sine of the angle  $E$  to the radius 1.

By the property of the hyperbola,  $ab + bb : ax + xx :: cc : yy = cc \times \frac{ax + xx}{ab + bb}$ ; and because parallel sections are similar, and similar figures as the squares of their like diameters, we obtain  $cc : yy :: B : B \times \frac{ax + xx}{ab + bb} = \text{the section at } ab.$

Hence the fluxion of the solid is  $Bsx \times \frac{ax + xx}{ab + bb}$ ; whose fluent  $Bsx \times \frac{\frac{1}{2}a + \frac{1}{3}x}{ab + bb} = \frac{Bsx}{2b} \times \frac{a + \frac{2}{3}x}{a + b}$  will be the content of  $aDb$ .

And when  $x$  or  $Dd$  becomes  $= b$  or  $DE$ , the rule will be  $\frac{1}{2}Bbs \times \frac{a + \frac{2}{3}b}{a + b} = \frac{1}{2}B \times DF \times \frac{a + \frac{2}{3}b}{a + b}$  for the solid  $ADB$ . Q.E.D.

## COROLLARY I.

When  $E$  is a right angle,  $DE$ ,  $DF$  and  $HG$  coincide, as do  $AB$  and  $BI$ ; and the rule will become  $\frac{1}{2} \text{base} \times HG \times \frac{2CH + \frac{2}{3}HG}{2CH + HG}$  for the right segment, the same as in problem 10.

## COROLLARY II.

If from  $\frac{1}{2} \text{base } BI \times HG \times \frac{2CH + \frac{2}{3}HG}{2CH + HG}$  the right segment  $IHB$ , be taken  $\frac{1}{2}B \times DF \times \frac{2CD + \frac{2}{3}DE}{2CD + DE}$  the oblique segment  $ADB$ , there will remain the part  $BAI$ .

## SCHOLIUM.

I might here proceed to find the slice parallel to the axe and the other parts dependent on it; but as there appears little occasion for it, and it would run into a complex infinite series, I have omitted it.



## PROBLEM XIII.

\* To find the content of an hyperbolic spindle.

## RULE I.

Proceed as in rule I for the elliptic spindle, and the content will be obtained.

That

## \* GENERAL INVESTIGATION.

Put  $x = DE$ , and  $y = EF$ ; then

By the property of the curve,  $c : t :: \sqrt{cc + xx} : \frac{t\sqrt{cc + xx}}{c} = FI$ ; hence  $y = DG - IF = C - \frac{t\sqrt{cc + xx}}{c}$ , and the fluxion of the solid will be  $p\dot{y}y\dot{x} = \dot{p}\dot{x} \times CC - \frac{2Ct\sqrt{cc + xx}}{c} + tt + \frac{tt\dot{x}x}{cc} = \dot{p}\dot{x} \times \frac{tt - CC + \frac{tt\dot{x}x}{cc} + 2C \times C - \frac{t\sqrt{cc + xx}}{c}}{1} = \dot{p}\dot{x} \times \frac{tt - CC + \frac{tt\dot{x}x}{cc} + 2Cy}{1} = 2pC\dot{y}\dot{x} - \frac{p\dot{t}t}{cc} \times \frac{1}{4}LL - \frac{1}{3}xx$ , whose fluent gives  $2pC \times \text{area } EFBD - \frac{p\dot{t}t}{cc} \times \frac{1}{4}LL - \frac{1}{3}xx = 2pC \times \text{area } EFBD - \frac{p\dot{t}t}{cc} \times \frac{3LL - 4xx}{12}$  for the content of the frustum  $BFML$ . Or  $2pC \times \text{area } EFBNO - \frac{p\dot{t}t}{cc} \times \frac{3LL - ll}{12} =$  the content of the middle zone  $FBNPLM$ , putting  $l = 2x = EO$  its length.

## COROLLARY I.

When  $l$  becomes  $= L$ , the last theorem becomes  $2pCS - \frac{p\dot{t}^2 L^3}{6cc}$  for the whole spindle  $ABCLA$ .

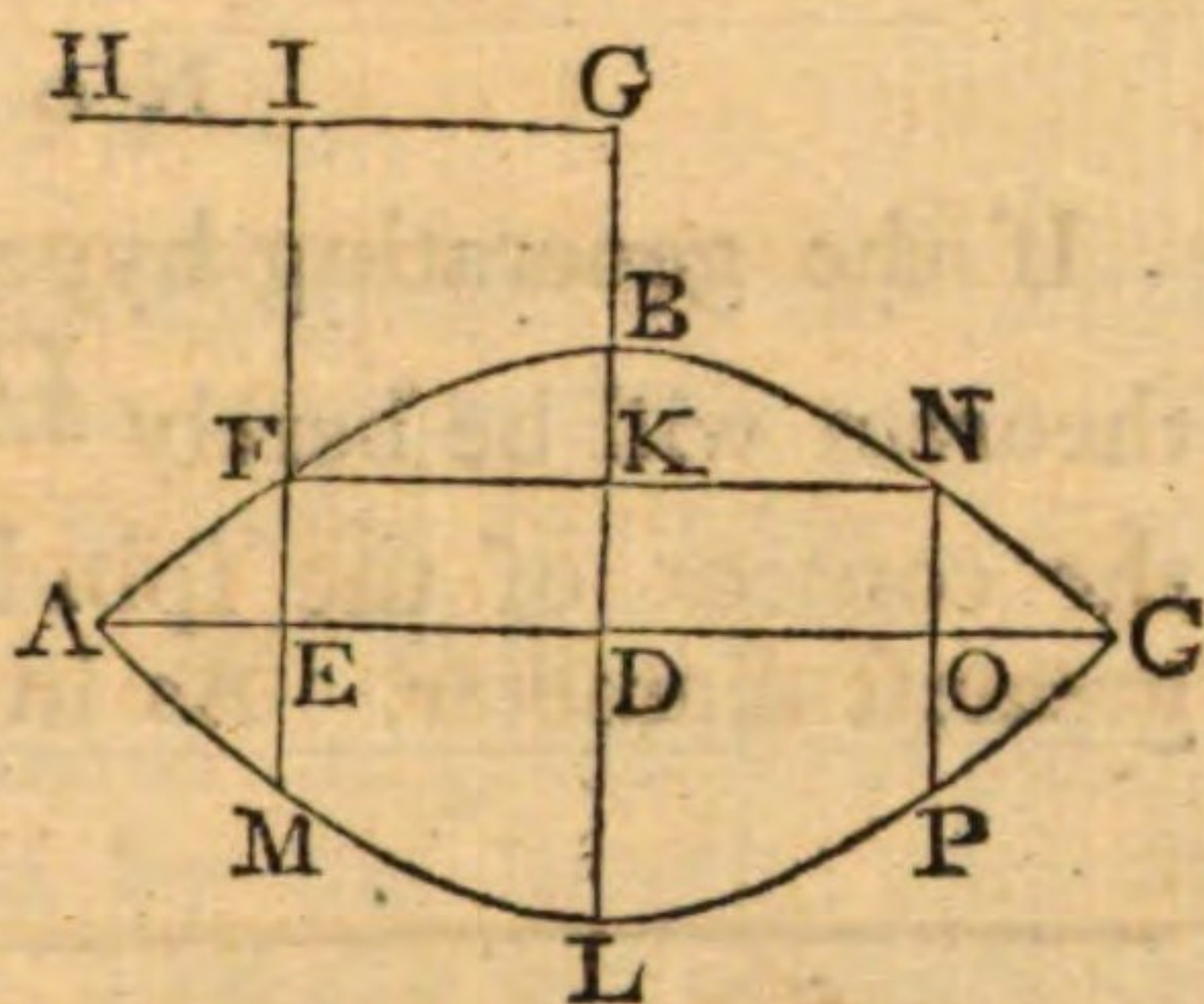
## COROLLARY II.

If from the semi-spindle be subtracted the frustum, there will remain  $2pC \times \text{area } AFE - \frac{p\dot{a}att}{cc} \times \frac{3L - 2a}{6}$  for the segment  $MAF$ , putting  $a$  for  $AE$  the altitude.

Co-



That is,  $2CS - \frac{t^2 L^3}{6cc}$   
 $\times p =$  the content of the  
 spindle  $ABCLA$ ; where  
 $C$  is  $= GD$  the central  
 distance or the distance  
 of the centers of the  
 spindle and generating  
 area  $ABC$ ,  $S =$  the ge-  
 neral area  $ABC$ ,  $t =$   
 $BG$  the semi-transverse,  $c = GH$  the semi-conjugate,  
 $L = AC$  the length of the spindle, and  $p = 3.14159$   
 &c. As in corollary I.



5 G

NOTE

## COROLLARY III.

When the generating hyperbola is equilateral, we shall have  $LL = 4CC - 4tt = 4CC - 2C - D|^2 = 4CD - DD$ , and hence  $C = \frac{LL + DD}{4D}$ , which being substituted in the foregoing rules, we obtain  $\frac{LL + DD}{2D} \times pS - \frac{1}{6}pL^3$  for the whole spindle,  $\frac{LL + DD}{2D} \times p \times \text{area } EFBNO - pl \times \frac{3LL - LL}{12}$  for the middle zone  $FBNPLM$ , and  $\frac{LL + DD}{2D} \times p \times \text{area } AFE - paa \times \frac{3L - 2a}{6}$  for the segment  $MAF$ .

## COROLLARY IV.

Putting  $d$  for  $MF$  the least diameter of the frustum or zone,  $n = \frac{1}{4}p = .785398$  &c. and the rest of the quantities as above; since  $LL$  is  $= ll \times \frac{DC - \frac{1}{4}DD}{DC - dC - \frac{1}{4}DD + \frac{1}{4}dd}$ ,  $\frac{tt}{cc} = \frac{DC - dC - \frac{1}{4}DD + \frac{1}{4}dd}{\frac{1}{4}ll}$ , and the area  $EFBNO = \frac{1}{2}dl + s$ , putting  $s$  for the area  $FBN$ ; if these values be substituted in the general forms, we shall obtain  $\frac{1}{4}nl \times 2DD + dd + 8C \times -D + d + \frac{3s}{l}$  for the value of the middle zone  $FBNPLM$ , and  $\frac{1}{4}nL \times 2DD + 8C \times -D + \frac{3S}{L}$  for that of the whole spindle.

Co-



## NOTE.

If the generating hyperbola be equilateral, the theorem will be barely  $\frac{LL + DD}{D} \times S - \frac{1}{3}L^3 \times \frac{1}{2}p$  for the content of the spindle, putting  $D = BL$  the greatest diameter. As in corollary 3.

## RULE

## COROLLARY V.

And if here, again, the generating hyperbola be supposed to be equilateral, since in that case  $ll$  is  $= \overline{2C-d}^2 - 4tt = \overline{2C-d}^2 - \overline{2C-D}^2 = -4Cd + dd + 4CD - DD$ , and hence  $C = \frac{ll + DD - dd}{4D - 4d} = \frac{LL + DD}{4D}$ ; we shall then have  $\frac{1}{3}nL \times$

$2DD + \frac{2LL + 2DD}{D} \times -D + \frac{3S}{L}$  or  $\frac{1}{8}pL \times 3S \times \frac{LL + DD}{LD} - LL$  for the whole spindle, and

$\frac{1}{3}nl \times 2DD + dd + \frac{2ll + 2DD - 2dd}{D - d} \times -D + d + \frac{3s}{l}$

or  $\frac{1}{8}pl \times \frac{1}{2}dd - ll + \frac{3s}{l} \times \frac{ll + DD - dd}{D - d}$  for the zone.

Hence it may be observed that the content of an equilateral-hyperbolic-spindular cask may be found from having only its length with the bung and head diameters given.

## COROLLARY VI.

The fluent of  $pyy\dot{x}$  or  $p\dot{x} \times CG + tt + \frac{ttxx}{cc} - \frac{2Ct\sqrt{cc + xx}}{c}$  is also  $= p\dot{x} \times CG + tt + \frac{ttxx}{3cc} - \frac{Ct\sqrt{cc + xx}}{c} - Ctc \times \text{hyp. log. of } \frac{x + \sqrt{cc + xx}}{c}$ , which is another expression of the content of the frustum  $FBLM$ .

## COROLLARY VII.

Putting  $m$  for the diameter in the middle of the frustum or semi-spindle; then, by the nature of the hyperbola, we shall have:



## RULE II.

Divide 3 times the generating area by the length of the spindle, from the quotient subtract the greatest diameter of the spindle; multiply the remainder by four times the central distance, and to the product add the square of the greatest diameter; then the sum multiplied by the length, and the product by 3.14159 &c.  $\frac{1}{8}$  of the last product will be the content.

That

have  $\overline{C - \frac{1}{2}d|^2} - \overline{C - \frac{1}{2}D|^2} : \overline{C - \frac{1}{2}m|^2} - \overline{C - \frac{1}{2}D|^2} :: 4 : 1$ , hence  $\overline{C - \frac{1}{2}d|^2} = 4 \times \overline{C - \frac{1}{2}m|^2} - 3 \times \overline{C - \frac{1}{2}D|^2}$ , and  $C = \frac{1}{4} \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d}$  in the frustum, or  $C = \frac{1}{4} \times \frac{4m^2 - 3D^2}{4m - 3D}$  in the whole spindle; and hence the rules in corollary 4 will become

$DD + \frac{4m^2 - 3D^2}{4m - 3D} \times \frac{3S}{L} - D \times \frac{1}{8}pL$  for the whole spindle; and

$DD + \frac{1}{2}dd + \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d} \times \frac{3s}{l} + d - D \times \frac{1}{8}pl$  for the zone.

## COROLLARY VIII.

And if the generating hyperbola be equilateral, then  $\frac{1}{4}ll = \overline{C - \frac{1}{2}d|^2} - \overline{C - \frac{1}{2}D|^2} =$  (by writing for  $C$  its value found in the last corollary)  $\frac{\overline{m - d|^2} - \frac{3}{4} \times \overline{D - d|^2} - \overline{D - m|^2} - \frac{1}{4} \times \overline{D - d|^2}}{4m - 3D - d}$ ;

hence  $2\sqrt{\frac{\overline{m - d|^2} - \frac{3}{4} \times \overline{D - d|^2} - \overline{D - m|^2} - \frac{1}{4} \times \overline{D - d|^2}}{4m - 3D - d}} = l$

the length of the zone; and, when  $d$  vanishes, we have

$2\sqrt{\frac{m^2 - \frac{3}{4}D^2 - \overline{D - m|^2} - \frac{1}{4}D^2}{4m - 3D}} = L$  the length of the whole spindle.

So that when the hyperbola is equilateral, the length of the spindle need not be given if the diameters  $D$  and  $m$  be given.



That is,  $DD + 4C \times \frac{3S}{L} - D \times \frac{1}{6}pL =$  the content; the letters as before.—By corollary 4.

N O T E.

If the generating hyperbola be supposed to be equilateral, 4 times the central distance will be  $= \frac{LL + DD}{D}$ , and then  $3S \times \frac{LL + DD}{LD} - LL \times \frac{1}{6}pL =$  the content.—By corollary 5.

R U L E III.

Divide the difference between 4 times the square of the middle diameter and 3 times the square of the greatest, by the difference between 4 times the said middle diameter and 3 times the greatest, and  $\frac{1}{4}$  of the quotient will be the central distance.

Then proceed as in the last rule.

That is,  $DD + \frac{4mm - 3DD}{4m - 3D} \times \frac{3S}{L} - D \times \frac{1}{6}pL =$  the content.—By corollary 7.

N O T E.

When the generating hyperbola is equilateral, the content may be found without having the length  $L$  given, for then  $L$  is =

$$\frac{2\sqrt{mm - \frac{1}{4}DD}^2 - D - m}^2 - \frac{1}{4}DD}^2}{4m - 3D}, \text{ by corollary 8.}$$

Another rule might be drawn from corollary 6.

PRO-



## PROBLEM XIV.

*To find the content of a zone or double frustum of an hyperbolic spindle.*

## RULE I.

Use here rule I for the zone of an elliptic spindle, and the content will be obtained.

That is,  $2pC \times \text{area } EFBNO - \frac{p^2 l^2}{cc} \times \frac{3LL - ll}{12} =$  the content of the zone  $FBNP LM$ ; where  $p = 3.14159 \text{ \&c.}$   $C = GD$  the central distance,  $t = BG$  the semi-transverse axe,  $c = GH$  the semi-conjugate,  $l = EO$  the length of the zone, and  $L = AC$  the length of the whole spindle. — By the general investigation of the last problem.

## NOTE.

When the generating hyperbola is equilateral, the rule will be barely  $p \times \frac{LL + DD}{2D} \times \text{area } EFBNO - pl \times \frac{3LL - ll}{12}$ . — By corollary 3 to the last problem. Putting  $D = BL$  the greatest diameter.

## RULE II.

Use the second rule for the zone of an elliptic spindle, only here *add* the first general product instead of subtracting it, and the content will be got.

That is,  $2DD + dd + 8C \times \frac{3^s}{l} + d - D \times \frac{1}{12} pl =$   
 $FBNP LM.$  Where  $d = FM$  the least diameter,  
5 H  $s =$



$s$  = the area  $FBN$ , and the other letters as before.—By corollary 4 to the last problem.

## NOTE.

If the generating hyperbola be equilateral, 4 times the central distance will be  $= \frac{ll + DD + dd}{D - d}$ , and then the content of the zone will be

$\frac{3}{2}dd - ll + \frac{3s}{l} \times \frac{ll + DD + dd}{D - d} \times \frac{1}{6}pl$ .—By corollary 5 to the last problem.

## RULE III.

From 4 times the square of the diameter equidistant from the greatest and least subtract the sum of the square of the least and 3 times the square of the greatest diameter, and from 4 times the said middle diameter take the sum of the least and 3 times the greatest; then divide the former difference by the latter, and  $\frac{1}{4}$  of the quotient will be the central distance.

That is,  $\frac{1}{4} \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d} = C$ ,  $m$  being the middle diameter.—By corollary 7 to the last problem.

Then proceed as in the last rule.

## NOTE.

When the generating hyperbola is equilateral,  $l$  is  $= \frac{2\sqrt{m-d|^2 - \frac{3}{4} \times D-d|^2 - D-m|^2 - \frac{1}{4} \times D-d|^2}}{4m - 3D - d}$  by

corollary 8 to the last problem; so that the content may then be found without having the length given.

Another rule might be drawn from corollary 6 to the last problem.

PRO-



PROBLEM XV.

*To find the content of the segment of an hyperbolic spindle.*

RULE.

Use here the rule for the segment of an elliptic spindle, only instead of what is there called the less axe take here the transverse, and instead of the greater the conjugate axe; and the content will be obtained.—By corollary 2 to the 13th problem.

That is,  $2pC \times \text{area } AFE - \frac{paatt}{cc} \times \frac{3L - 2a}{6} = \text{the segment } MAF$ . Where  $a$  is  $= AE$  its altitude, and the other letters as in the last problems.

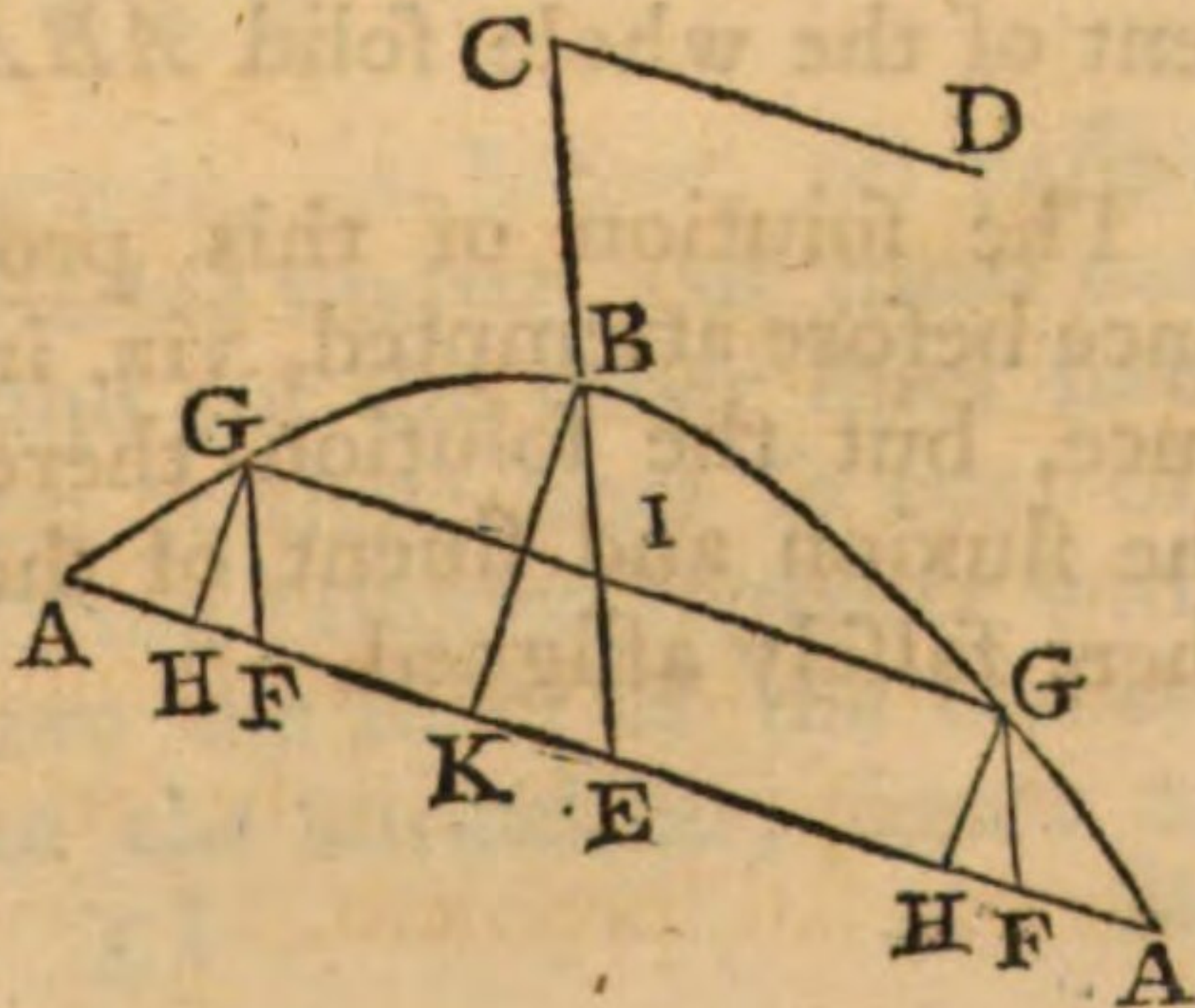
Other rules might be found by subtracting the frustum from the semi-spindle according to the corresponding rules of the last two problems.

PROBLEM XVI.

*To find the content of an universal hyperbolic spindle; that is, of a solid generated from the revolution of an hyperbola about an ordinate to any diameter.*

INVESTIGATION.

Put  $a = BC$  the semi-diameter to which belongs the double ordinate  $AA$  about which the figure  $ABA$  revolves,  $c = CD$  its semi-conjugate,  $b = AE$ ,  $C = EC$ ,  $z = EF$ ,  $y = FG$  parallel to  $CE$ ,  $m = \text{fine of}$





of the angle  $E$  or  $F$ . and  $n$  = its cosine to the radius 1, also  $p = 3.14159$  &c. and let  $GH$  be perpendicular to  $AE$ .

Then, by the nature of the hyperbola,  $c : a :: \sqrt{cc + zz} : \frac{a\sqrt{cc + zz}}{c} = CI$ ; hence  $y = C - CI = C - \frac{a\sqrt{cc + zz}}{c}$ ; but  $GH$  is  $= my$ , and  $HF = ny$ ; hence  $EH$  is  $= z \pm ny$ , and the fluxion of the solid is  $p \times GH^2 \times HK = p \times GH^2 \times HE = pmm \times y \times z \pm ny = pmm \times : \pm ny^2 y + C^2 z - \frac{2Caz\sqrt{cc + zz}}{c} + a^2 z + \frac{a^2 z^3}{cc}$ ; whose correct fluent is  $pmm \times : \mp \frac{1}{3}ny^3 + C^2 z + a^2 z + \frac{a^2 z^3}{3c^2} - \frac{Caz\sqrt{cc + zz}}{c} - Cac \times \text{hyp. log. of } \frac{z + \sqrt{cc + zz}}{c} = \text{the solid generated by } KBGH$ .

And when  $z$  becomes  $= b$ , the above fluent becomes  $pmm \times : \mp C^2 b + a^2 b + \frac{a^2 b^3}{3c^2} - \frac{Cab\sqrt{cc + bb}}{c} - Cac \times \text{hyp. log. of } \frac{b + \sqrt{cc + bb}}{c}$  for half the content of the whole solid  $ABA$ .

The solution of this problem was never but once before attempted, viz. in a periodical performance, but the solution there given is wrong, for the fluxion and fluent of the general frustum was there falsely assigned.



## SECTION VI.

*Of the regular bodies.*

**A** Regular solid or body, is a solid contained under some number of like, equal, and regular plane figures.

The plane figures, under which the solid is contained, are the faces of the solid. And the sides of the plane figures are the edges or linear sides of the solid.

There are only five sorts of regular solids, viz. the tetraedron, or regular triangular pyramid, having four triangular faces; the hexaedron, or cube, having six square faces; the octaedron, having eight triangular faces; the dodecaedron, having twelve pentagonal faces; and the icosaedron, having twenty triangular faces.

## PROBLEM I.

*To describe or form the regular solids.*

## RULE.

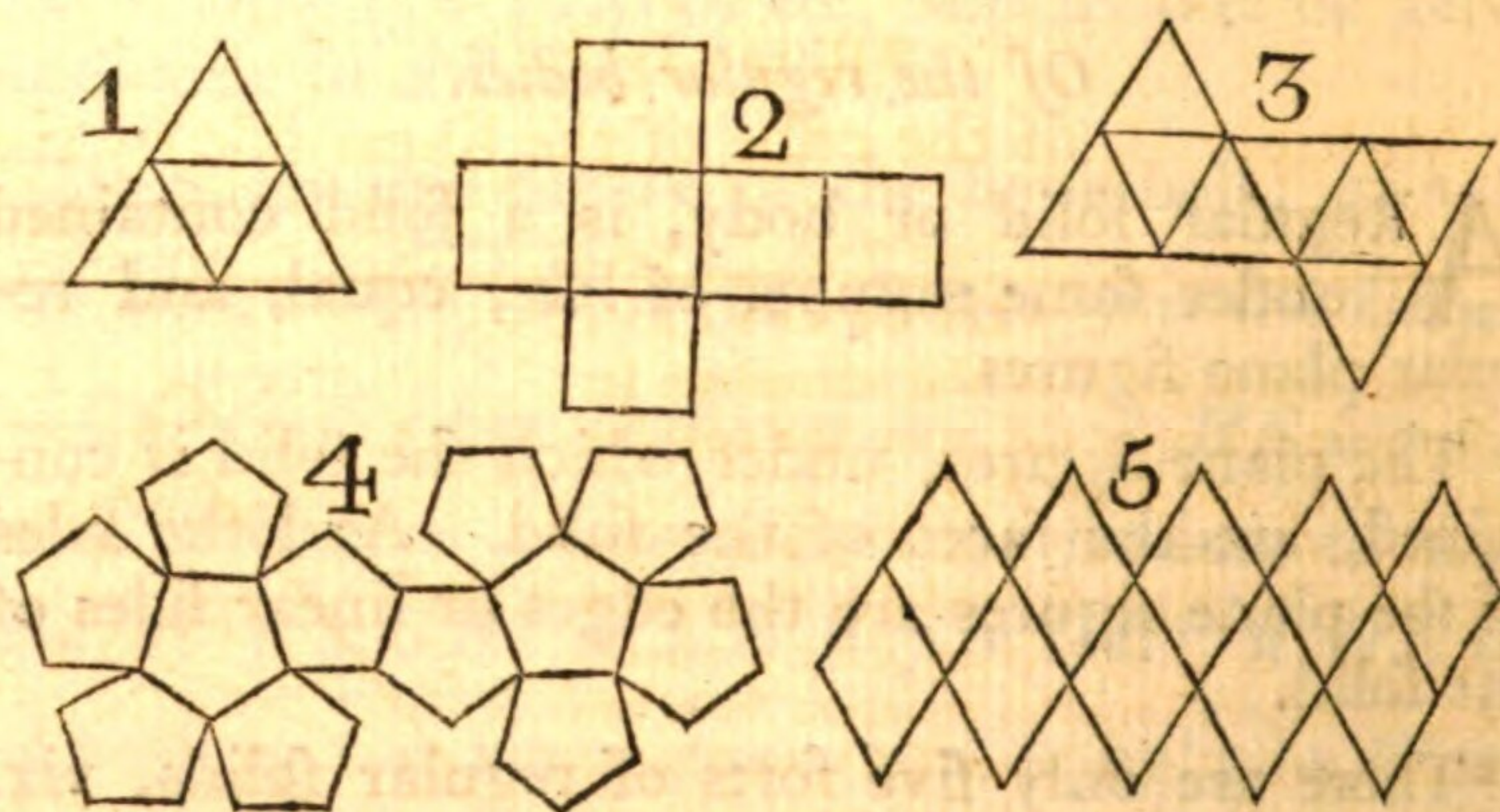
Having described figures, as below, on paste-board or some other pliable matter, cut them out by the extreme sides, and cut the other lines half through, then fold them at these lines so cut, till the sides meet, which being fastened together with glue, &c. you will have the form of the bodies. Viz. figure 1 will form the tetraedron; figure 2,

5 I

the



the hexaedron; figure 3, the octaedron; figure 4, the dodecaedron; and figure 5, the icosaedron.



### PROBLEM II.

*To find the surface of a tetraedron.*

#### R U L E.

Multiply the square of the linear edge by the root of 3, and the product will be the whole surface, or the sum of the four faces.

That is,  $A^2 \sqrt{3} =$  the whole surface;  $A$  being the linear edge, or a side of one of the faces.\*

#### E X A M P L E.

If each side of a face of a tetraedron be 1, required the whole surface.

Here  $A = 1$ ; and therefore  $A^2 \sqrt{3} = \sqrt{3} = 1.7320508 =$  the surface required.

PRO-

#### \* DEMONSTRATION.

For, by rule 2 prob. 4 sect. 1 part 2, one of the faces will be  $= \frac{1}{4} \sqrt{3} \times A^2$ ; which being multiplied by 4, gives  $A^2 \sqrt{3}$  for all the four faces. Q.E.D.



## PROBLEM III.

To find the solidity of a tetraedron.

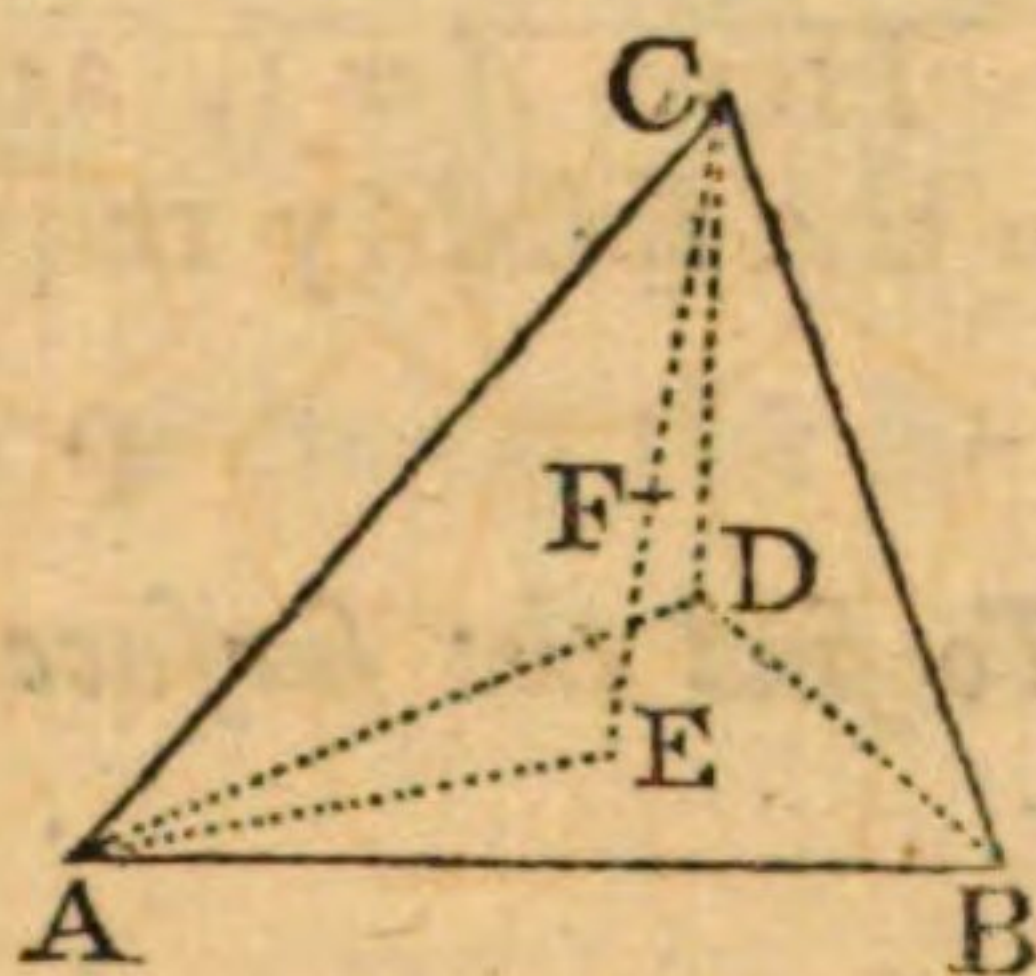
## \*RULE.

Multiply  $\frac{1}{12}$  of the cube of the linear side by the  
root

## \* DEMONSTRATION.

From one angle  $C$  of the tetraedron let fall a perpendicular  $CE$  upon the opposite side  $ADB$ , and draw  $EA$ .

Then  $AC^2 = CE^2 + EA^2$ ; but, from the tables in pages 81 and 84, it appears that  $\frac{1}{3}AC^2 = \frac{1}{3}AB^2 = AE^2$ ; wherefore, by subtracting this equation from the former,  $\frac{2}{3}AC^2$  will be  $= CE^2$ , and hence  $CE = AC\sqrt{\frac{2}{3}}$ ; but  $ABD$  is  $= \frac{1}{4}AB^2\sqrt{3} = \frac{1}{4}AC^2\sqrt{3}$ : Then  $\frac{1}{3}CE \times ABD = \frac{1}{3}AC\sqrt{\frac{2}{3}} \times \frac{1}{4}AC^2\sqrt{3} = \frac{1}{12}AC^3\sqrt{2} =$  the solidity. *Q.E.D.*



## COROLLARY I.

A cube is to a tetraedron of the same linear side, as 1 is to  $\frac{1}{12}\sqrt{2}$ , or as 12 is to  $\sqrt{2}$ .

## COROLLARY II.

Put  $r$  for the radius  $EF$  of the inscribed sphere,  $A$  for the linear side  $AC$ ,  $B$  for the whole surface, and  $C$  for the solidity; and we shall have  $r = \frac{3C}{B} = \left( \frac{\frac{1}{12}A^3\sqrt{2}}{A^2\sqrt{3}} = \frac{1}{4}A\sqrt{\frac{2}{3}} = \right) \frac{1}{12}A\sqrt{6}$ .

## COROLLARY III.

Hence also  $r$  or  $FE$  is  $= \frac{1}{4}EC$ ; for, by the demonstration,  $CE$  is  $= A\sqrt{\frac{2}{3}}$ .

## COROLLARY IV.

And hence the radius of the circumscribed sphere is equal to 3 times that of the inscribed; that is,  $R = FC = CE - EF = 4FE - EF = 3FE = 3r = \frac{1}{4}A\sqrt{6}$ .

## SCHOLIUM.

From this problem and its corollaries, together with the former problem, are deduced these equations.

1.  $A = \sqrt{\frac{1}{3}B\sqrt{3}} = \sqrt[3]{6C\sqrt{2}} = 2r\sqrt{6} = \frac{2}{3}R\sqrt{6}$ .
2.  $B = A^2\sqrt{3} = 6\sqrt{C^2\sqrt{3}} = 24r^2\sqrt{3} = \frac{8}{3}R^2\sqrt{3}$ .
3.  $C = \frac{1}{12}A^3\sqrt{2} = \frac{1}{36}B\sqrt{2B\sqrt{3}} = 8r^3\sqrt{3} = \frac{8}{27}R^3\sqrt{3}$ .
4.  $r = \frac{1}{12}A\sqrt{6} = \frac{1}{12}\sqrt{2B\sqrt{3}} = \frac{1}{2}\sqrt[3]{\frac{1}{3}C\sqrt{3}} = \frac{1}{3}R$ .
5.  $R = \frac{1}{4}A\sqrt{6} = \frac{1}{4}\sqrt{2B\sqrt{3}} = \frac{3}{2}\sqrt[3]{\frac{1}{3}C\sqrt{3}} = 3r$ .



root of 2, and the product will be the solidity of the tetraedron required.

That is,  $\frac{1}{12} A^3 \sqrt{2} = \text{the solidity.}$

#### EXAMPLE.

If the linear side of a tetraedron be 1; required the solidity.

Here  $A = 1$ , and  $\frac{1}{12} A^3 \sqrt{2} = \frac{1}{12} \sqrt{2} = .11785113 = \text{the solidity required.}$

#### PROBLEM IV.

*To find the surface or solidity of a hexaedron or cube.*

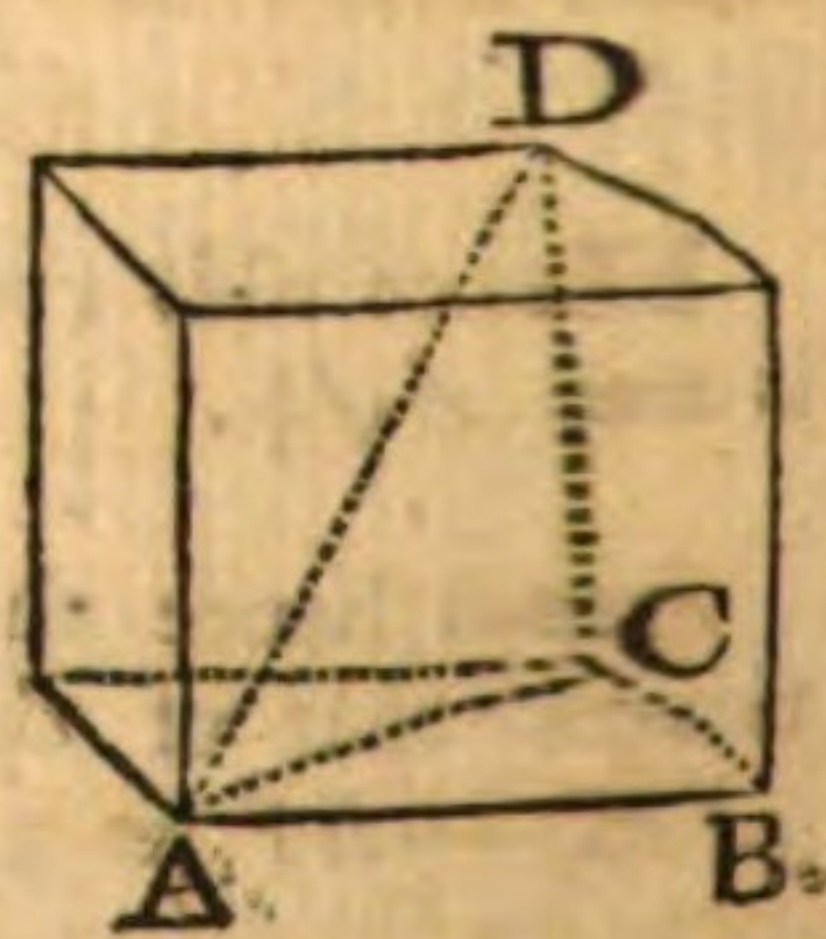
#### RULE.

It is evident that 6 times the square of the linear side will be equal to the whole surface, and the cube of the linear side equal to the solidity of the hexaedron or cube.\*

PRO-

#### \* SCHOLIUM.

Since it is evident that the diameter of the inscribed sphere is equal to the linear side of the hexaedron, and that the diameter of the circumscribed sphere is  $AD = \sqrt{DC^2 + CA^2} = \sqrt{DC^2 + CB^2 + BA^2} = \sqrt{3} AB = AB\sqrt{3}$ ; if  $A$ ,  $B$ , and  $C$  be put to denote the linear side, the surface, and the solidity of a hexaedron, and  $r$ ,  $R$  for the radius of the inscribed and circumscribed sphere; then in the hexaedron or cube we shall have these equations.



1.  $A = \sqrt{\frac{1}{6} B} = \sqrt[3]{C} = 2r = \frac{2}{3} R \sqrt{3}.$
2.  $B = 6 A^2 = 6 \sqrt{C^2} = 24 r^2 = 8 R^2.$
3.  $C = A^3 = \frac{1}{6} B \sqrt{\frac{1}{6} B} = 8 r^3 = \frac{8}{9} R^3 \sqrt{3}.$
4.  $r = \frac{1}{2} A = \frac{1}{2} \sqrt{\frac{1}{6} B} = \frac{1}{2} \sqrt[3]{C} = \frac{1}{3} R \sqrt{3}.$
5.  $R = \frac{1}{2} A \sqrt{3} = \frac{1}{2} \sqrt{\frac{1}{2} B} = \frac{1}{2} \sqrt{3} \times \sqrt[3]{C} = r \sqrt{3}.$



## PROBLEM V.

*To find the surface of an octaedron.*

## RULE.

Multiply 2 times the square of the linear side by the root of 3, and the product will be the surface required.\*

That is,  $2A^2\sqrt{3}$  = the surface;  $A$  being the linear side.

## EXAMPLE.

The linear side of an octaedron being 1, it is required to find the solidity.

Here  $A = 1$ , and  $2A^2\sqrt{3} = 2\sqrt{3} = 3.4641016$  = the surface required.

## PROBLEM VI.

*To find the solidity of an octaedron.*

## RULE.

Multiply  $\frac{1}{3}$  of the cube of the linear side by the root of 2, and the product will be the content of the octaedron.

5 K

That

## \* DEMONSTRATION.

By rule 2 prob. 4 sect. 1 part 2, one face is  $= \frac{1}{4}A^2\sqrt{3}$ ; consequently 8 faces, or the whole surface, will be  $= 2A^2\sqrt{3}$ . Q.E.D.



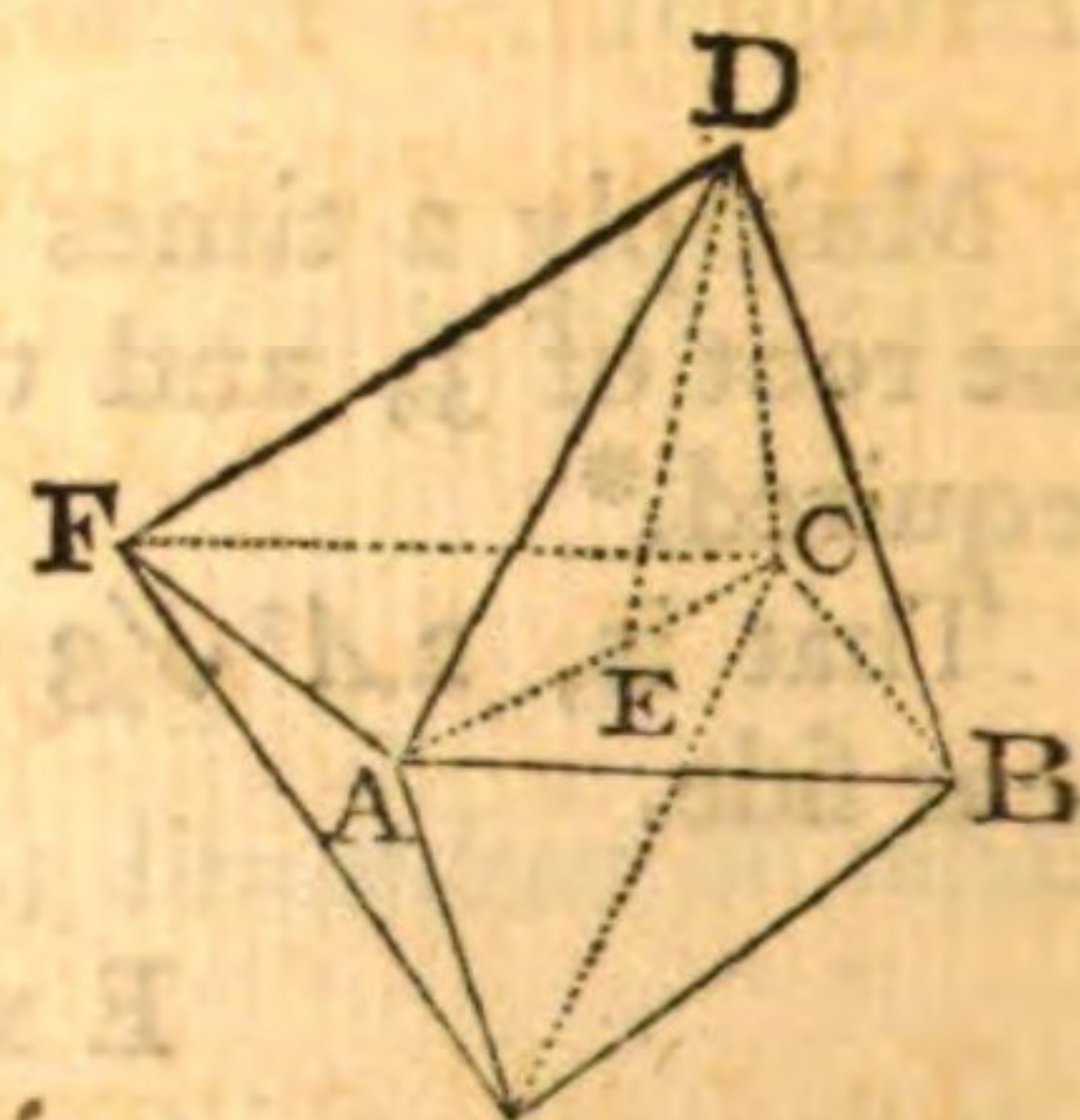
That is,  $\frac{1}{3}A^3\sqrt{2} = C = \text{the content; } A \text{ being the linear side.}$

Ex-

## \* DEMONSTRATION.

Let  $E$  be the middle of the diagonal  $AC$  of the solid, or the center of the solid; and draw  $DE$  which will be equal to  $EA$ .

Then, since the solid is evidently composed of two equal square pyramids, each base of which  $ABCF$  is equal to the square of the linear side of the solid, and the altitude of each equal to  $DE$  or  $EA$  half the diagonal of that square; we shall have  $ABCF \times \frac{2}{3}AE = AB^2 \times \frac{2}{3}AC = \frac{2}{3}AB^2\sqrt{AB^2 + BC^2} = \frac{2}{3}AB^2\sqrt{2AB^2} = \frac{2}{3}AB^3\sqrt{2}$  for the solidity.  $\text{Q.E.D.}$



## COROLLARY I.

The radius  $AE$  of the circumscribed sphere is  $= \frac{1}{2}AC = \frac{1}{2}\sqrt{2}AB^2 = \frac{1}{2}AB\sqrt{2} = AB\sqrt{\frac{1}{2}}$ .

## COROLLARY II.

The radius of the inscribed sphere is equal to the quotient arising from the division of 3 times the solidity by the whole surface  $=$  (from this problem and the last)  $AB^3\sqrt{2} \div 2AB^2\sqrt{3} = \frac{1}{2}AB\sqrt{\frac{2}{3}} = \frac{1}{6}AB\sqrt{6} = AB\sqrt{\frac{1}{6}}$ .

## SCHOLIUM.

From the two last problems and the corollaries are deduced the following equations; in which  $A, B, C, r, R$ , denote the linear side of an octaedron,  $B$  the whole surface,  $C$  the solidity,  $r$  the radius of the inscribed sphere, and  $R$  the radius of the circumscribed sphere.

$$1. A = \sqrt{\frac{B\sqrt{3}}{6}} = \sqrt[3]{\frac{3C\sqrt{2}}{2}} = r\sqrt{6} = R\sqrt{2}.$$

$$2. B = 2A^2\sqrt{3} = 6\sqrt{\frac{C^2\sqrt{3}}{2}} = 12r^2\sqrt{3} = 4R^2\sqrt{3}.$$

$$3. C = \frac{1}{3}A^3\sqrt{2} = \frac{B\sqrt{B\sqrt{3}}}{18} = 4r^3\sqrt{3} = \frac{4}{3}R^3.$$

$$4. r = \frac{1}{6}A\sqrt{6} = \frac{1}{6}\sqrt{B\sqrt{3}} = \sqrt[3]{\frac{C\sqrt{3}}{12}} = \frac{1}{3}R\sqrt{3}.$$

$$5. R = \frac{1}{2}A\sqrt{2} = \sqrt{\frac{B\sqrt{3}}{12}} = \sqrt[3]{\frac{3}{4}C} = r\sqrt{3}.$$



## EXAMPLE.

If the linear side of an octaedron be 1, what is its solidity.

Here  $A = 1$ , and therefore  $\frac{1}{3}A^3\sqrt{2} = \frac{1}{3}\sqrt{2} = .47140452079 =$  the content required.

## PROBLEM VII.

*To find the surface of a dodecaedron.*

## RULE.

To 5 add 2 times the root of 5, and divide the sum by 5; multiply the root of the quotient by 15 times the square of the linear side, and the product will be the surface of the dodecaedron.

That is,  $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} =$  the surface;  $A$  being the linear side.\*

## EXAMPLE.

If the linear side be 1, required the surface.

Here  $A = 1$ , and therefore  $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} = 15\sqrt{\frac{5+2\sqrt{5}}{5}} = 20.6457788075 =$  the surface required.

PRO-

## \* DEMONSTRATION.

By rule 2 prob. 4 sect. 1 part 2, the area of one face is  $\frac{5}{4}A^2\sqrt{1+\frac{2}{5}\sqrt{5}}$ , and therefore 12 faces or the whole surface will be  $12 \times \frac{5}{4}A^2\sqrt{1+\frac{2}{5}\sqrt{5}} = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}}$ . Q.E.D.



## PROBLEM VIII.

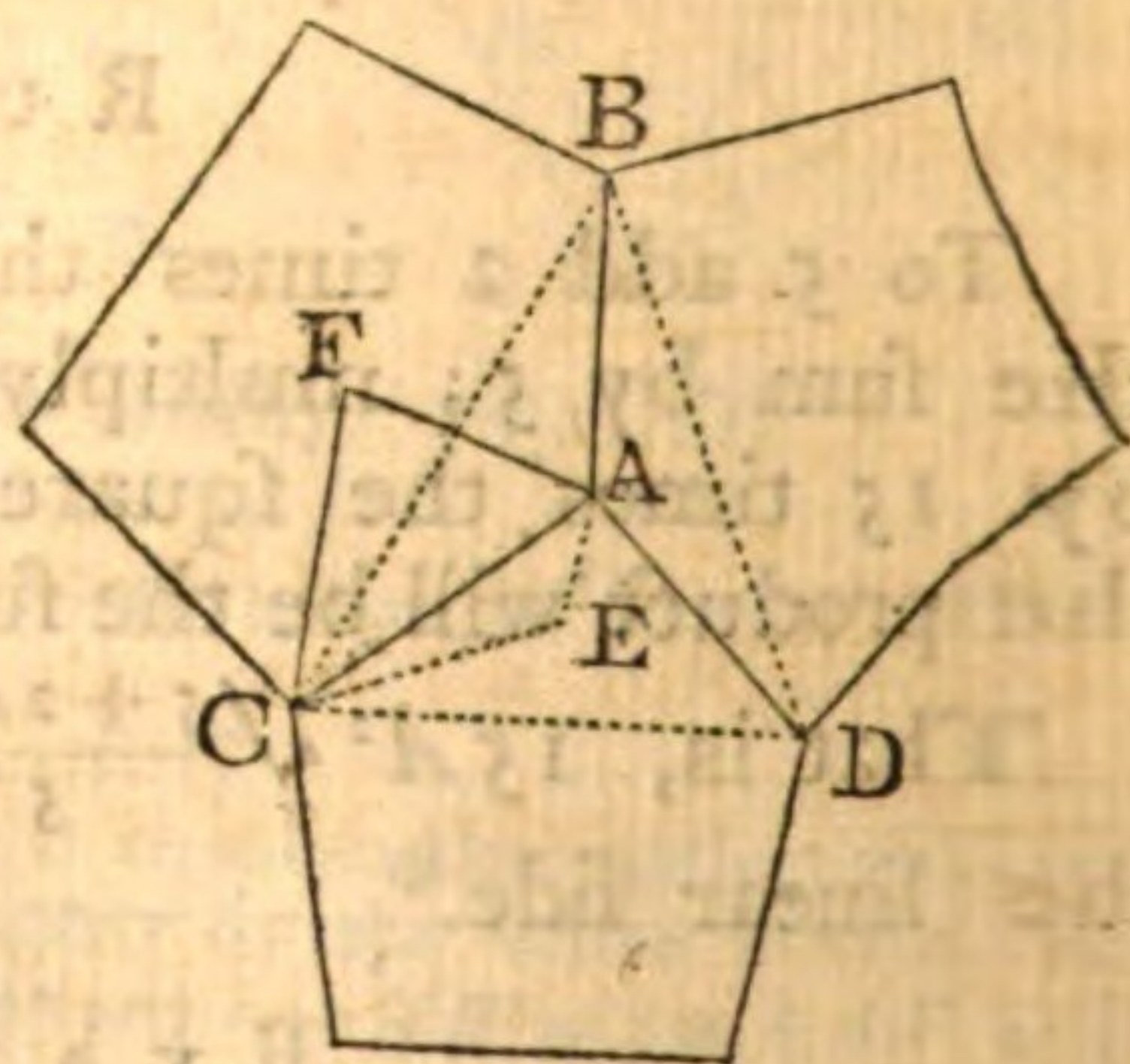
*To find the solidity of a dodecaedron.*

## R U L E.

To 21 times the root of 5 add 47, and divide the sum by 40; multiply the root of the quotient by

## \* DEMONSTRATION.

Let  $A$  be a solid angle of the dodecaedron, and let the extremities of the sides  $AB$ ,  $AC$ ,  $AD$ , of the faces which form the angle be connected by the right lines  $BC$ ,  $CD$ ,  $DB$ , forming the equilateral triangle  $BCD$  within the solid; upon the center  $E$  of which triangle let fall the perpendicular  $AE$ , and to the center  $F$  of one of the faces draw the lines  $AF$ ,  $CF$ .



The angle  $CAD$  contains 108 degrees, whose sine is  $\frac{1}{4}\sqrt{10+2\sqrt{5}}$ , to the radius 1; and the angle  $ADC$  contains 36 degrees, whose sine is  $\frac{1}{4}\sqrt{10-2\sqrt{5}}$ ; wherefore as  $\sqrt{10-2\sqrt{5}} : \sqrt{10+2\sqrt{5}} ::$

$$AC : CD = AC \sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}} = AC \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = AC \sqrt{\frac{5+\sqrt{5}}{25-5}}$$

$$= \frac{5+\sqrt{5}}{2\sqrt{5}} AC = \frac{1+\sqrt{5}}{2} AC : \text{And much after the same manner,}$$

we find  $EC = \frac{1}{3}CD\sqrt{3} = CD\sqrt{\frac{1}{3}} = \frac{1+\sqrt{5}}{2\sqrt{3}} AC$ . Hence  $EA =$

$$\sqrt{AC^2 - CE^2} = \sqrt{AC^2 - \left(\frac{1+\sqrt{5}}{2\sqrt{3}}\right)^2 AC^2} = AC \sqrt{1 - \frac{3+\sqrt{5}}{6}}$$

$$= AC \sqrt{\frac{3-\sqrt{5}}{6}}. \text{ Then because the chord of an arc is a mean}$$

proportional between its versed sine and the diameter, and  $AE$  is the versed sine of the arc whose chord is  $AC$  and diameter equal to that of



by 5 times the cube of the linear side, and the product will be the solidity.

That is,  $5A^3 \sqrt{\frac{47+21\sqrt{5}}{40}}$  = the solidity;  $A$  being the linear side.

5 L

Ex-

of the circumscribed sphere; we shall have  $\frac{CA^2}{2AE} = \frac{CA^2}{2AC\sqrt{\frac{3-\sqrt{5}}{6}}}$

$$= \frac{1}{2}AC\sqrt{\frac{6}{3-\sqrt{5}}} = \frac{1}{2}AC\sqrt{6 \times \frac{3+\sqrt{5}}{9-5}} = \frac{1}{2}AC\sqrt{3 \times \frac{6+2\sqrt{5}}{4}}$$

$$= AC \times \frac{1+\sqrt{5}}{4}\sqrt{3} = \frac{\sqrt{3+\sqrt{15}}}{4}AC = R \text{ the radius of the circumscribed sphere.}$$

Again, the angle  $AFG$  contains 72 degrees, whose sine is  $\frac{1}{4}\sqrt{10+2\sqrt{5}}$ ; and the angle  $ACF$  54 degrees, whose sine is  $\frac{1+\sqrt{5}}{4}$ ; wherefore as  $\sqrt{10+2\sqrt{5}} : 1+\sqrt{5} :: CA : AF =$   
 $\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}AC = AC\sqrt{\frac{5+\sqrt{5}}{10}}.$

But, since the radius of the circumscribed sphere is the hypotenuse of a right-angled triangle whose two legs are  $AF$  and the radius of the inscribed sphere, we shall have  $\sqrt{R^2 - AF^2} =$

$$\sqrt{\left[\frac{\sqrt{3+\sqrt{15}}}{4}AC\right]^2 - \frac{5+\sqrt{5}}{10}AC^2} = AC\sqrt{\frac{18+6\sqrt{5}}{16} - \frac{5+\sqrt{5}}{10}}$$

$$= AC\sqrt{\frac{25+11\sqrt{5}}{40}} = r \text{ the radius of the inscribed sphere.}$$

Then, because the solidity of any regular solid is equal to the surface drawn into  $\frac{1}{3}$  of the radius of the inscribed sphere, and the surface  $B$  equal to  $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}}$  by the last problem, we shall

$$\text{have } B \times \frac{1}{3}r = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} \times \frac{1}{3}A\sqrt{\frac{25+11\sqrt{5}}{40}} =$$

$$5A^3\sqrt{\frac{47+21\sqrt{5}}{40}} = C \text{ the solidity. } \textit{Q.E.D.}$$

Co-



## EXAMPLE.

If the linear side be 1, required the solidity of the dodecaedron.

Here  $A = 1$ , and therefore  $5A^3 \sqrt{\frac{47 + 21\sqrt{5}}{40}} = 5\sqrt{\frac{47 + 21\sqrt{5}}{40}} = 7.6631189605 =$  the content required.

PRO-

## COROLLARY.

From the demonstration it appears that the radius of the circumscribed sphere is  $R = \frac{\sqrt{3 + \sqrt{15}}}{4} A$ , and that of the inscribed sphere is  $r = A \sqrt{\frac{25 + 11\sqrt{5}}{40}}$ .

## SCHOLIUM.

Putting, as before,  $A, B, C$  for the linear side, surface, and solidity, and  $R, r$  for the radius of the circumscribed and inscribed sphere respectively; we shall have in the dodecaedron

$$\begin{aligned}
 1. \quad A &= \sqrt{\frac{B\sqrt{5-2\sqrt{5}}}{15}} = \sqrt[3]{\frac{C\sqrt{470-210\sqrt{5}}}{5}} = \\
 r\sqrt{50-22\sqrt{5}} &= \frac{\sqrt{15-\sqrt{3}}}{3} R. \\
 2. \quad B &= 15A^2 \sqrt{\frac{5+2\sqrt{5}}{5}} = 3\sqrt[3]{10C^2\sqrt{130-58\sqrt{5}}} = \\
 30r^2\sqrt{130-58\sqrt{5}} &= 10R^2 \sqrt{\frac{10-2\sqrt{5}}{5}}. \\
 3. \quad C &= 5A^3 \sqrt{\frac{47+21\sqrt{5}}{40}} = \frac{B}{30} \sqrt{\frac{B\sqrt{650+290\sqrt{5}}}{6}} = \\
 10r^3\sqrt{130-58\sqrt{5}} &= \frac{20R^3}{3} \sqrt{\frac{3+\sqrt{5}}{30}}. \\
 4. \quad r &= \frac{A\sqrt{250+110\sqrt{5}}}{20} = \frac{1}{20} \sqrt{\frac{2B\sqrt{650+290\sqrt{5}}}{3}} = \\
 \sqrt[3]{\frac{C\sqrt{650+290\sqrt{5}}}{200}} &= R \sqrt{\frac{5+2\sqrt{5}}{15}}. \\
 5. \quad R &= \frac{\sqrt{15+\sqrt{3}}}{4} A = \sqrt{\frac{B\sqrt{10+2\sqrt{5}}}{40}} = \sqrt[3]{\frac{3C\sqrt{90-30\sqrt{5}}}{40}} \\
 &= r\sqrt{15-6\sqrt{5}}.
 \end{aligned}$$



## PROBLEM IX.

*To find the surface of an icosaedron.*

## RULE.

\* Multitiply 5 times the square of the linear fide by the root of 3, and the product will be the surface.

That is,  $5A^2\sqrt{3}$  = the surface;  $A$  being the linear fide.

## EXAMPLE.

The linear fide of an icosaedron being 1; required the surface.

Here  $A = 1$ , and therefore  $5A^2\sqrt{3} = 5\sqrt{3} = 8.66025403$ .

## PROBLEM X.

*To find the solidity of an icosaedron.*

## RULE.

To 7 add 3 times the root of 5, divide the sum by 2, multiply the root of the quotient by five-sixths of

## \* DEMONSTRATION.

For one face is  $= \frac{1}{4}A^2\sqrt{3}$ , and therefore twenty faces or the whole surface will be  $20 \times \frac{1}{4}A^2\sqrt{3} = 5A^2\sqrt{3}$ . Q.E.D.



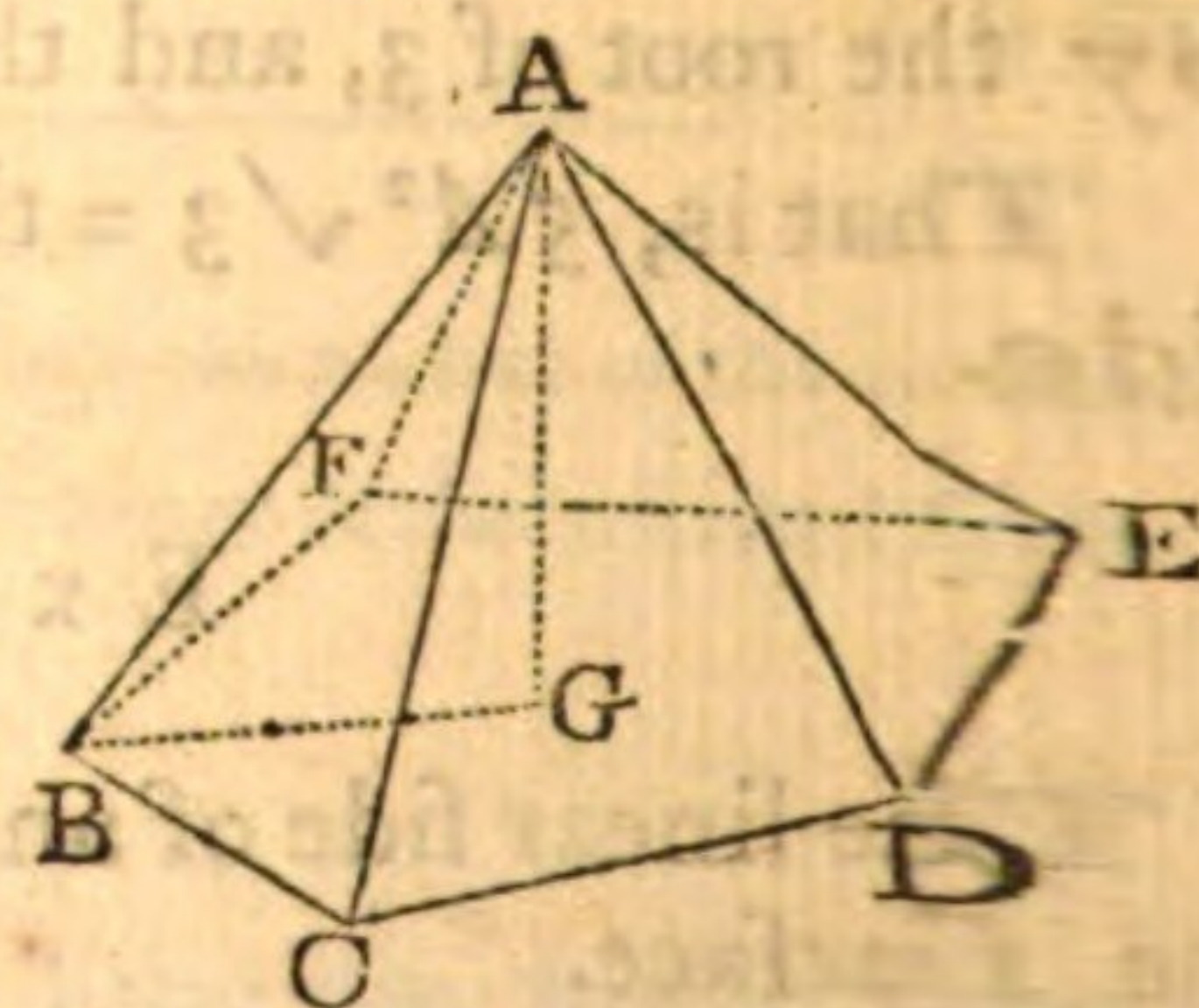
of the cube of the linear side, and the product will be the content of the icosaedron.\*

That is,  $\frac{5}{6} A^3 \sqrt{\frac{7+3\sqrt{5}}{2}}$  = the content;  $A$  being the linear side.

Ex-

## \* DEMONSTRATION.

Let  $A$  be a solid angle, of the icosaedron, formed by five faces or triangles whose bases form the pentagon  $BCDEF$ , upon whose center  $G$  let fall the perpendicular  $AG$ , and draw  $GB$ .



In the demonstration of problem 8 it was found that  $GB$  is  $= BA \sqrt{\frac{5+\sqrt{5}}{10}}$ , and that the radius  $\rho$  of the circle circumscribing one face  $ABC$  is  $= AB \sqrt{\frac{1}{3}}$ . But the radius of the circumscribed sphere is  $R = \frac{BA^2}{2AG} = \frac{BA^2}{2\sqrt{AB^2 - BG^2}} =$

$$\begin{aligned} & \frac{BA^2}{2\sqrt{AB^2 - \frac{5+\sqrt{5}}{10} AB^2}} = \frac{BA}{2\sqrt{1 - \frac{5+\sqrt{5}}{10}}} = \frac{BA}{2\sqrt{\frac{5-\sqrt{5}}{10}}} = \\ & \frac{1}{2} AB \sqrt{\frac{10}{5-\sqrt{5}}} = \frac{1}{2} AB \sqrt{\frac{10}{5-\sqrt{5}}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{1}{2} AB \sqrt{\frac{10 \times (5+\sqrt{5})}{25-5}} \\ & = AB \sqrt{\frac{5+\sqrt{5}}{8}}; \text{ and } R \text{ is the hypotenuse of a right-angled triangle of which the one leg is the radius } \rho \text{ of the circle circumscribing the face, and the other the radius } r \text{ of the inscribed sphere;} \\ & \text{wherefore } r \text{ is } = \sqrt{R^2 - \rho^2} = \sqrt{\frac{5+\sqrt{5}}{8} AB^2 - \frac{1}{3} AB^2} = \\ & AB \sqrt{\frac{15+3\sqrt{5}-8}{8 \times 3}} = AB \sqrt{\frac{7+3\sqrt{5}}{24}}. \text{ Consequently the solidity } C = \frac{1}{3} r B \text{ (} B \text{ being the whole surface) will be } \frac{1}{3} AB \sqrt{\frac{7+3\sqrt{5}}{24}} \\ & \times 5 AB^2 \sqrt{3} = \frac{5}{3} AB^3 \sqrt{\frac{7+3\sqrt{5}}{8}} = \frac{5}{6} AB^3 \sqrt{\frac{7+3\sqrt{5}}{2}}. \end{aligned}$$

Q.E.D.

SCHO



## EXAMPLE.

The linear side of an icosaedron being 1, required the solidity.

Here  $A = 1$ , and therefore  $\frac{5}{8}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{5}{8}\sqrt{\frac{7+3\sqrt{5}}{2}} = 2.1816949905 =$  the content required.

5 M

SEC-

## SCHOLIUM.

From this problem and the last may be deduced the following equations in the icosaedron, the letters being as before.

$$1. A = \sqrt{\frac{B\sqrt{3}}{15}} = \sqrt[3]{\frac{6}{5}C\sqrt{\frac{7-3\sqrt{5}}{2}}} = r\sqrt{42-18\sqrt{5}} = R\sqrt{\frac{10-2\sqrt{5}}{5}}.$$

$$2. B = \frac{5}{2}A^2\sqrt{3} = \sqrt[3]{70\sqrt{3}-30\sqrt{15}} = 3r^2 \times \sqrt{7\sqrt{3}-3\sqrt{15}} = 2R^2 \times 5\sqrt{3}-\sqrt{15}.$$

$$3. C = \frac{5}{8}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{B}{18}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}}B = 10r^3 \times \sqrt{7\sqrt{3}-3\sqrt{15}} = \frac{2}{3}R^3\sqrt{10+2\sqrt{5}}.$$

$$4. r = \frac{1}{2}A\sqrt{\frac{7+3\sqrt{5}}{6}} = \frac{1}{6}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}}B = \frac{1}{2}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{30}}C = R\sqrt{\frac{5+2\sqrt{5}}{15}}.$$

$$5. R = \frac{1}{2}A\sqrt{\frac{5+\sqrt{5}}{2}} = \frac{1}{2}\sqrt{\frac{5\sqrt{3}+\sqrt{15}}{30}}B = \sqrt{\frac{3}{4}C\sqrt{\frac{5-\sqrt{5}}{10}}} = r\sqrt{15-6\sqrt{5}}.$$



## SECTION VII.

*Of solid Rings.*

## DEFINITIONS.

**B**Y a ring, in general, is meant a solid returning into itself, of which every section perpendicular to the axe, or line passing through the middle of the solid, is every-where the same figure and of the same magnitude.

## PROBLEM I.

*To find the surface of a solid ring.*

## RULE.

Multiply the axe by the perimeter of a section perpendicular to it, and the product will be the surface.

## EXAMPLE.

A workman having made for a jeweller a circular ring, or a ring whose axe forms the circumference of a circle; it is required to find the expence of the gilding at a penny the square inch, the thickness of the ring, or the diameter of a section of it, being 4 inches, and the inner diameter, a-cross from side to side 18 inches.

Here  $18 + 2 = 20$  = the diameter of the circle formed by the axe; consequently  $20 \times 3.14159$  &c. = the length of the axe. But  $4 \times 3.14159$  &c. = the circumference of a section of it. Therefore  $20 \times 3.14159 \times 4 \times 3.14159 = 80 \times 3.14159^2 = 789.57$  square inches, nearly, = 789.57 pence = 3*l.* 5*s.* 9½*d.* nearly = the expence required.

PRO-



## P R O B L E M II.

*To find the solidity of a ring.*

## R U L E.

Multiply the axe by a section perpendicular to it, and the product will be the solidity.

## E X A M P L E.

Required the price of a ring of iron, whose dimensions are the same with that in the example to the last problem, at 4 pence a pound; a cubic inch of iron weighing 4.423 ounces averdupois.

Here the area of a section being  $4^2 \times .785398 \&c.$  =  $4 \times 3.14159 \&c.$  which is the same number as that expressing the circumference, and the axe being the same as before, it is evident that the solidity will be expressed by the same number as the surface in the last example, viz. the solidity = 789.57 cubic inches; which multiplied by 4.423 give 3492.2605 ounces = 218.26628 pounds; which, at 4 pence each, come to 3*l.* 12*s.* 9*d.* = the price required.

## S C H O L I U M.

I omit any more examples as the manner of operation is the same in all forms with those for prisms, both with regard to the surfaces and solidities; for, since it is evident that any ring is equal to a prism whose altitude and end are respectively equal to the axe and section of the ring, both in surface and solidity, the rules for them both must be the same; and for this reason also any demonstration of the rules for rings was quite unnecessary in this place.

S E C-



## SECTION VIII.

*Promiscuous Questions concerning solids.*

## QUESTION I.

**R**equired the content of a tub whose greater diameter is 60, diagonal 66, and the length of the stave 30 inches.

$$AB = 60 : BC + CA = 96 ::$$

$$BC - CA = 36 : \frac{36 \times 96}{60} = 6 \times$$

$$9.6 = 57.6 = BE - EA = DC.$$

$$AE = \frac{AB - CD}{2} = \frac{60 - 57.6}{2} = \frac{2.4}{2} =$$

$$1.2.$$

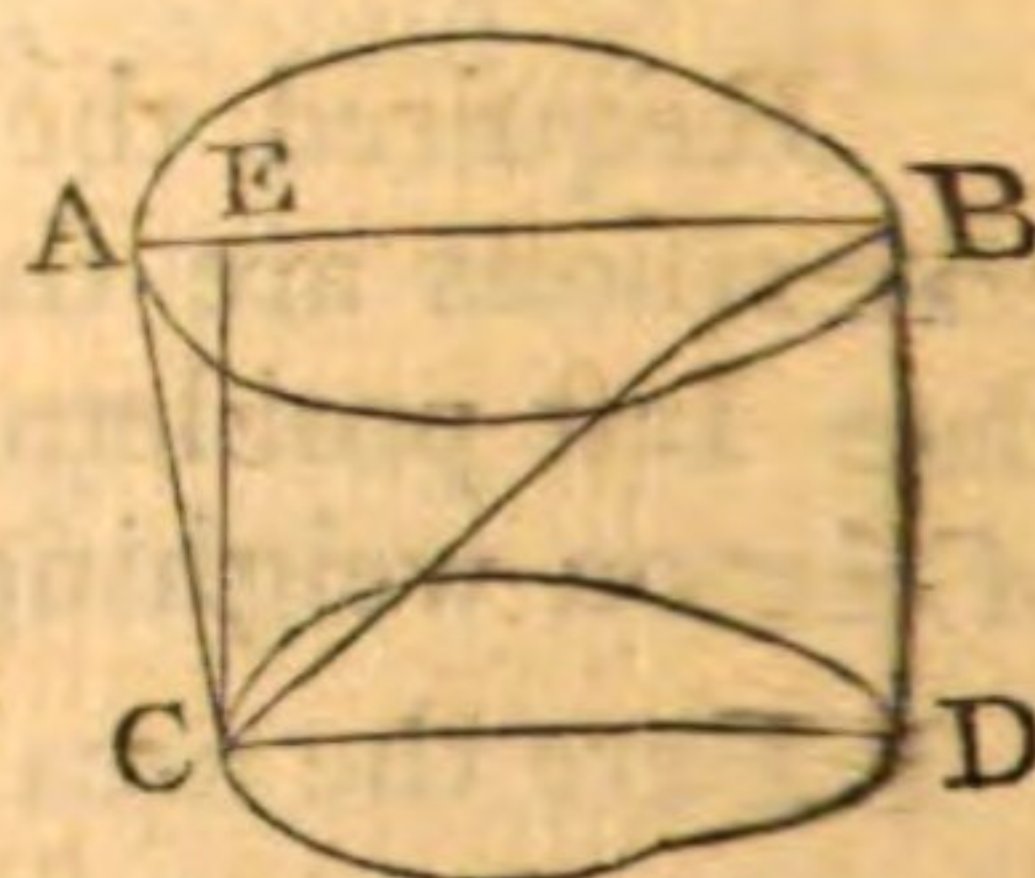
$$EC = \sqrt{CA^2 - AE^2} = \sqrt{30^2 - 1.2^2} = 6\sqrt{5^2 - .2^2} \\ = 6\sqrt{25 - .04} = 6\sqrt{24.96}.$$

Then  $AB^2 + AB \times DC + DC^2 \times \frac{1}{3}CE \times .785398$   
 $\&c. = 60^2 + 60 \times 57.6 + 57.6^2 \times 2\sqrt{24.96} \times .785398$   
 $\&c. = 10373.76 \times 4\sqrt{.39} \times 3.14159 \&c. = 81410.112$   
 cubic inches = 288.688 ale gallons = the content re-  
 quired.

## QUESTION II.

Three persons having bought a fugar loaf, would divide it equally among them by sections parallel to the base; it is required to find the altitude of each person's share, supposing the loaf to be a cone whose height is 20 inches.

Similar cones being as the cubes of their altitudes, we shall have as  $\sqrt[3]{3} : \sqrt[3]{1} :: 20 : 20\sqrt[3]{\frac{1}{3}} = \frac{20\sqrt[3]{9}}{3} =$   
 $13.8672247 =$  the height of the upper part, and  
 as





as  $\sqrt[3]{3} : \sqrt[3]{2} :: 20 : 20\sqrt[3]{\frac{2}{3}} = \frac{20\sqrt[3]{18}}{3} = 17.4716107 =$   
 that of the upper and middle part together; consequently  $17.4716107 - 13.8672247 = 3.604386 =$   
 the height of the middle part, and  $20 - 17.4716107 = 2.5283893 =$  that of the lower part.

### QUESTION III.

A silver cup, in the form of the frustum of a cone, whose top diameter is 3 inches, its bottom diameter 4, and its altitude 6 inches, being filled with liquor, a person drank out of it until he could see the middle of the bottom; it is required to find how much he drank.

By problem 14 of section 1 we have  $D = 4$ ,  $d = 3$ ,  $h = 6$ , and  $BD = \frac{1}{2}D = 2$ ; hence  $D^3 - d^3 = 64 - 27 = 37$ ,  $P =$  (the tabular area whose versed sine is  $\frac{BD}{D}$  or  $\frac{1}{2} =$ )  $\frac{1}{2}n = \frac{1}{2}$  of  $.78539816$  &c.  $Q =$  (the tabular area whose versed sine is  $\frac{BD - D + d}{d}$  or  $\frac{1}{3} =$ )  $.22945528$ ,  $\frac{BD}{BD - D + d} = 2$ , and  $\frac{\frac{1}{2}h}{D - d} = 2$ ; consequently  $\frac{37n - 32n + 54Q\sqrt{2} \times 2}{5n + 54Q\sqrt{2} \times 2} =$   
 $\frac{5n + 54Q\sqrt{2} \times 2}{5n + 54Q\sqrt{2} \times 2} = 7.8539816 + 35.0458624 = 42.899844$  cubic  
 inches  $= .152127$  ale gallons  $=$  the quantity required, or 1 gill and  $\frac{1}{5}$  nearly.

### QUESTION IV.

I have a right cone which cost me 5*l.* 13*s.* 7*d.*  
 at 10*s.* a cubic foot, the diameter of its base being  
 to its altitude as 5 to 8; and would have its  
 5 N convex



convex surface divided in the same ratio by a plane parallel to the base, the upper part to be the greater: Required the slant height of each part.

Here  $\frac{5l. 13s. 7d.}{10s.} = \frac{1363}{120}$  = the solidity of the cone in feet, and  $.785398 \&c. \times 5^2 \times \frac{8}{3} = 200 \times \frac{.785398 \&c.}{3} = \frac{200n}{3}$  = that of a cone similar to it whose altitude is 8.

Now, the surfaces of similar solids being as the squares of their like dimensions, we have  $\sqrt{5+8} : \sqrt{8} :: \sqrt{2 \cdot 5^2 + 8^2} =$  the side of the said similar cone :  $\sqrt{\frac{562}{13}} =$  the slant height of the upper part of this cone when its surface is divided in the proposed ratio; and consequently  $\sqrt{70\frac{1}{4}} - \sqrt{\frac{562}{13}} = \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}} =$  that of the under part of it.

Then, because similar solids are as the cubes of their like dimensions, we shall have  $\sqrt[3]{\frac{200n}{3}} : \sqrt[3]{\frac{1363}{120}} :: \sqrt{\frac{562}{13}} : \frac{1}{20} \sqrt[3]{\frac{1363}{n}} \times \sqrt{\frac{562}{13}} = 3.9506486 =$  the slant height of the upper part required, and  $\sqrt[3]{\frac{200n}{3}} : \sqrt[3]{\frac{1363}{120}} :: \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}} : \frac{1}{20} \sqrt[3]{\frac{1363}{n}} \times \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}} = \frac{1}{20} \sqrt[3]{\frac{1363}{n}} \times \sqrt{\frac{562}{8}} - 3.9506486 = 5.0361098 - 3.9506486 = 1.0854612 =$  that of the under part.

QUEST.



## QUESTION V.

There is a mill-hopper in the form of a square pyramid, whose solid content is  $13\frac{1}{2}$  feet; but one foot is cut off its perpendicular altitude, to make a passage for the grain from the frustum or hopper to the mill-stone: The sides of its greater and less end are in proportion as  $4\frac{1}{2}$  to 1. Required the content of the frustum, in corn measure, in which 268.8025 cubic inches make a gallon, or 2150.42 = the bushel.

Since similar solids are as the cubes of their like dimensions, we shall have  $4.5^3 : 1^3 :: 13\frac{1}{2} : \frac{13\frac{1}{2}}{4.5^3} = \frac{3}{4.5^2} = \frac{12}{81} = \frac{4}{27}$  = the content of the part cut off.

Wherefore  $13\frac{1}{2} - \frac{4}{27} = \frac{27}{2} - \frac{4}{27} = \frac{27^2 - 8}{54} = \frac{729 - 8}{54} = \frac{721}{54}$  cubic feet =  $\frac{721}{54} \times 1728 = \frac{721 \times 288}{9} = 721 \times 32 = 23072$  cubic inches = the content of the hopper. Which being divided by 2150.42 gives 10.729067 bushels for the content in corn measure.

## QUESTION VI.

A piece of round tapering timber, whose top and bottom diameters are 40 and 50 inches, and its height 6 feet, is to be cut through the extremity of the less diameter and parallel to the tapering direction; required the content of the two parts or hoofs into which it is cut.

By prob. 14 sect. 1,  $D = 50$ ,  $d = 40$ ,  $h = 6$  feet = 72 inches; then  $A = 50^2 \times \text{tabular segment}$  whose



whose height is  $\left(\frac{50-40}{50} = \frac{10}{50} =\right) \cdot 2 = 50^2 \times \cdot 1118238$   
 $= 279\cdot5595$ .

Hence  $\frac{A \times D}{D-d} - \frac{4}{3}d\sqrt{D-d} \times d \times \frac{1}{3}b =$   
 $\frac{279\cdot5595 \times 50}{50-40} - \frac{160}{3}\sqrt{50-40} \times 40 \times 24 =$   
 $279\cdot5595 \times 5 - \frac{160-20}{3} \times 24 = 279\cdot5595 \times 3 - 160 \times 4$   
 $\times 40 = 198\cdot6785 \times 40 = 7947\cdot14 = \text{the content}$   
of the hoof cut off.

But, by prob. 8 sect. 1,  $\frac{50^3-40^3}{50-40} \times \frac{7^2}{3} \times \cdot 785398$   
&c.  $= 5^3 - 4^3 \times 100 \times 24 \times \cdot 785398$  &c.  $= 166400$   
 $\times \cdot 785398$  &c.  $= 250690\cdot2543893 = \text{the content of}$   
the whole piece.

Consequently  $130690\cdot2543893 - 7947\cdot14 =$   
 $122743\cdot1143893 = \text{the solidity of the complementary}$   
hoof.

#### QUESTION VII.

“Two Oxonians, meeting at an inn, encountered with a tankard of negus; the one, being pot-valiant, gave it a black-eye, as it is called, that is, he drank till he could see the center of the bottom of the tankard; the other drank the rest: Now, if the liquor cost 1s. 6d. and the tankard measure 4 inches diameter at the top and bottom, and 6 inches in depth, what must each person pay proportionable to the liquor he drank.”

By prob. 4 sect. 1, we have  $\frac{1}{8}ddh$  for the quantity left by the first person; and, by prob. 2 sect. 1,  $n ddh =$  the content of the whole tankard,  $d$  being the  
the

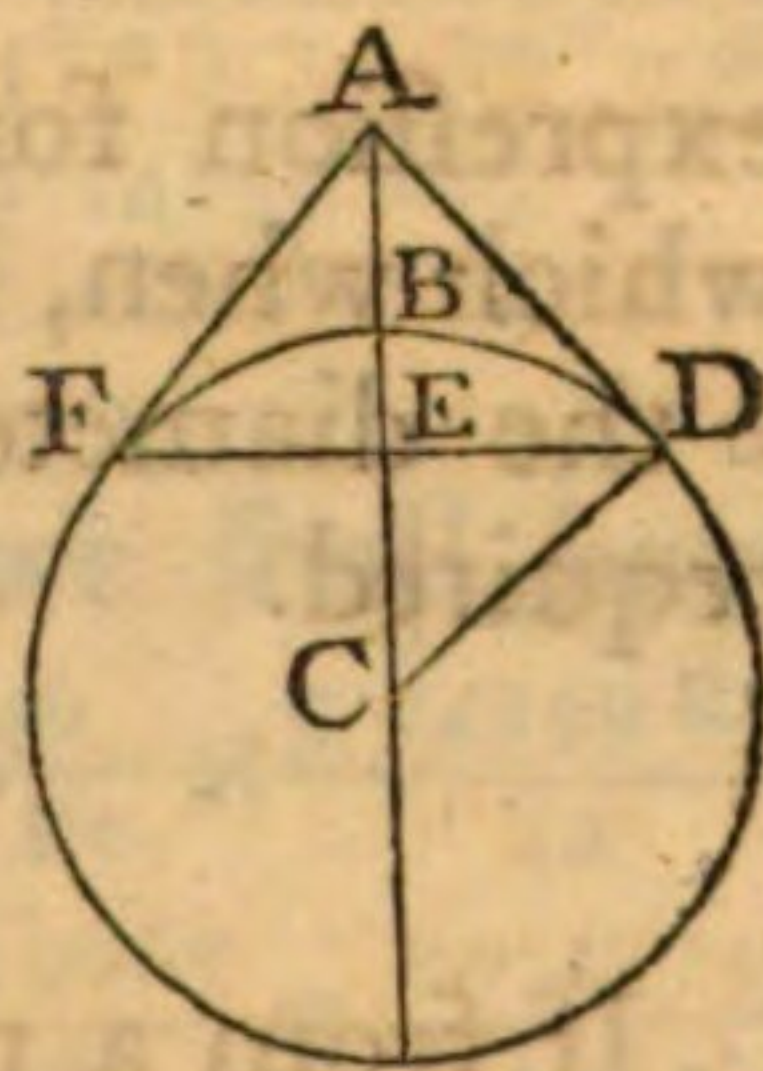


the diameter,  $h$  the height, and  $n = .785398 \text{ \&c.}$  that is, the whole is to the less share as  $n$  to  $\frac{1}{6}$ ; consequently as  $n : \frac{1}{6} :: 18d. : \frac{3}{n} = \frac{3}{.785398 \text{ \&c.}} = 3.8197185$  pence = the sum the latter person must pay; and hence  $18d. - 3.8197185d. = 14.1802815$  pence = the sum the first drinker must pay.

### QUESTION VIII.

How many acres of the earth's surface may be seen from the top of a steeple whose height is 400 feet; the earth being supposed a perfect sphere whose circumference is 25000 miles.

Having drawn from  $A$ , the top of the steeple, two lines  $AD$ ,  $AF$ , to touch the earth whose center is  $C$ , and the other lines as in the figure;  $FBD$  will be the surface required.



By similar triangles,  $AC : CD$  or  $CB :: CD$  or  $CB : CE$ , hence  $AC : AC - CB = BA :: CB : CB - CE =$

$$EB = \frac{CB \times BA}{AC} = \frac{CB \times BA}{CB + BA} = \frac{\frac{25000}{3.14159 \text{ \&c.}} \times \frac{400}{5280}}{\frac{25000}{3.14159 \text{ \&c.}} + \frac{800}{5280}} = \frac{25000 \times 4}{2500 \times 528 + 3.14159 \text{ \&c.} \times 8} = \frac{100000}{1320025.1327412287} = .07575613336416945.$$

Then, by problem 19 section 1, we have  $.07575613336416945 \times 250000 \times 640$  (the acres in a square mile) =  $12120981.338267112$  = the number of acres required.



## QUESTION IX.

How high above the surface of the earth must a person be raised, that he may see one-third of its surface.

The surfaces of segments being, by prob. 19 sect. 1, as their altitudes, the altitude of the segment in the question must be one-third of the diameter or two-thirds of the radius of the sphere; that is,  $EB = \frac{2}{3}BC$ ; and consequently  $EC = \frac{1}{3}CB$ . [See the last figure]

But, by similar triangles, as  $EC : CD$  or  $CB :: CD$  or  $CB : CA$ , and hence  $EC : CB - CE = EB :: BC : CA - CB = BA = \frac{CB \times BE}{EC}$ , which is a general expression for the height above the surface, and which when, as above,  $BE$  is  $= 2EC$ , becomes  $2BC =$  the diameter of the earth; which is the height required.

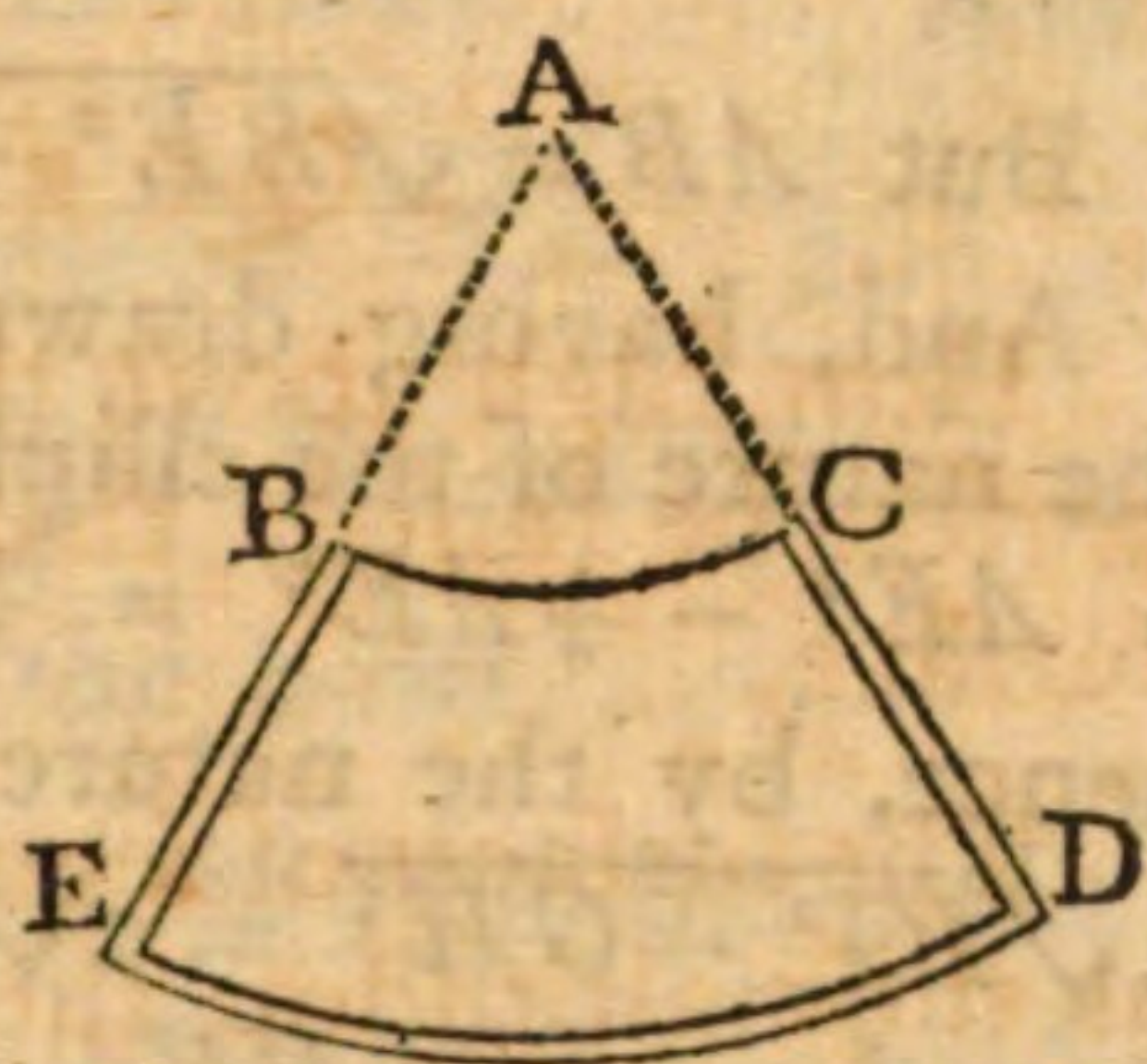
## QUESTION X.

If from a piece of tin  $AED$ , in the form of the sector of a circle, whose radius  $AE$  or  $AD$  is 30 inches, and the length of its arc  $ED$  36 inches, be cut another sector  $ABC$  whose radius  $AB$  or  $AC$  is 20 inches; and if then the remaining frustum  $BCDE$  be rolled up so as to form the frustum of a cone, it is required to find its content, supposing one-eighth of an inch to be allowed of its slant height  $BE$  for the bottom, and the same allowance off the circumference of both top and bottom for what the sides  $BE$ ,  $CD$ , fold over each other, in order to their being foldered together.

By



By similar figures, as  $EA = 30 : AB = 20 :: ED = 36 : \frac{36 \times 20}{30} = 24 = BC$ . Then the circumferences of the top and bottom are  $(36 - \frac{1}{8} = 35\frac{7}{8} =) 35.875$ , and  $23.875$ ; and the side of the vessel is  $(30 - 20 - \frac{1}{8} = 9\frac{7}{8} =) 9.875$ .



Hence the diameters of the ends will be  $D = \frac{35.875}{3.14159}$ , and  $d = \frac{23.875}{3.14159}$ ; half their difference is  $\frac{6}{3.14159}$ ; consequently the perpendicular altitude will be found, by the property of right-angled triangles, to be  $\sqrt{9.875^2 - \frac{6^2}{3.14159^2}} = \frac{\sqrt{9.875^2 \times 3.14159^2 - 6^2}}{3.14159 \text{ \&c.}}$   
 $= \frac{30.43749}{3.14159 \text{ \&c.}}$

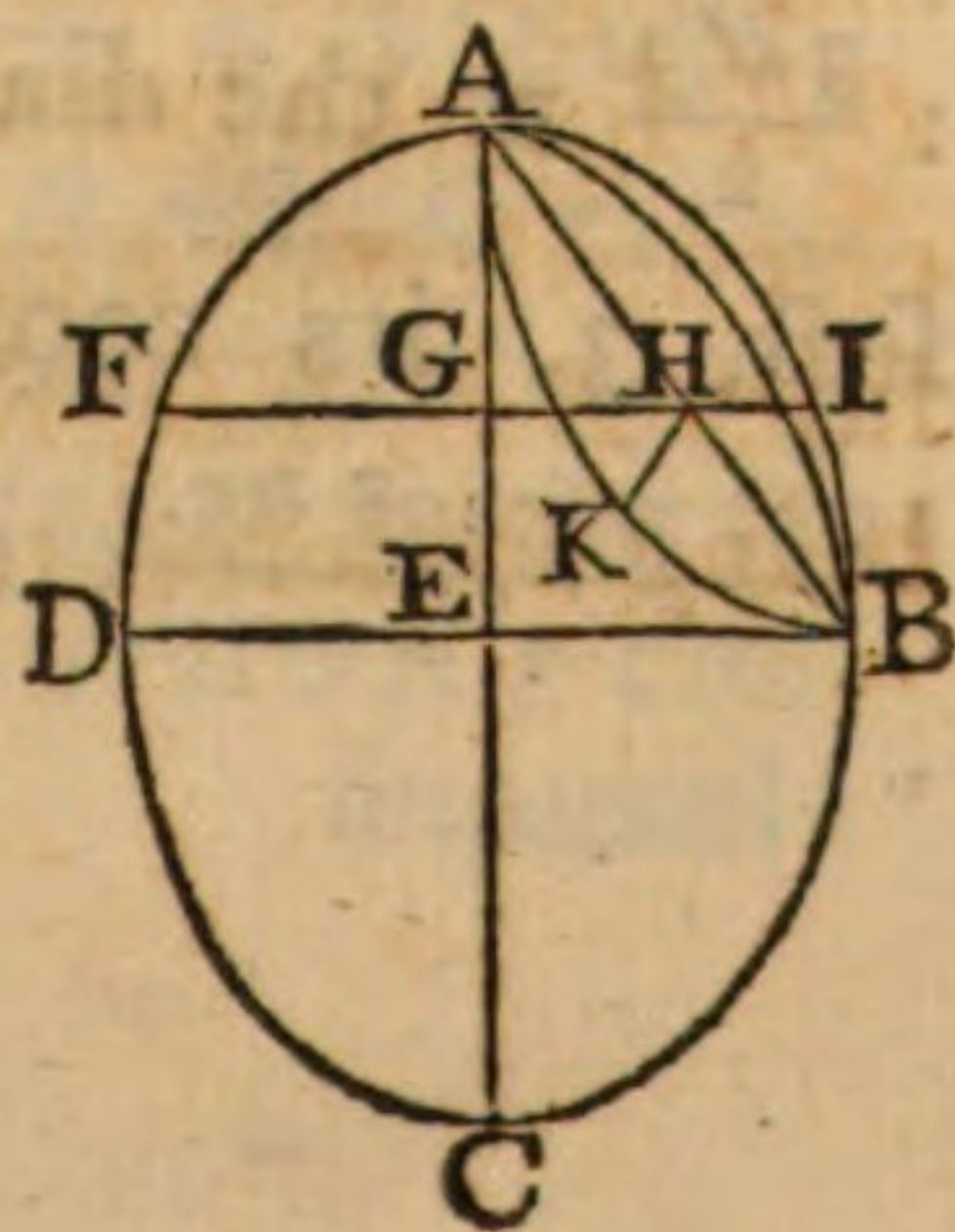
Then, by problem 8 section 1, we shall have  
 $\frac{35.875^2 + 35.875 \times 23.875 + 23.875^2}{3.14159^2} \times \frac{30.43749}{3.14159 \text{ \&c.}} \times \frac{3.14159 \text{ \&c.}}{12}$   
 $= \frac{287^2 + 287 \times 191 + 191^2}{8^2 \times 3.14159^2} \times \frac{30.43749}{12} = \frac{170667}{768} \times \frac{30.43749}{3.14159^2}$   
 $= 685.3263 \text{ cubic inches} = \text{the content required.}$

### QUESTION XI.

It is required to find the area of the section of any spheroid, formed by a plane passing through the extremities of the two axes, the one axe being 80 and the other 60.

By prop. 1 sect. 2 the section  $AKB$  will be an ellipse whose axes are  $AB$  and  $2HK$ ,  $H$  being the middle of  $AB$ .

But





But  $AB = \sqrt{BE^2 + EA^2} = \sqrt{30^2 + 40^2} = 50$ .

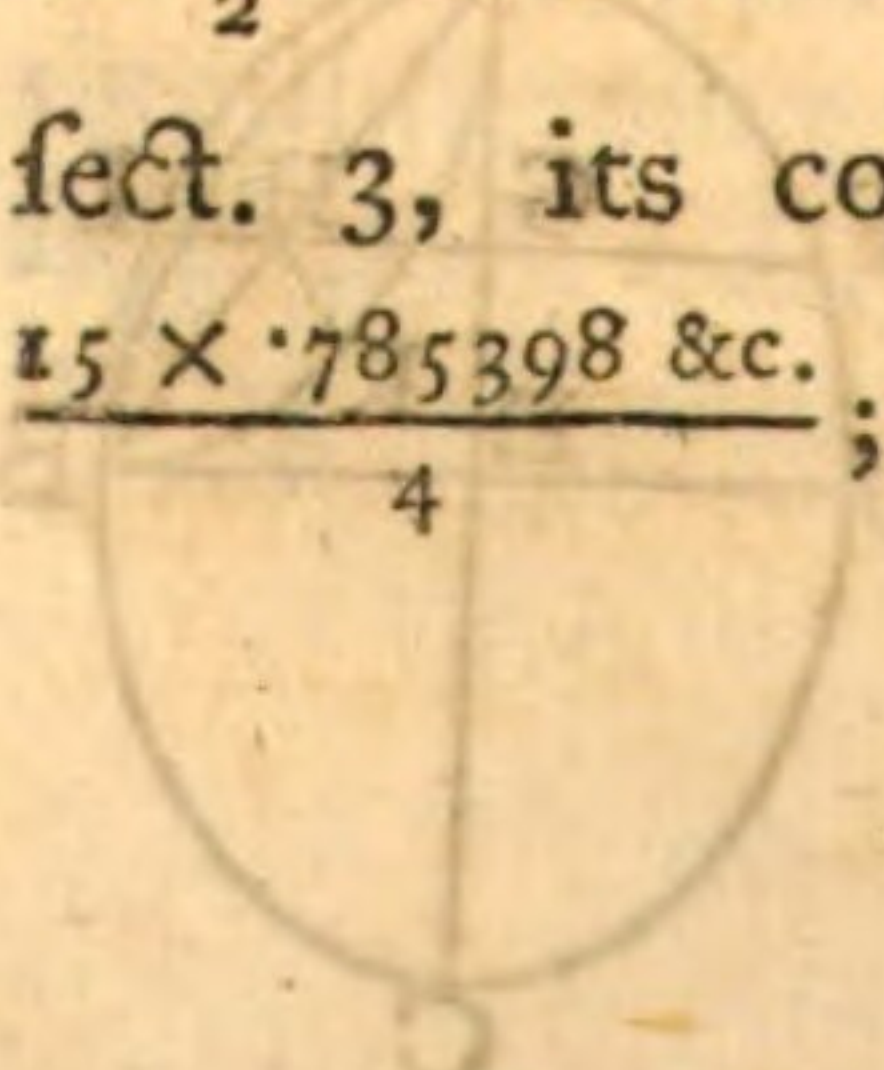
And, having drawn  $FGHI$  parallel to  $DB$ , by the nature of the ellipse,  $AE^2 : EB^2 :: AE^2 - EG^2 = AE^2 - \frac{1}{4}AE^2 = \frac{3}{4}AE^2 : GI^2 = \frac{3}{4}EB^2$ ; and hence, by the nature of the circle,  $2\sqrt{FH \times HI} = 2\sqrt{IG^2 - GH^2} = 2\sqrt{\frac{3}{4}EB^2 - \frac{1}{4}EB^2} = EB\sqrt{2} =$  the other axe  $2HK$ .

Consequently  $AB \times 2HK \times .785398 \&c. = 50 \times 30\sqrt{2} \times .785398 \&c. = 1666.081101807 =$  the area of the section when the spheroid is oblong. And  $50 \times 40\sqrt{2} \times .785398 \&c. = 2221.441469076 =$  the area when oblate.

### QUESTION XII.

There is a punch bowl in the form of the segment of an oblong spheroid whose axes are to each other in the proportion of 3 to 4, the depth of the bowl being one-fourth of the whole transverse axe, and the diameter of its top 20 inches: it is required to determine what number of rounds a company of 10 men may take out of it, when filled with liquor, using a conical glass whose depth is 2 inches, and the diameter of its top one inch and a half.

The segment, whose altitude is 1, of the spheroid whose axes are 3 and 4 is similar to the proposed one; and, by the nature of the ellipse,  $4:3 :: 2\sqrt{3 \times 1} : \frac{3\sqrt{3}}{2} =$  the diameter of its top; but by prob. 14 sect. 3, its content is  $\frac{6-1}{3 \times 4^2} \times 3^2 \times 3.14159 \&c. = \frac{15 \times .785398 \&c.}{4}$ ; and similar solids are as the cubes of





of their like dimensions; wherefore  $\frac{3\sqrt{3}}{2}^3 : 20^3 ::$   
 $\frac{15 \times .785398 \&c.}{4} : \frac{40^3}{3^4 \times \sqrt{3}} \times \frac{15 \times .785398 \&c.}{4} = \frac{80000\sqrt{3}}{81} \times$   
 $.785398 \&c. =$  the content of the bowl.

But  $1.5^2 \times 2 \times \frac{.785398 \&c.}{3} = \frac{3}{2} \times .785398 \&c. =$  the  
 content of the glass, and  $15 \times .785398 \&c. =$  each  
 round.

Wherefore  $\frac{80000\sqrt{3} \times .785398 \&c.}{81 \times 15 \times .785398 \&c.} = \frac{16000\sqrt{3}}{243} =$   
 $114.0444976 =$  the number of rounds required.

## QUESTION XIII.

Two persons would divide between them, by a  
 plane perpendicular to the base, an hay-rick in the  
 form of a paraboloid whose altitude is 40 and the  
 diameter of its base 30 feet; it is required to find  
 the difference between the solidities of the parts,  
 supposing the altitude of the section to be 28 feet.

Here  $DH = 40$  [See fig. to prob. 13 sect. 4],  
 $EA = 30$ , and  $BC = HL = 28$ .—Hence  $LD$   
 $= DH - HL = 40 - 28 = 12$ ; and, by prob. 3  
 sect. 4, as  $\sqrt{HD} : \sqrt{DL} :: HA : LC = HB =$   
 $\frac{15\sqrt{12}}{\sqrt{40}} = \frac{3\sqrt{30}}{2}$ ; hence  $\sqrt{AH^2 - HB^2} = \sqrt{15^2 - \frac{9 \times 30}{4}}$   
 $= \frac{315}{2}$ , and  $\frac{BA}{AE} = \frac{15 - \frac{3\sqrt{30}}{2}}{30} = \frac{10 - \sqrt{30}}{20} = .2261387$   
 $=$  the tabular versed sine similar to  $AB$ , whose tabu-  
 lar area is .13322474, which taken from .39269908,  
 the tabular semi-circle, leaves .25947434  $=$  the ta-  
 bular area similar to double the area  $HBGK$ ; con-  
 5 P frequently



frequently  $\frac{1}{2} \times 30^2 \times .25947434 = 116.763453 =$  the area *HBGK*.

Then, by cor. 4 to prob. 13 sect. 1, we shall have  
 $116.763453 \times 40 \times 2 + \frac{315 \sqrt{28 \times 12}}{2} = 9341.07624 +$   
 $630 \sqrt{21} = 9341.07624 + 2887.022691 = 12228.09893$   
 $=$  the difference required.

#### QUESTION XIV.

The curve surface of a paraboloid being  
 $109 \sqrt{109 - 27} \times 2 \times 3.14159 \text{ \&c.} = 6980.57746 \text{ \&c.}$   
 it is required to find its altitude and the diameter  
 of its base, supposing them in proportion as 5 to 6.

By prob. 7 sect. 4,  $\frac{3^2 + 10^2 \sqrt{\frac{3}{2}} - 3^3}{3^2 + 10^2 - 3^2} \times 3 \times \frac{2}{3} \times 3.14159$   
 $\text{\&c.} = \frac{109 \sqrt{109 - 27}}{100} \times 2 \times 3.14159 \text{ \&c.}$  is the surface  
 of a paraboloid whose altitude is 5 and base dia-  
 meter 6, which is similar to the proposed parabo-  
 loid; but the surfaces of similar bodies are as the  
 squares of their lineal dimensions, or the **demen-**  
**sions** are as the roots of the surfaces; wherefore  
 as  $\sqrt{\frac{109 \sqrt{109 - 27}}{100}} \times 2 \times 3.14159 \text{ \&c.}$  is to  $\sqrt{\frac{109 \sqrt{109 - 27}}{1}}$   
 $\times 2 \times 3.14159 \text{ \&c.}$  or as  $1 : 10 :: 5 : 50 =$  the alti-  
 tude, and  $:: 6 : 60 =$  the base-diameter required.

#### QUESTION XV.

If a cubical foot of brass were to be drawn into  
 wire, of one-fortieth of an inch in diameter, it is  
 required to determine the length of the said wire,  
 allowing no loss in the metal.

Here



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Here  $1 \times 12^3 = 1728$  = the solidity of the wire in inches; and  $\frac{1}{40}^2 \times .785398 \text{ \&c.} = \frac{1}{1600} \times .785398$  &c. the area of its end; consequently  $\frac{1728}{\frac{1}{1600} \times .785398 \text{ \&c.}}$   
 $= \frac{1728 \times 1600}{.785398 \text{ \&c.}} = 3520252.69329$  inches = the length required.

QUESTION XVI.

A gentleman having a bowling green, 300 feet long and 200 feet broad, which he would raise one foot higher by means of the earth to be dug out of a ditch around it; it is required to find to what depth the ditch must be dug, its breadth being every where 8 feet.

Here the ditch is a kind of ring whose breadth is 8, and length  $= 300 \times 2 + 200 \times 2 + 8 \times 4 = 516 \times 2 = 1032$ , and consequently the area of its bottom  $= 8 \times 1032 = 8256$ ; but  $300 \times 200 \times 1 = 60000$  is its content; consequently  $\frac{60000}{8256} = 7\frac{2}{3}$  feet = the depth required.

QUESTION XVII.

Required the weight of a bomb-shell, or hollow sphere of cast iron, whose out-side diameter is 15, and the thickness of the metal  $2\frac{1}{2}$  inches; the gravity of water being to that of cast iron as 1 to 7, and a cubic inch of water weighing .5787 ounces avoirdupois.

Here  $15^3 \times \frac{3.14159 \text{ \&c.}}{6} =$  the solidity of the whole sphere, supposing it all solid; and, since  $15 - 2\frac{1}{2} \times 2 = 15 - 5 = 10$  is the diameter of the cavity



cavity,  $10^3 \times \frac{3 \cdot 14159 \text{ \&c.}}{6}$  will be the content of the cavity; consequently their difference  $15^3 - 10^3 \times \frac{3 \cdot 14159 \text{ \&c.}}{6} = 3^3 - 2^3 \times 5^3 \times \frac{3 \cdot 14159 \text{ \&c.}}{6} = 19 \times 125 \times \frac{3 \cdot 14159 \text{ \&c.}}{6}$  = the solidity of the metal in inches.

But  $1 : 7 :: \cdot 5785 : 4 \cdot 0509$  ounces = the weight of a cubic inch of the metal.

Consequently  $4 \cdot 0509 \times 19 \times 125 \times \frac{3 \cdot 14159 \text{ \&c.}}{6} = 1 \cdot 3503 \times 19000 \times \frac{\cdot 785398 \text{ \&c.}}{4} = 6413 \cdot 925 \times \cdot 785398 \text{ \&c.} = 5037 \cdot 4849152$  ounces =  $314 \cdot 8428072$  pounds avoirdupois = the weight required.

### QUESTION XVIII.

Of what diameter must the bore of a cannon be cast for a ball of 24 pound weight, so that the diameter of the bore may be one tenth of an inch more than that of the ball.

By the last, the weight of a cubic inch of cast iron is  $4 \cdot 0509$  ounces, consequently as  $4 \cdot 0509$  ounces : 1 inch :: 24 lb. or  $24 \times 16$  oz. :  $\frac{24 \times 16}{4 \cdot 0509} = \frac{128}{1 \cdot 3503}$  inches = the solidity of the ball; but the solidity is equal to the cube of the diameter multiplied by  $\frac{3 \cdot 14159 \text{ \&c.}}{6}$ ; wherefore  $\sqrt[3]{\frac{128 \times 6}{1 \cdot 3503 \times 3 \cdot 14159 \text{ \&c.}}} = \sqrt[3]{\frac{256}{\cdot 4501 \times 3 \cdot 14159 \text{ \&c.}}} = 8 \times \sqrt[3]{\frac{1}{\cdot 9002 \times 3 \cdot 14159 \text{ \&c.}}} = 5 \cdot 657098$  inches = the diameter of the ball; to which adding  $\cdot 1$  makes  $5 \cdot 757098$  inches = the diameter required.

PART



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## P A R T IV.

Having given at large the mensuration of all figures which are generally thought to be met with in real practice, viz. the figures formed by right or circular lines, or by the conic sections; I shall, under this part, give a *very brief* treatise of such other things as belong to mensuration in general; such as the quadrature and cubature of figures from general equations expressing their nature or property, the method of equidistant ordinates, and the relation between the areas or solidities of figures and their centers of gravity.

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### S E C T I O N I.

*Of the true QUADRATURE and CUBATURE of CURVES in GENERAL.*

#### P R O P O S I T I O N I.

*I*F  $z$  be the abscissa of any curve, and the ordinate  $y$  be equal to  $z^{p-1} \times \alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n} + \&c.$   
 $\times : a + bz^n + cz^{2n} + dz^{3n} + ez^{4n} \&c.$  and if  $\frac{p}{n}$  be put equal to  $r$ ,  $r+q=s$ ,  $s+q=t$ ,  $t+q=v$ ,  $v+q=w$ ,  $\&c.$  I say the area will be equal to  
 $z^p \times \alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n} + \&c.$  drawn into the series

5Q



$$\begin{aligned}
& + \frac{\frac{a}{n}}{\alpha r} \\
& + \frac{\frac{b}{n} - A\beta s}{\alpha \cdot r + 1} z^n \\
& + \frac{\frac{c}{n} - A\gamma t - B\beta \cdot s + 1}{\alpha \cdot r + 2} z^{2n} \\
& + \frac{\frac{d}{n} - A\delta v - B\gamma \cdot t + 1 - C\beta \cdot s + 2}{\alpha \cdot r + 3} z^{3n} \\
& + \frac{\frac{e}{n} - A\epsilon w - B\delta \cdot v + 1 - C\gamma \cdot t + 2 - D\beta \cdot s + 3}{\alpha \cdot r + 4} z^{4n}
\end{aligned}$$

Where  $A, B, C, \&c.$  are the whole coefficients of the preceding terms with their signs  $+$  or  $-$ ; viz.  $A = \frac{\frac{a}{n}}{\alpha r}$ ,

$$B = \frac{\frac{b}{n} - A\beta s}{\alpha \cdot r + 1}, \&c.$$

#### DEMONSTRATION.

The fluxion of the area is  $y\dot{z} = z^{p-1} \dot{z} \times$   
 $\frac{\alpha + \beta z^n + \gamma z^{2n} \&c.}{\alpha + \beta z^n + \gamma z^{2n} \&c.} \times : \alpha + \beta z^n + \gamma z^{2n} + \&c.$

Let the fluent of this expression, or the area required, be represented by  $z^p \times \frac{\alpha + \beta z^n + \gamma z^{2n} + \&c.}{\alpha + \beta z^n + \gamma z^{2n} + \&c.} \times : A + Bz^n + Cz^{2n} + \&c.$  where  $A, B, C, \&c.$  are not supposed to represent the same quantities as in the proposition, but other quantities yet to be determined.

Then



Then let the fluxion of this last assumed area or fluent be taken, and compared with the given fluxion, so shall we have equations for determining the values of the assumed letters  $A, B, C$ , &c.

Thus the fluxion of the assumed area being  $z^{p-1} \dot{z} \times \overline{\alpha + \beta z^n + \gamma z^{2n} + \&c.}^q \times : Ap + \overline{p+n} \cdot Bz^n + \overline{p+2n} \cdot Cz^{2n} + \&c. + nqz^{p-1} \dot{z} \times \overline{\alpha + \beta z^n + \gamma z^{2n} + \&c.}^{q-1} \times : \beta z^n + 2\gamma z^{2n} + 3\delta z^{3n} + \&c. \times : A + Bz^n + Cz^{2n} + \&c.$

Or, by proper multiplication, &c.

$$z^{p-1} \dot{z} \times \overline{\alpha + \beta z^n + \gamma z^{2n} + \&c.}^{q-1} \text{ drawn into}$$

$$\begin{aligned} p\alpha A + \overline{p+qn} \cdot \beta Az^n + \overline{p+2qn} \cdot \gamma Az^{2n} + \overline{p+3qn} \cdot \delta Az^{3n} + \&c. \\ + \overline{p+n} \cdot \alpha Bz^n + \overline{p+qn+n} \cdot \beta Bz^{2n} + \overline{p+2qn+n} \cdot \gamma Bz^{3n} \\ + \overline{p+2n} \cdot \alpha Cz^{2n} + \overline{p+qn+2n} \cdot \beta Cz^{3n} \\ + \overline{p+3n} \cdot \alpha Dz^{3n} \end{aligned}$$

If the several terms of this series be compared with the corresponding terms of the series in the given fluxion, we shall obtain these equations, viz.

$$\begin{aligned} a &= p\alpha A, \\ b &= \overline{p+qn} \cdot \beta A + \overline{p+n} \cdot \alpha B, \\ c &= \overline{p+2qn} \cdot \gamma A + \overline{p+qn+n} \cdot \beta B + \overline{p+2n} \cdot \alpha C, \\ d &= \overline{p+3qn} \cdot \delta A + \overline{p+2qn+n} \cdot \gamma B + \overline{p+qn+2n} \cdot \beta C + \overline{p+3n} \cdot \alpha D, \\ e &= \overline{p+4qn} \cdot \epsilon A + \overline{p+3qn+n} \cdot \delta B + \overline{p+2qn+2n} \cdot \gamma C + \overline{p+qn+3n} \cdot \beta D + \overline{p+4n} \cdot \alpha E, \\ &\&c. \&c. \end{aligned}$$

From



From whence we obtain

$$A = \frac{a}{\alpha p},$$

$$B = \frac{b - \overline{p + qn} \cdot \beta A}{\alpha \cdot \overline{p + n}},$$

$$C = \frac{c - \overline{p + 2qn} \cdot \gamma A - \overline{p + qn + n} \cdot \beta B}{\alpha \cdot \overline{p + 2n}},$$

$$D = \frac{d - \overline{p + 3qn} \cdot \delta A - \overline{p + 2qn + n} \cdot \gamma B - \overline{p + qn + 2n} \cdot \beta C}{\alpha \cdot \overline{p + 3n}},$$

$$E = \frac{e - \overline{p + 4qn} \cdot \epsilon A - \overline{p + 3qn + n} \cdot \delta B - \overline{p + 2qn + 2n} \cdot \gamma C - \overline{p + qn + 3n} \cdot \beta D}{\alpha \cdot \overline{p + 4n}},$$

&c.

Where now the letters  $A, B, C$ , &c. in the terms on the right side of these equations denote the preceding terms, as specified in the proposition.

And if  $\frac{p}{n}$  be put  $= r$ ,  $r + q = s$ ,  $s + q = t$ ,  $t + q = v$ , &c. these last equations will become

$$A = \frac{\frac{a}{n}}{\alpha r},$$

$$B = \frac{\frac{b}{n} - A\beta s}{\alpha \cdot r + 1},$$

$$C = \frac{\frac{c}{n} - A\gamma t - B\beta \cdot \overline{s + 1}}{\alpha \cdot r + 2},$$

$$D = \frac{\frac{d}{n} - A\delta v - B\gamma \cdot \overline{t + 1} - C\beta \cdot \overline{s + 2}}{\alpha \cdot r + 3},$$

$$E = \frac{\frac{e}{n} - A\epsilon w - B\delta \cdot \overline{v + 1} - C\gamma \cdot \overline{t + 2} - D\beta \cdot \overline{s + 3}}{\alpha \cdot r + 4},$$

&c.

Which



Which values of  $A, B, C, \&c.$  being substituted for them in the assumed fluent or area, will give the area as in the proposition. *Q.E.D.*

And much after the same manner we may find the area of the curve whose abscissa is  $z$  and ordinate

$$\frac{\alpha + \beta z^n + \gamma z^{2n} + \&c. \times z^{p-1} \times \alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \&c.}{\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \&c.} \times A + Bz^n + Cz^{2n} + Dz^{3n} + \&c. \times \&c. \text{ whatever be the number of series.}$$

When, after some of the first terms, the numerators of each of the following terms of the series

$$\frac{\frac{a}{n}}{\alpha r} + \frac{\frac{b}{n} - A\beta s}{\alpha, r+1} z^n + \&c. \text{ become equal to nothing, the series will break off and terminate, and then the curve is said to be quadrable; if otherwise, it is said to be non-quadrable.} \text{---If } r \text{ be either nothing or a negative integer number, it is evident that the denominator of one of the terms of the above series will become equal to nothing, and then that term will be infinite; and if this happen before the series terminate by means of the numerators becoming equal to nothing, the value of the area comes out infinite, in which case the series is said to fail.}$$

The curve is denominated from the number of terms contained in the quantity

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \&c. \text{ ; so, if it contain only one term } \alpha, \text{ it is a simple nominal; if it contain two terms } \alpha + \beta z^n, \text{ it is a binomial; if three}$$



$\alpha + \beta z^n + \gamma z^{2n}$ , it is a trinomial; &c. But in what follows I shall consider none beyond the trinomial.

The area might be expressed by a descending series, and from thence other cases of the termination of the series might be pointed out; but this I shall do in the particular forms, as it will in them be done with greater ease.

Nor should it be wondered at that the area admits of two different values; for when an ordinate flows, the area on one side of it will increase as fast as that on the other decreases, consequently the fluxions of those two areas will be equal to each other, that is, the fluxions of the areas are both expressed by the same quantity, and it is therefore but right that the fluxion should admit of two different fluents answering to the two areas on each side of the ordinate. When the expression comes out affirmative, it denotes the area lying on that side of the ordinate from which it is supposed to move; but when it comes out negative, it denotes the area on the other side of the ordinate. When it comes out infinite, it denotes the area lying along the abscissa infinitely produced.

If all the terms of the series  $a + bz^n + cz^{3n} + \&c.$  after the first vanish, and by that means the expression for the ordinate become only

$az^{p-1} \times \alpha + \beta z^n + \gamma z^{2n} + \&c.$ , the curve or area is said to be simple; but if there be more terms than one, it is said to be compound.



In what follows I shall consider chiefly those cases in which the curve is said to be simple; not only because those are the cases that generally happen in practice, but because every compound case may be resolved into so many simple ones as there are terms in the series  $a + bz^n + cz^{2n} + \&c.$  and then by finding the area for every simple case, the aggregate of those areas will be the area for the whole compound case.

In the quantity expressing any area write that particular value of the abscissa which it is supposed to have where the area commences or when it is equal to nothing, and the value of the area resulting from that substitution will be equal to nothing; if the first area were rightly assigned, and then it needs no correction; but if the area by this substitution come out of some value, by just so much will the first area differ from the truth, and must be corrected by subtracting the said value from it.

It may also be observed that when the ordinate is oblique to the abscissa, the area as found by the series must be drawn into the sine of the angle of inclination of the ordinate to the abscissa.

#### EXAMPLE.

If an ordinate  $y$  be  $= \frac{3a - bzz}{zz\sqrt{az - bz^3 + cz^4}}$ .

To have this in the same form with the general series, we must express it thus

$$y = z^{-\frac{3}{2}-1} \times \frac{3a - bzz}{\sqrt{az - bz^3 + cz^4}}^{\frac{1}{2}-1}; \text{ or it may be expressed thus } y = z^{-1-1} \times \frac{-b + 3az^{-2}}{\sqrt{c - bz^{-1} + az^{-3}}}^{\frac{1}{2}-1}.$$

Now



Now by comparing the first of these forms with the general expression of the area, we obtain  $a = 3a$ ,  $b = 0$ ,  $c = -b$ ;  $\alpha = a$ ,  $\beta = 0$ ,  $\gamma = -b$ ,  $\delta = c$ ;  $p = -\frac{3}{2}$ ,  $n = 1$ ,  $q = \frac{1}{2}$ .

Hence  $r = \frac{p}{n} = -\frac{3}{2}$ ,  $s = r + q = -1$ ,  $t = s + q = -\frac{1}{2}$ ,  $v = t + q = 0$ ,  $w = v + q = \frac{1}{2}$ , &c.

Then, by substituting these values in the general series, we have  $z^{-\frac{3}{2}} \times \sqrt{a - bz^2 + cz^3}^{\frac{1}{2}} \times -2$  (all the rest of the terms after the first vanishing)  $= -2\sqrt{\frac{a - bz^2 + cz^3}{z^3}}$ . And because this expression is negative, it denotes the area on the other side of the ordinate.

Again, by comparing the latter form with the general series, we obtain  $a = -b$ ,  $b = 0$ ,  $c = 3a$ ;  $\alpha = c$ ,  $\beta = -b$ ,  $\gamma = 0$ ,  $\delta = a$ ;  $p = -1$ ,  $n = -1$ ,  $q = \frac{1}{2}$ .

Hence  $r = \frac{p}{n} = 1$ ,  $s = r + q = \frac{3}{2}$ ,  $t = s + q = 2$ , &c.

Then substituting these values in the general series, we obtain  $\sqrt{\frac{a - bz^2 + cz^3}{z^3}} \times : \frac{b}{c} + \frac{3bA}{4cz} + \frac{5bB - 6a}{6cz^2} + \frac{7bC - 5aA}{8cz^3} + \frac{9bD - 7aB}{10cz^4} + \&c.$  for the area in this case; where the law of the progression is manifest, and where  $A, B, C, \&c.$  denote the whole coefficients of the first, second, third, &c. terms.

#### COROLLARY I.

When  $b, c, d, \&c.$  are each equal to nothing, the curve will be simple, and the general expression for simple



ple areas becomes  $\frac{az^p}{an} \times \alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \&c. \Big|^q \times$   
 $+\frac{1}{r}$   
 $-\frac{s\beta A}{r+1.\alpha} z^n$   
 $-\frac{t\gamma A + \overline{s+1.\beta B}}{r+2.\alpha} z^{2n}$   
 $-\frac{v\delta A + \overline{t+1.\gamma B} + \overline{s+2.\beta C}}{r+3.\alpha} z^{3n}$   
 $-\frac{w\epsilon A + \overline{v+1.\delta B} + \overline{t+2.\gamma C} + \overline{s+3.\beta D}}{r+4.\alpha} z^{4n}$   
 $\&c.$

## COROLLARY II.

If the curve be only a trinomial, as  $y = az^{p-1} \times$   
 $\alpha + \beta z^n + \gamma z^{2n} \Big|^{q-1}$ , that is,  $\delta, \epsilon, \&c.$  each equal to  
 nothing, the area in the last corollary will become  
 $\frac{az^p}{an} \times \alpha + \beta z^n + \gamma z^{2n} \Big|^q \times : \frac{1}{r} - \frac{s\beta A}{r+1.\alpha} z^n - \frac{\overline{s+1.\beta B} + \overline{t\gamma A}}{r+2.\alpha} z^{2n}$   
 $-\frac{\overline{s+2.\beta C} + \overline{t+1.\gamma B}}{r+3.\alpha} z^{3n} - \frac{\overline{s+3.\beta D} + \overline{t+2.\gamma C}}{r+4.\alpha} z^{4n} - \&c.$

But to find another expression of the area in  
 a descending series, we have the ordinate  $y$  or  
 $az^{p-1} \times \alpha + \beta z^n + \gamma z^{2n} \Big|^{q-1} = az^{p-1+2nq-2n} \times \gamma + \beta z^{-n} + \alpha z^{-2n} \Big|^{q-1}$ ;  
 then by comparing this with the original series,  
 5 S the



the area will come out  $\frac{az^{\frac{p-2n}{n}} \times \alpha + \beta z^n + \gamma z^{2n}}{\gamma n} \times \frac{1}{t-2}$   
 $-\frac{s-2.\beta A}{t-3.\gamma z^n} - \frac{s-3.\beta B + r-2.\alpha A}{t-4.\gamma z^{2n}} - \frac{s-4.\beta C + r-3.\alpha B}{t-5.\gamma z^{3n}}$   
 - &c.

When any particular example is proposed, if, after having substituted in the former or ascending form, it come out an infinite series, let the latter or descending form be tried; for sometimes an area is quadrable by the one series when it is not so by the other.

## EXAMPLE I.

Let the ordinate  $y$  be  $= \frac{a}{\sqrt{z^6 - 2z^7 + 3z^8}}$ ; which reduced to form is  $az^{\frac{p-2n}{n}} \times \frac{1}{1-2z+3z^2}^{\frac{1}{2}}$ .

Here then we have  $a = a$ ;  $\alpha = 1$ ,  $\beta = -2$ ,  $\gamma = 3$ ;  $p = -2$ ,  $n = 1$ ,  $q = \frac{1}{2}$ . Whence  $r = \frac{p}{n} = -2$ ,  $s = r + q = -\frac{3}{2}$ ,  $t = s + q = -1$ ,  $v = t + q = -\frac{1}{2}$ ,  $w = v + q = 0$ , &c.

And by substituting these values in the ascending series, we obtain  $az^{-2} \times \frac{1}{1-2z+3z^2}^{\frac{1}{2}} \times -\frac{1}{2} - \frac{3z}{2}$  (all the rest of the terms vanishing)  $= \frac{1+3z}{2zz} \times -a\sqrt{1-2z+3zz}$  for the area required, and which therefore is quadrable.

But if the same values be substituted in the descending series, we get the area expressed by the infinite



$$\begin{aligned} \text{finite series } \frac{a\sqrt{1-2z+3z^2}}{9z^4} \times &: -1 - \frac{7A}{4z} - \frac{9B-4A}{5z^2} \\ - \frac{11C-5B}{6z^3} - \&c. = \frac{a\sqrt{1-2z+3z^2}}{9z^4} \times &: -1 + \frac{7}{4z} - \frac{9 \cdot 7 + 4 \cdot 4}{4 \cdot 5 z^2} \\ + \frac{11 \times 9 \cdot 7 + 4 \cdot 4 + 5 \cdot 5}{4 \cdot 5 \cdot 6 z^3} - \&c. = \frac{a\sqrt{1-2z+3z^2}}{9z^4} \times &: -1 + \frac{7}{4z} \\ - \frac{79}{4 \cdot 5 z^2} + \frac{894}{4 \cdot 5 \cdot 6 z^3} - \&c. \end{aligned}$$

## EXAMPLE H.

Suppose  $y = \frac{azz}{2+3z+3z^2}^{\frac{1}{3}}$ ; which reduced to the proper form is  $az^{\frac{1}{3}} \times \sqrt[3]{2+3z+3z^2}^{\frac{1}{3}-1}$ .

Here  $a = a$ ;  $\alpha = 2$ ,  $\beta = 3$ ,  $\gamma = 3$ ,  $p = 3$ ,  $n = 1$ ,  $q = \frac{1}{3}$ .

Then  $r = \frac{p}{n} = 3$ ,  $s = 3\frac{1}{3}$ ,  $t = 3\frac{2}{3}$ ,  $v = 4$ , &c.

And, by the ascending series, the area will be expressed by the infinite series  $\frac{1}{2}az^3\sqrt[3]{2+3z+3z^2}$   
 $\times: \frac{1}{3} - \frac{10A}{8}z - \frac{13B+11A}{10}z^2 - \frac{16C+14B}{12}z^3 - \frac{19D+17C}{14}z^4$   
 $- \&c. = az^3\sqrt[3]{2+3z+3z^2} \times: \frac{1}{6} - \frac{10}{6 \cdot 8} + \frac{42}{6 \cdot 8 \cdot 10}$   
 $+ \frac{728}{6 \cdot 8 \cdot 10 \cdot 12} - \&c.$  and therefore it is not quadrable by this method.

But by the descending series the area is quadrable, and comes out  $\frac{z-2}{5} \times a\sqrt{2+3z+3z^2}$ , all the terms after the second vanishing.

So that sometimes the area is quadrable by the one series, and sometimes by the other; but it is also sometimes quadrable by neither, and sometimes by them both.

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## COROLLARY III.

In the case of a binomial  $y = az^{p-1} \times \alpha + \beta z^n$ , beside the letters which in the last corollary were equal to nothing,  $\gamma$  also will be nothing; and by supposing it to vanish in the ascending series in the

last corollary, there will result  $\frac{az^p}{\alpha n} \times \alpha + \beta z^n \times \frac{1}{r}$

$$-\frac{s\beta}{r+1.\alpha}Az^n - \frac{s+1.\beta}{r+2.\alpha}Bz^{2n} - \frac{s+2.\beta}{r+3.\alpha}Cz^{3n} - \frac{s+3.\beta}{r+4.\alpha}Dz^{4n}$$

— &c. for the ascending series to be used in this case; where the several letters have the same value as before. And this series, it is evident, will always terminate when  $s$  is either nothing or a negative integer, and the number of terms will be one more than the number of units in  $s$ .

But the same ordinate  $y = az^{p-1} \times \alpha + \beta z^n$  is =  $az^{p-1+qn-n} \times \beta + \alpha z^{-n}$ . Then by writing, in the above ascending series,  $\alpha$  for  $\beta$ ,  $\beta$  for  $\alpha$ ,  $p+qn-n$

for  $p$ , and  $-n$  for  $+n$ , we shall have  $\frac{az^{p-n}}{\beta n} \times \alpha + \beta z^n$

$$\times: \frac{1}{s-1} - \frac{r-1.\alpha A}{s-2.\beta z^n} - \frac{r-2.\alpha B}{s-3.\beta z^{2n}} - \frac{r-3.\alpha C}{s-4.\beta z^{3n}} - \&c. \text{ for}$$

the area expressed by a descending series; and which, it is evident, will terminate and be quadrable when  $r$  is any affirmative integer number.

Ex-



## EXAMPLE I.

If the ordinate be  $y = \frac{azz}{bb + 2bcz^3 + c^2z^6} = \frac{azz}{b + cz^3}^2$

$$azz \times \overline{b + cz^3}^{-2} = az^{3-1} \times \overline{b + cz^3}^{-1-1}.$$

We shall have  $a = a$ ;  $\alpha = b$ ,  $\beta = c$ ;  $p = 3$ ,  $n = 3$ , and  $q = -1$ . Hence  $r = \frac{p}{n} = 1$ , and  $s = r + q = 1 - 1 = 0$ ; so that the curve is quadrable by both forms of the series. Thus,

By substituting in the ascending series, we shall have  $\frac{az^3}{3c} \times \overline{b + cz^3}^{-1} \times \frac{1}{1} = \frac{az^3}{3b \times b + cz^3}$  for the area on the one side of the ordinate.

And by using the descending series, we obtain  $\frac{az^0}{3b} \times \overline{b + cz^3}^{-1} \times -\frac{1}{1} = -\frac{a}{3c \times b + cz^3}$  for the area on the other side of the ordinate.

## EXAMPLE II.

Suppose the ordinate be  $y = \frac{az^3}{\sqrt{b + cz^2}} = az^3 \times \overline{b + cz^2}^{-\frac{1}{2}}$

$$az^3 \times \overline{b + cz^2}^{-\frac{1}{2}} = az^{4-1} \times \overline{b + cz^2}^{\frac{1}{2}-1}.$$

Then  $a = a$ ;  $\alpha = b$ ,  $\beta = c$ ;  $p = 4$ ,  $n = 2$ , and  $q = \frac{1}{2}$ . Hence  $r = \frac{p}{n} = 2$ , and  $s = r + q = 2\frac{1}{2}$ . So that it appears that the descending series will bring out a terminate, but the ascending one an interminate area. Thus,

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By



By substituting in the ascending series, we have  

$$\frac{az^4\sqrt{b+cz^2}}{2b} \times : \frac{1}{2} - \frac{5cz^2}{2.3.2b} - \frac{5.7c^2z^4}{2.3.4.2^2b^2} - \frac{5.7.9c^3z^6}{2.3.4.5.2^3b^3} - \&c.$$
 for the area. Or by writing  $d$  for  $2b$ , it will be  

$$\frac{az^4\sqrt{b+cz^2}}{d} \times : \frac{1}{2} - \frac{5cz^2}{2.3d} - \frac{5.7c^2z^4}{2.3.4d^2} - \frac{5.7.9c^3z^6}{2.3.4.5d^3} - \&c.$$

But by substituting in the descending series, the area is quadrable, and comes out  $\frac{az^2\sqrt{b+cz^2}}{2c} \times$   

$$\frac{2}{3} - \frac{4b}{3cz} = a\sqrt{b+cz^2} \times \frac{cz - 2b}{3c}.$$

## EXAMPLE III.

If the ordinate  $y$  be  $= \sqrt{a \times b + x} = \sqrt{ab + ax} = a^{\frac{1}{2}}x^0 \times \overline{b+x}^{\frac{1}{2}} = a^{\frac{1}{2}}x^{1-1} \times \overline{b+x}^{\frac{1}{2}-1}$ ; which expresses the common parabola,  $a$  being the parameter, and  $b+x$  the whole abscissa or distance of the ordinate from the vertex of the curve.

Here  $a = a^{\frac{1}{2}}$ ;  $\alpha = b$ ,  $\beta = 1$ ,  $z = x$ ;  $p = 1$ ,  $n = 1$ , and  $q = \frac{3}{2}$ . Hence  $r = \frac{p}{n} = 1$ , and  $s = r + q = \frac{5}{2}$ . So that the curve is quadrable by the descending series, but not by the ascending one. Thus,

By substituting in the descending series, we obtain  $a^{\frac{1}{2}}x^0 \times \overline{b+x}^{\frac{1}{2}} \times \frac{2}{3} = \frac{2}{3}\sqrt{a \times \overline{b+x}^{\frac{3}{2}}}$  for the area.—And since this area vanishes only when  $b+x$  or the whole abscissa is equal to nothing, it denotes the area of the whole parabola. But when  $x$  is  $= 0$ , this area becomes  $\frac{2}{3}\sqrt{a \times b^{\frac{3}{2}}}$  for the area to the abscissa  $b$ ; hence, by correction,  $2\sqrt{a \times \frac{\overline{b+x}^{\frac{3}{2}} - b^{\frac{3}{2}}}{3}}$  will



will be the area of the frustum included by the two ordinates answering to the two abscissas  $b+x$  and  $b$ .

And, after the same manner,  $2\sqrt{a} \times \frac{\overline{b+x}^{\frac{3}{2}} - \overline{b-x}^{\frac{3}{2}}}{3}$  will be found to denote the frustum contained by the two ordinates whose abscissas are  $b+x$  and  $b-x$ .

But by substituting in the ascending series, the area will come out the infinite series  $\frac{x\sqrt{a}}{b} \times \overline{b+x}^{\frac{3}{2}} \times$

$$: 1 - \frac{5x}{4b} + \frac{5.7x^2}{4.6b^2} - \frac{5.7.9x^3}{4.6.8b^3} + \&c.$$

As this series vanishes when  $x$  is put equal to nothing, it expresses the area beginning where  $x$  begins, viz. the frustum included by the two ordinates whose abscissas are  $b$  and  $b+x$ ; and there-

fore this series is equal to  $2\sqrt{a} \times \frac{\overline{b+x}^{\frac{3}{2}} - b^{\frac{3}{2}}}{3}$ , which was found above to express the same area. Then, by making this quantity equal to the said series,

and reducing, we obtain  $\frac{2b}{3x} \times 1 - \overline{\frac{b}{b-x}}^{\frac{3}{2}}$  for the va-

lue or sum of the infinite series  $1 - \frac{5x}{4b} + \frac{5.7x^2}{4.6b^2} - \frac{5.7.9x^3}{4.6.8b^3} + \&c.$  And by supposing  $b$  and  $x$  in these last expressions to be equal to each other, there will result

$$\frac{2\sqrt{2}-1}{3\sqrt{2}} \text{ for the sum of the infinite series } 1 - \frac{5}{4} + \frac{5.7}{4.6} - \frac{5.7.9}{4.6.8} + \frac{5.7.9.11}{4.6.8.10} - \&c. \text{ or } \frac{1+\sqrt{2}}{3\sqrt{2}} = \frac{5}{4} - \frac{5.7}{4.6} + \frac{5.7.9}{4.6.8} - \frac{5.7.9.11}{4.6.8.10} + \&c.$$

Now such kinds of diverging series have generally been thought to be of an infinite magnitude, and their sums absolutely unassignable; but that there is nothing absurd in asserting the foregoing series,



series, and consequently any other of the same kind, to have a determinate finite sum, may be easily made evident thus: Notwithstanding any term of this series doth exceed any of the preceding terms, yet it is evident that it exceeds the term immediately preceding it by a less quantity than *that* by which this last term exceeds *its* preceding one, and therefore the differences of every two adjacent terms being taken, those differences will form an infinite converging series, of which the first or greatest term will be the difference between the first two adjacent terms of the original series, and consequently the sum of this series of differences will be of a finite magnitude; but the sum of this series is equal to the sum of the original series, and therefore that also hath a finite sum.

## COROLLARY IV.

In the case of a simple nominal  $y = az^{p-1} \times a^{q-1} = a\alpha^{q-1} z^{p-1}$ , supposing  $\beta$  in the last corollary to become nothing, the area will be  $\frac{a\alpha^{q-1} z^p}{p} = \frac{mz^p}{p}$ , putting  $m = a\alpha^{q-1}$ .

Or if instead of  $mz^{p-1}$  be written its value  $y$ , the area will be  $\frac{yz}{p}$ .

And the curve in this case is always quadrable except when  $p$  is  $= 0$ ; for then the area  $\frac{mz^p}{p}$  becomes  $\frac{mz^0}{0} = \frac{m}{0}$ , viz. infinitely great.

To



To this case of areas belong all simple parabolic spaces, and the spaces lying between the curves and asymptotes of all hyperbolas. And all such are, evidently, to the parallelograms of the same base and altitude as 1 to  $p$ .

## EXAMPLE I.

The general equation to parabolas is  $p^m z^n = y^{m+n}$ , or  $y = p^{\frac{m}{m+n}} z^{\frac{n}{m+n}}$ ; where  $y$  is the ordinate,  $z$  the abscissa, and  $p$  a given quantity.

Here then  $m = \frac{m}{m+n}$ , and  $p = \frac{n}{m+n} + 1 = \frac{m+2n}{m+n}$ ; which value of  $p$  being of some magnitude, we conclude that all parabolas are quadrable, and the area will be, by substituting in the two forms above, either  $\frac{m+n}{m+2n} \times p^{\frac{m}{m+n}} z^{\frac{n}{m+n}}$ , or  $\frac{m+n}{m+2n} \times yz$ .

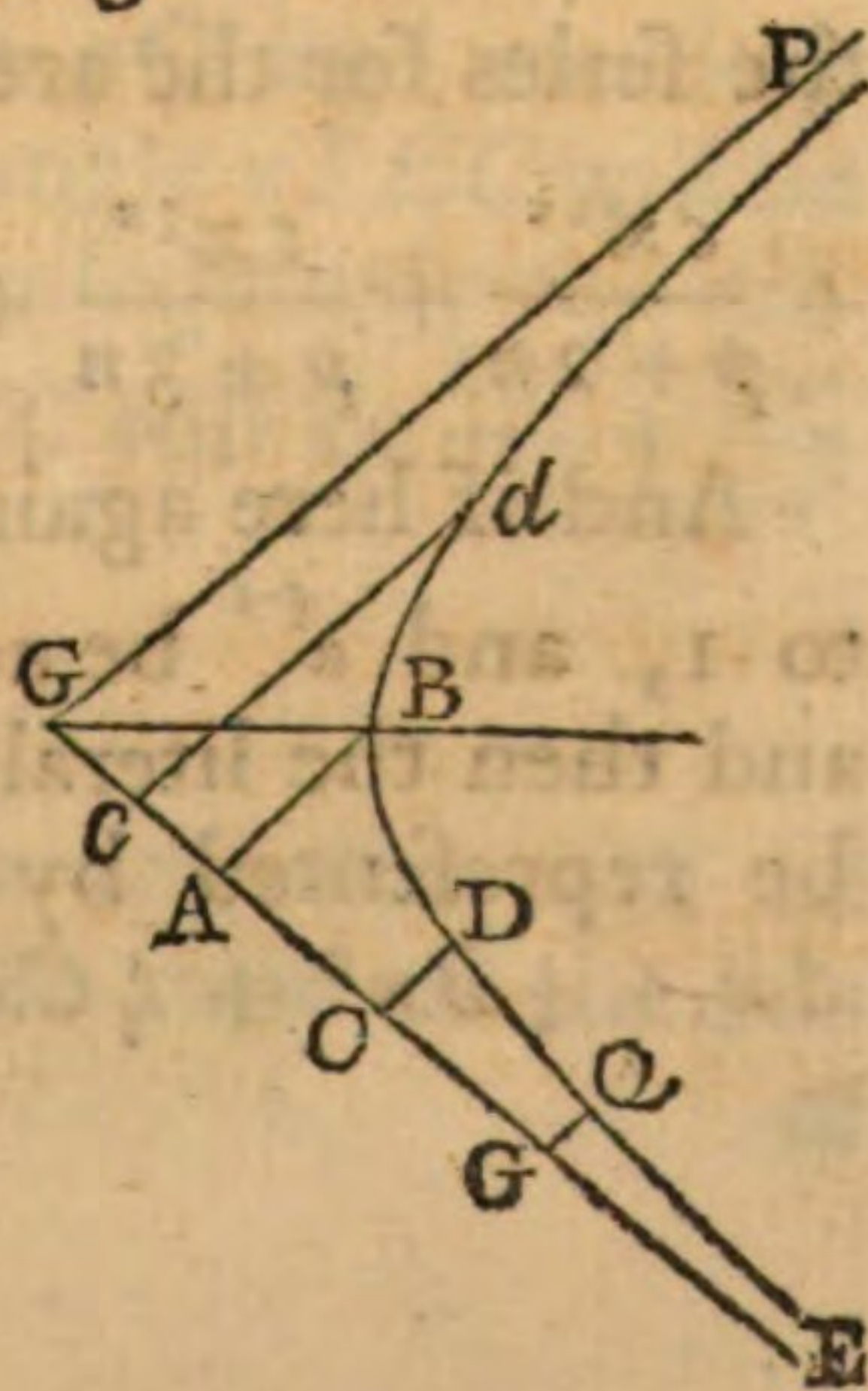
So that the area of any parabola is to that of its circumscribed parallelogram, as  $m+n$  is to  $m+2n$ .

In the common parabola  $m$  and  $n$  are each equal to 1, or  $pz = yy$ , and the general area becomes  $\frac{2}{3} yz$  or  $\frac{2}{3}$  of the circumscribed parallelogram.

## EXAMPLE II.

The general equation to hyperbolas is  $z^m y^n = p^{m+n}$ , or  $y = p^{\frac{m+n}{n}} z^{\frac{m}{n}}$ ; where  $z$  is = one asymptote  $GC$ , and  $y$  = an ordinate  $CD$  drawn parallel to the other  $GP$ .

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Here then  $m = p^{\frac{m+n}{n}}$ , and  $p = 1 - \frac{m}{n} = \frac{n-m}{n}$ ; so that the space will always be quadrable unless when  $n$  and  $m$  are equal to each other, as in the case of the conic parabola; and by substituting in the two foregoing forms, we obtain either  $\frac{np^{\frac{n+m}{n}} z^{\frac{n+m}{n}}}{n-m}$  or  $\frac{nyz}{n-m}$  for the area  $GPdDC$ .

So that hyperbolic spaces are to their inscribed parallelograms as  $\frac{n}{n-m}$  to 1, or as  $n$  to  $n-m$ ; and since when  $n$  is  $= m$ , as in the conic parabola, this ratio becomes that of  $n$  to 0; it appears that the said hyperbolic space  $GPdDC$  is in this case infinitely great; but in any other, of a finite magnitude, notwithstanding it is of an infinite length towards  $P$ .

#### COROLLARY V.

If in the original series, in the proposition, only  $\beta$ ,  $\gamma$ ,  $\delta$ , &c. vanish, by becoming equal to nothing, the series for the area will become  $\alpha^{q-1} z^p \times \frac{a}{p} + \frac{bz^n}{p+n} + \frac{cz^{2n}}{p+2n} + \frac{cz^{3n}}{p+3n} + \&c.$

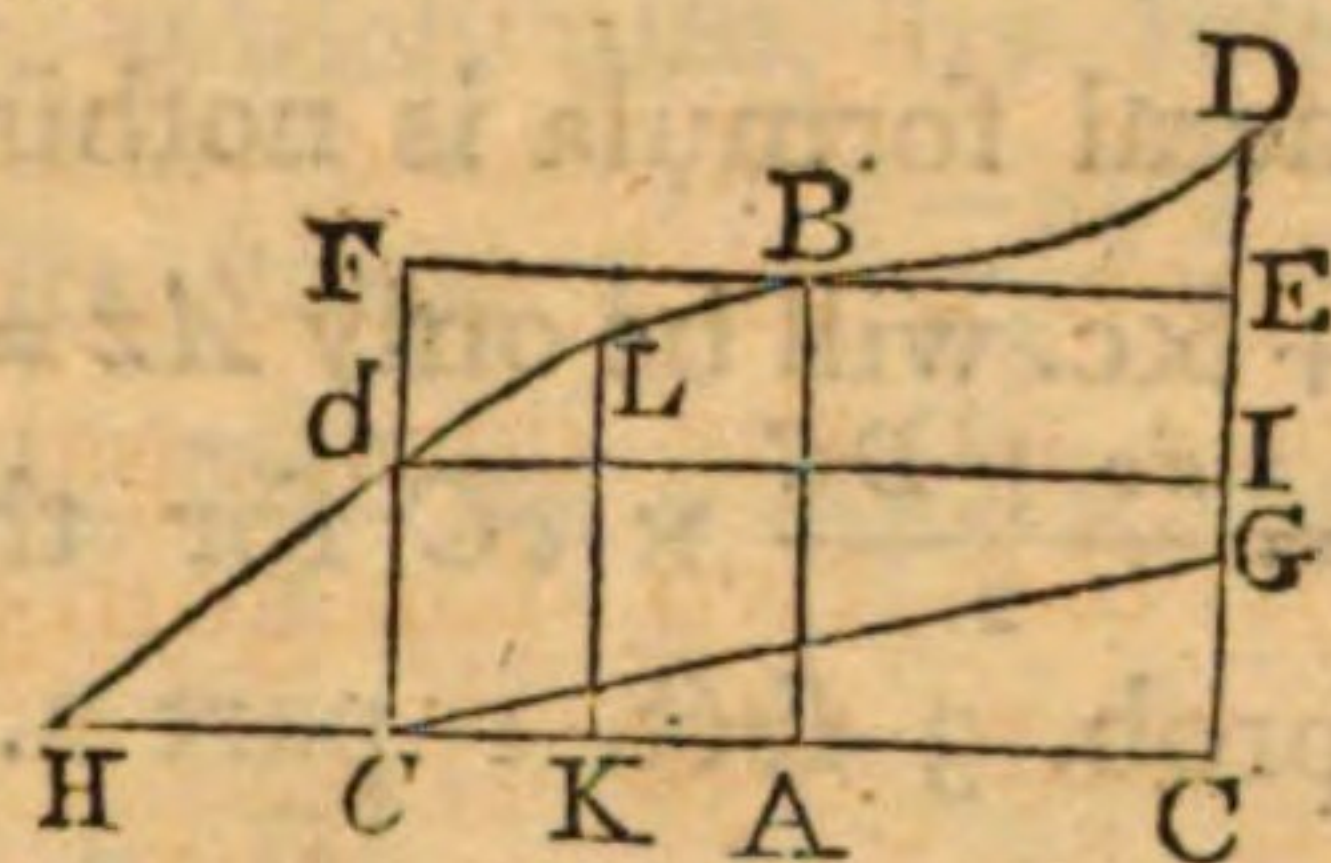
And if here again  $p$  and  $n$  be supposed each equal to 1, and  $\alpha^{q-1}$  be actually multiplied into the series, and then the literal coefficients of the several terms be represented by  $A$ ,  $B$ ,  $C$ , &c. we shall obtain  $Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4 + \&c.$  for the area

cor.



corresponding to the ordinate  $y = A + Bx + Cx^2 + Dx^3 + \&c.$  putting  $x$  for  $z$ .

And by this series we might find a great variety of areas; but to reduce it to another series converging quicker; since, if  $x$  be  $= AC$ ,  $y = CD$ , the above series will represent the area  $ABDC$  of the curve  $dBD$ ; so likewise if  $y$  flow back from  $A$  to  $cd$ , making the ordinate  $y = A - Bx + Cx^2 + Dx^3 + \&c.$   $x$  being now negative, the area  $ABdc$  will be  $Ax - \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 - \frac{1}{4}Dx^4 + \&c.$



Then, supposing the former  $x$  or  $AC$  to be equal to this latter  $x$  or  $Ac$ , and adding this series to the former one, there will result  $2Ax + \frac{2}{3}Cx^3 + \frac{2}{5}Ex^5 + \&c.$  for the area  $cdDC$ . Or if now  $z$  be put  $= cC = 2x$ , this area will be equal to  $Az + \frac{Cz^3}{3 \cdot 4} + \frac{Ez^5}{5 \cdot 4^2} + \frac{Gz^7}{7 \cdot 4^3} + \&c.$  where  $c$  may either be at the vertex of the curve or not.

When  $x$  is  $= 0$ , then  $y = A = AB$ , so that  $AB$  is always the value of  $A$ , or  $A$  denotes the middle ordinate of the area  $cdDC$ .

If  $FBE$  be drawn parallel to  $cC$ , and meeting  $CD$  and  $cd$  in  $E$  and  $F$ ; the rectangle  $cFEC$  will be equal to  $cC \times AB = Az$ , and consequently the difference between  $cdDC$  and  $cFEC$  will be  $\frac{Cz^3}{3 \cdot 4} + \frac{Ez^5}{3 \cdot 4^2} + \&c.$

#### EXAMPLE I.

If  $dBD$  be a right line, its equation will be  $y = A + Bx$ ;  $B$  here denoting the ratio of the sine of



of the angle  $dBA$  to its cosine. Then, since the value of each of the letters  $C, D, E, \&c.$  in the general formula is nothing, the general area  $Az + \frac{Cz^3}{3 \cdot 4} + \&c.$  will be only  $Az =$  the rectangle  $CF = AB \times cC = \frac{dc + DC}{2} \times cC$  for the area  $dDCc$ , as in rule 6 prob. 3 sect. 1 part 2.

## EXAMPLE II.

If  $dBD$  be a parabola, its equation will be  $y = \sqrt{a \times b + x} =$  (by extracting the root)  $\sqrt{ab} \times : 1 + \frac{x}{2b} - \frac{x^2}{2 \cdot 4 b^2} + \frac{1 \cdot 3 x^3}{2 \cdot 4 \cdot 6 b^3} - \frac{1 \cdot 3 \cdot 5 x^5}{2 \cdot 4 \cdot 6 \cdot 8 b^4} + \&c.$

Here  $A = \sqrt{ab}$ ,  $C = -\frac{A}{2 \cdot 4 b^2}$ ,  $E = -\frac{1 \cdot 3 \cdot 5 A}{2 \cdot 4 \cdot 6 \cdot 8 b^4}$ ,  $\&c.$  and, by substituting, the area  $cdDC$  will be  $= 2A$  (viz.  $2\sqrt{ab}$ )  $\times : x - \frac{x^3}{4 \cdot 6 b^2} - \frac{1 \cdot 3 \cdot 5 x^5}{4 \cdot 6 \cdot 8 \cdot 10 b^4} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 x^7}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 b^6} - \&c. = Az \times : 1 - \frac{x^2}{4 \cdot 6 \cdot 2^2 b^2} - \frac{1 \cdot 3 \cdot 5 x^4}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 2^4 b^4} - \&c.$  And therefore the rectangle  $Az$  or  $BA \times cC$  exceeds the parabolic area by all the terms of the series after the first.

But by example 3 to corollary 3, the same area  $cdDC$  is  $= 2\sqrt{a} \times \frac{\overline{b+x}^{\frac{3}{2}} - \overline{b-x}^{\frac{3}{2}}}{3}$ ; consequently  $\frac{\overline{b+x}^{\frac{3}{2}} - \overline{b-x}^{\frac{3}{2}}}{3}$  is  $= \sqrt{b} \times : x - \frac{x^3}{4 \cdot 6 b^2} - \frac{1 \cdot 3 \cdot 5 x^5}{4 \cdot 6 \cdot 8 \cdot 10 b^4} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 x^7}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 b^6} - \&c.$  And when  $b$  and  $x$  are equal to each other in this equation, we obtain  $\frac{3 - 2\sqrt{2}}{3}$  for the value of the infinite series  $\frac{1}{4 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8 \cdot 10} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} + \&c.$  When



When  $c$  is the vertex of the curve, then  $b$  is  $= x$ , and the series for the area  $cDC$  becomes  $Az \times : 1 - \frac{1}{4.6} - \frac{1.3.5}{4.6.8.10} - \&c.$  which also, by substituting for this last series its value  $1 - \frac{3-2\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}$  as found above, will be  $= \frac{2\sqrt{2}}{3} Az = \frac{2\sqrt{2}}{3} \times$  the rectangle  $cFEC$ .

## EXAMPLE III.

If the curve be a circle, and  $A$  were any point between the circumference and the center, the series would be very complex; but if  $A$  be the center, and  $r$  the radius, then will  $y$  be  $= \sqrt{rr - xx}$   $= r \times : 1 - \frac{x^2}{2r^2} - \frac{x^4}{2.4r^4} - \frac{1.3x^6}{2.4.6r^6} - \&c.$  and consequently the area will be  $2rx \times : 1 - \frac{x^2}{2.3r^2} - \frac{x^4}{2.4.5r^4} - \frac{1.3x^6}{2.4.6.7r^6} - \frac{1.3.5x^8}{2.4.6.8.9r^8} - \&c.$  which series converges very fast when  $x$  is small in comparison with  $r$ .

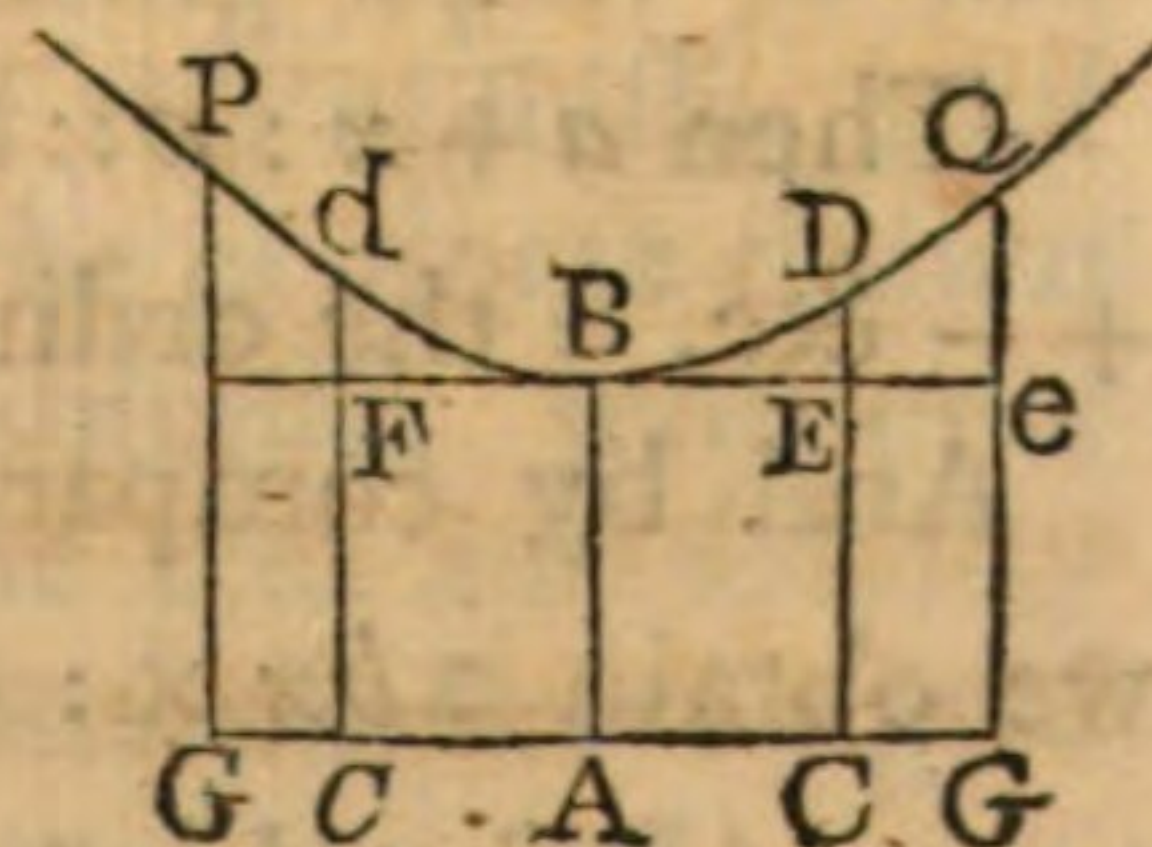
When  $x$  becomes equal to  $r$ , we obtain  $4rr$  (or the square of the diameter)  $\times : 1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7} - \frac{1.3.5}{2.4.6.8.9} - \&c.$  for the whole circle.

Hence  $1 - .785398 \&c.$  is  $= \frac{1}{2.3} + \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7} + \&c.$

## EXAMPLE IV.

If  $dBD$  be an hyperbola; put its semi-transverse  $AB = t$ , semi-conjugate  $AG = c$ , abscissa  $AC = x$ , and ordinate  $CD = y$ ; then its equation will be  $yy = tt \times \frac{cc + xx}{cc}$ , or

5 X

 $y =$



$y = t\sqrt{1 + \frac{xx}{cc}} = tx : 1 + \frac{x^2}{2c^2} - \frac{x^4}{2.4c^4} + \frac{1.3x^6}{2.4.6c^6} - \frac{1.3.5x^8}{2.4.6.8c^8}$   
 + &c. and, by substituting in the general series,  
 the area  $cdDC$  will be  $= 2tx \times : 1 + \frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4}$   
 $+ \frac{1.3x^6}{2.4.6.7c^6} - \&c.$  the same with the series in the last  
 example for the area of a circle, differing only in  
 the signs of the second, fourth, sixth, &c. terms.

When  $x$  is  $= c$ , the series becomes  $2tc$  (or  $AB \times GG$ )  
 $\times : 1 + \frac{1}{2.3} - \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7} - \&c.$  for the area  $GPQG$ .

The rectangle  $ABEC$  is  $= tx$ , which taken from  
 the area  $ABDC$  will leave  $AB \times BE \times : \frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4}$   
 $+ \frac{1.3x^6}{2.4.6.7c^6} - \&c.$  for the area  $BDE$ . So that the  
 rectangle  $ABEC$  is to the trilineal  $BDE$ , as  $1$  is  
 to the series  $\frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6} - \&c.$  And  
 when  $x$  or  $AC$  is  $= c$  or  $AG$ , the rectangle  $ABeG$   
 will be to the trilineal  $BeQ$ , as  $1$  is to  $\frac{1}{2.3} - \frac{1}{2.4.5}$   
 $+ \frac{1.3}{2.4.6.7} - \&c.$

Parallel to one asymptote  $GP$  (figure to ex-  
 ample 2 cor. 4) of an hyperbola draw ordinates  
 $BA, CD$ , to the other  $GG$ ;  $B$  being the vertex  
 of the curve: And put  $GA = a$ ,  $AB = b$ ,  $AC = x$ ,  
 and  $CD = y$ .

Then  $a + x : a :: b : y = \frac{ab}{a+x} = b \times : 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3}$   
 $+ - \&c. =$  the ordinate.

And by comparing this with the general series,  
 we obtain  $2bx \times : 1 + \frac{x^2}{3a^2} + \frac{x^4}{5a^4} + \frac{x^6}{7a^6} + \&c.$  for the  
 area  $cdDC$ , taking  $Ac = AC$ .

And



And when  $x = a$ , or  $AC = AG$ , the above expression will become  $2GA \times AB \times : 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c.$  for the area  $GPBQG$ .

## PROPOSITION II.

*If there be any solid, in which all the parallel sections, either right or oblique to the axe, are like and similar figures, and if, when the abscissa, or part of the axe drawn through the centers of the parallel sections, is represented by  $z$ , the value of the section be expressed by any series of terms involving  $z$  and known quantities, after the same manner as is expressed the ordinate in the last proposition; then will the solidity be expressed by the same quantity as the area in the last proposition: That is, supposing the relation between the abscissa and section to be expressed by any equation of which one side is the section, after the manner of the relation between the abscissa and ordinate in the last proposition; whenever the value of the ordinate agrees with that of the section, then will the value of the solidity be the same with that of the area as found by the said proposition.*

## DEMONSTRATION.

For, the fluxion of the solid being equal to the section drawn into the fluxion of the abscissa, and the fluxion of the area equal to the ordinate drawn into the same fluxion of the abscissa, whenever the ordinate and section agree, the area and solidity must likewise agree, since equal fluxions have equal fluents. *Q.E.D.*

SCHO-



## SCHOLIUM.

All the examples that have been given to the corollaries to the last proposition may be considered as examples to this, the quadrature in those being considered as the cubature here; and it is evident that curves will be cubable not only whenever they are quadrable, but will often be cubable when the area cannot be expressed but in an infinite series, because that the section of a solid (using the area of a given circle as a given number) is often a terminate expression when the ordinate is denoted by an infinite series; and this is the case in the conic sections; for those curves (the parabola excepted) are not duadrable, yet all the solids generated by their revolution have finite expressions, as we have already found in the foregoing part of this work, and as will more generally appear in what follows; which I have put down not merely to shew that the same conclusions may be brought out by different means, but chiefly for the sake of some easy, curious, and general rules with which this method of investigation so easily supplies us.

## COROLLARY.

Thus, by applying here what is done in the last corollary of the last proposition to the curves of the second kind, we shall obtain general and terminate expressions for their solidities. For, by prop. 2 sect. 2 part 3, the flowing section being always as  $A + Bx + Cxx$ , if  $p$  denote a given quantity,  $p \times A + Bx + Cxx$  will denote the section itself; and



and consequently  $px \times \overline{A + \frac{1}{2}Bx + \frac{1}{3}Cx^2}$  or  $p \times \overline{Az + \frac{1}{12}Cz^3}$  will denote the solidity  $cdDC$  generated by the section of a solid in flowing from the place  $cd$  to the place  $CD$ ,  $z$  being  $= cC$ . [Fig. to corollary 5 to the last prop.]

Since  $pA$  represents the middle section  $AB$ , the first term  $pAz$  in the above expression will be equal to the cylinder  $cFBEc$  whose base is equal to the middle section  $AB$ , and altitude  $cC = z$ ; and consequently  $\frac{1}{12}pCz^3$  will always be the difference between the solid  $cdBDC$  and the cylinder  $cFBEc$ .

Or if there be taken  $CG : Cc :: \sqrt{C} : 1$ , and the three areas, viz. the curve  $cdBDC$ , the rectangle  $cFBEc$ , and the right-angled triangle  $cGC$ , be supposed to revolve together about the common axe  $cC$ ; then will the difference between the solid generated by  $cdBDC$  and the cylinder generated by  $cFBEc$  be constantly equal to one-fourth of the cone generated by  $cGC$ .

The frust. generated by  $cdBDC$  is  $\left\{ \begin{array}{l} \text{greater than} \\ \text{equal to} \\ \text{or less than} \end{array} \right\}$   
the cylinder generated by  $cFBEc$  according as the  
value of  $C$  is  $\left\{ \begin{array}{l} \text{affirmative} \\ \text{nothing} \\ \text{or negative} \end{array} \right\}$ , that is, according as  
the curve  $dBd$  is  $\left\{ \begin{array}{l} \text{an hyperbola} \\ \text{a parabola} \\ \text{or an ellipse} \end{array} \right\}$ .

In the parabola  $C$  is  $= 0$ , the equation to the curve being  $yy = A + Bx$ ; and therefore the parabolic conoid, or any frustum of it, is equal to a



cylinder of the same height with it, and whose end is equal to the middle section of the conoid, viz. the section parallel to and equally distant from its two ends.

In the hyperbola  $\sqrt{C}$  is  $= \frac{n}{m}$ , putting  $m$  for the whole axe of which  $cC$  is the continuation, and  $n$  for its conjugate axe; and therefore the hyperbolic conoid, or its frustum, exceeds the cylinder by one-fourth of a cone of which the radius of the base is to the common altitude ( $cC$  of all the three solids) as  $n$  is to  $m$ .

But in the ellipse the value of  $C$  is  $-\frac{n^2}{m^2}$ ,  $m$  and  $n$  being the axes as above; and therefore the semi-spheroid, or its frustum, is less than the cylinder by one-fourth of the said cone of which the radius of the base is to the common altitude as  $n$  is to  $m$ .

When the ellipse becomes a circle,  $m$  will be equal to  $n$ ; and consequently the semi-sphere, or its frustum, will be less than the cylinder by one-fourth of the cone of which the radius of the base is equal to the common altitude of all the solids.

When  $dBD$  is a right line, or the solid a cone, then the value of  $\sqrt{C}$  or  $\frac{GC}{Cc}$  will be equal to  $\frac{DC}{CH}$ , producing the right lines  $Dd$ ,  $Cc$  to meet in  $H$ ; consequently the triangles  $DHC$ ,  $GcC$  are similar, and the conic frustum exceeds the cylinder by one-fourth of a similar cone, the solids being still all of the same altitude. The radius  $CG$  of the base of the said similar cone will evidently be  $= \frac{DC \times Cc}{CH}$  a fourth proportional to  $HC$ ,  $CD$  and  $Cc$ : Or, if  $dI$  be



be drawn parallel to  $cC$ , the triangle  $dID$  will be equal to the triangle  $cCG$  in all respects, or  $CG = ID = DC - cd$ , that is, the radius  $CG$  of the base of the similar cone is equal to the difference between the greatest  $CD$  and least  $cd$  radius of the frustum.—When the frustum becomes a compleat cone, it is evident that it will exceed the cylinder by one-fourth of itself, or the cone will be to the cylinder as 4 is to 3.

From the general manner of considering the generation of the frustum, in the beginning of this corollary, by the parallel motion of a flowing section, it is evident that the above properties will obtain in the same solids whether the ends are perpendicular or oblique to the axe; and also that the general method will include the frustum of any pyramid whether right or oblique; and such a frustum will exceed the prism, of the same altitude and whose base is equal to the middle section of the frustum, by one-fourth of a cone of the same altitude also and of which the radius of the base

is  $\frac{DC \times Cc}{CH}$ .

It may farther be observed in general that in the same or in similar solids, when the altitude and the inclination of the ends to the axe are the same, the cone, or difference between the frustum of the solid and the cylinder or prism, will be constantly the same quantity, whatever the magnitude of the ends may be. When the altitude is constant, and the inclinations of the ends vary, the said difference is reciprocally as the cube of the diameter of the generating plane which is conjugate



gate to  $Cc$  the axe of the frustum or diameter connecting the centers of its ends. When both the altitude and inclination vary, the difference is as the cube of the altitude directly and cube of the said conjugate diameter reciprocally; but when they vary so as that the altitude is always reciprocally as that diameter, then the difference is a constant quantity.

### PROPOSITION III.

*In the frustum of any solid generated by the revolution of any conic section about its axe, if to the sum of the two ends be added four times the middle section or section parallel to and equally distant from the two ends, one-sixth of the last sum will be a mean area, and being drawn into the altitude of the solid, the product will be equal to the content.*

### DEMONSTRATION.

By the corollary to the last prop. the content is  $px \times A + \frac{1}{2}Bx + \frac{1}{3}Cx^2 = px \times \frac{6A + 3Bx + 2Cx^2}{6}$ ; but by prop. 2 sect. 2 part 3, the one end  $pD^2$  is  $= p \times A + Bx + Cx^2$ , then by writing  $\frac{1}{2}x$  for  $x$  the middle section will be  $p\delta^2 = p \times A + \frac{1}{2}Bx + \frac{1}{4}Cx^2$ , and by writing 0 for  $x$  the other end will be  $pd^2 = pA$ ; now it is evident that the sum of the two ends with four times the middle section is equal to the numerator of the quantity expressing the content; consequently  $px \times \frac{D^2 + 4\delta^2 + d^2}{6} (= px \times \frac{6A + 3Bx + 2Cx^2}{6})$  will be equal to the solidity of the frustum. Q.E.D.

Co-



## COROLLARY I.

When the frustum becomes the compleat solid, the less end vanishes, and the content is barely

$$px \times \frac{D^2 + 4d^2}{6}.$$

## COROLLARY II.

This proposition, it is evident, includes all frustums (as well as the compleat solids) whether right or oblique, not only of the solids generated from the revolution of the conic sections, but also of all pyramids, cones, and in short of any solid whose parallel sections are similar figures. The same theorem may also be applied to the areas of all curves whose equation is of this form  $y = A + Bx + Cx^2$ , calling  $D$  and  $d$  the extreme ordinates, and  $\delta$  the middle one.

And of this form is the equation to the parabola  $dBD$  (figure to corollary 5 to prop. 1)  $CD$  being the axe of the curve, and putting  $cA = x$ , and  $AB = y$ ; and consequently any parabolic portion  $cdBA$  bounded by the curve  $dB$ , the two lines  $AB$ ,  $cd$  parallel to the axe  $CD$ , and the line  $Ac$  perpendicular to the same, is truly expressed by  $\frac{AB + 4KL + cd}{6} \times Ac$ ,  $KL$  being the middle ordinate.



## SECTION II.

*Of the approximate QUADRATURE and CUBATURE of CURVES by means of EQUIDISTANT ORDINATES or SECTIONS.*

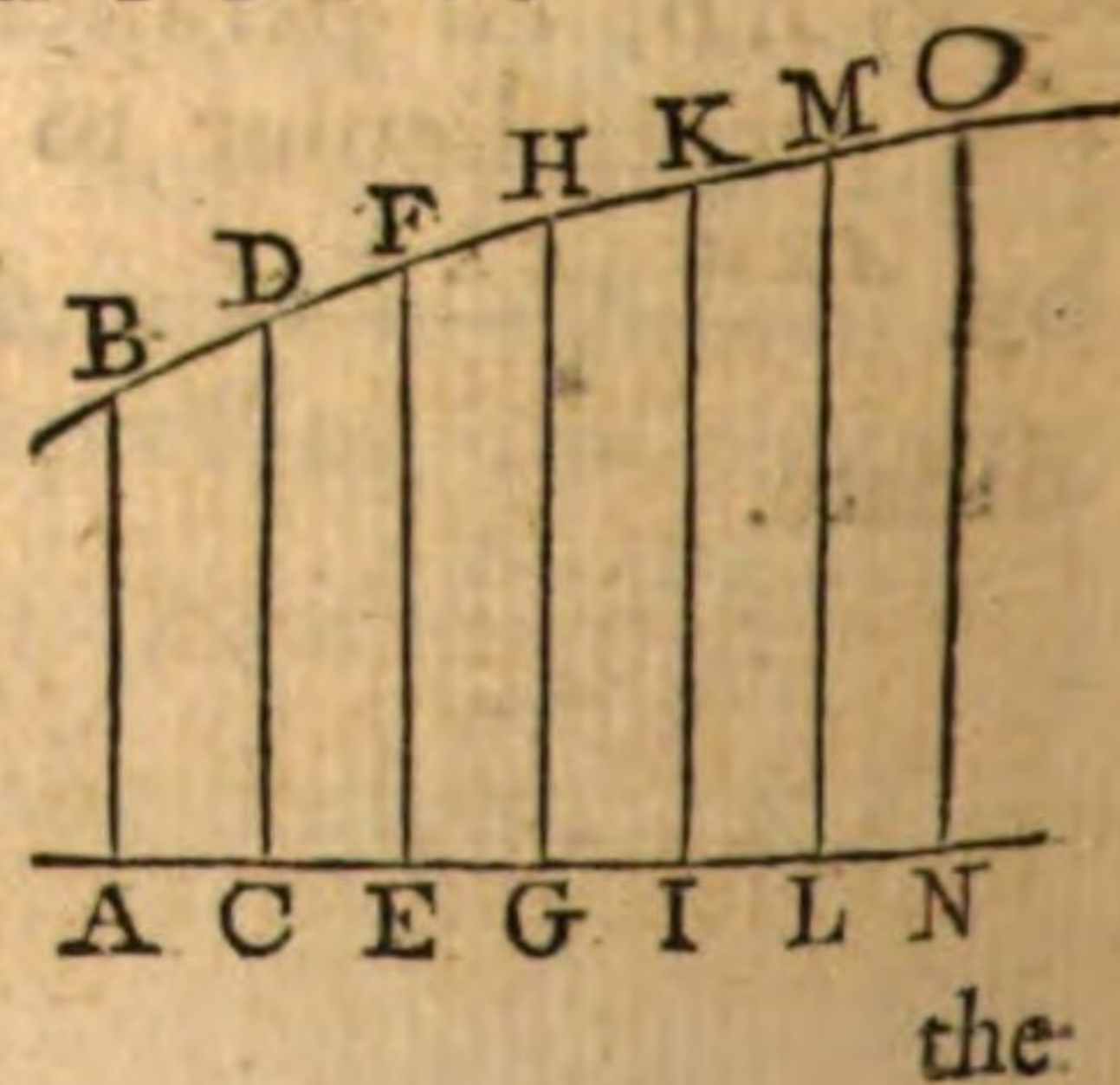
## PROPOSITION I.

*IF any right line AN be divided into any even number of equal parts AC, CE, EG, &c. and at the points of division be erected perpendicular ordinates AB, CD, EF, &c. terminated by any curve BDF &c. and if A be put for the sum of the extreme or first and last ordinates AB, NO; B for the sum of the even ordinates CD, GH, LM, &c. viz. the second, fourth, sixth, &c. and C for the sum of all the rest EF, IK, &c. viz. the third, fifth, &c. or the odd ordinates wanting the first and last: Then I say that the common distance AC or CE, &c. of the ordinates being drawn into the sum arising from the addition of A, 4 times B, and 2 times C, one-third of the product will be the area ABON very nearly.*

*That is,  $\frac{A + 4B + 2C}{3} \times D = \text{the area, putting } D = \text{AC the common distance of the ordinates.}$*

## DEMONSTRATION.

If through the first three points B, D, F, of the proposed curve a parabola be conceived to be drawn, having its axe parallel to the ordinates; the parabolic area, by the second corollary of



the



the last proposition will be  $\frac{AB + 4CD + EF}{6} \times AE$   
 $= \frac{AB + 4CD + EF}{3} \times AC$ ; but when the points  $B$ ,  
 $D$ ,  $F$ , are at no great distance from each other,  
the parabolic curve will nearly coincide with the  
curve proposed, and consequently the area of the  
one equal to that of the other very nearly; hence  
 $\frac{AB + 4CD + EF}{3} \times AC$  will be equal to the area  
 $ABFE$  very nearly. After the same manner  
 $\frac{EF + 4GH + IK}{3} \times AC$  or  $(EG)$  will be equal to  
the area  $EFGI$ ; and  $\frac{IK + 4LM + NO}{3} \times AC =$  the  
area  $IKON$ ; and so on.

Wherefore, by taking the sum of these areas, we  
shall have  $\frac{AB + 4CD + 2EF + 4GH + 2IK + 4LM + NO}{3}$   
 $\times AC = \frac{A + 4B + 2C}{3} \times AC$  for the whole area  $ABON$   
very nearly. Q.E.D.

## COROLLARY.

The same theorem will also obtain for the con-  
tents of all solids, by using the sections perpendi-  
cular to the axe instead of the ordinates, as will  
appear from the last proposition of the last section;  
for the frustum of the solid there may be supposed  
to coincide in the extremes and middle with any  
other frustum, after the same manner as the para-  
bola with any other curve. The proposition is ac-  
curately true for all parabolic and right-lined areas,  
as also for all solids generated from the revolution of  
conic sections or right lines, with all kinds of py-  
ramids;



ramids ; and for all other kinds of areas and solidities it is a very good approximation.

#### SCHOLIUM.

It is very evident that the greater the number of ordinates or sections are used, the more accurately will the area or solidity be determined ; but in a case of real practice, such as cask-gauging, three sections are sufficient ; and because of the simplicity and accuracy of this method, it is by far the most proper for practical gaugers of any thing that can well be devised ; for by only taking the bung and head diameters and a diameter in the middle between them, the sum of the bung, head, and 4 times the middle circle drawn into half the length of the cask will be six times the content extremely near. And the same method may be used to good purpose in all cases of ullaging either standing or lying casks, by taking the area at the top, bottom, and middle of the liquor.

I shall subjoin two or three examples to shew the ease and accuracy of the method, referring those who would see it more fully illustrated by examples &c. to *Mr Moss's Gauging*.

#### EXAMPLE I.

Let it be required to find the area of the quadrant of a circle whose radius is 1.

Let the radius be divided into 10 equal parts by 11 ordinates, the first being nothing and the last equal to the radius, and the intermediate 9 ordinates the sines of the arcs whose versed sines are respectively  $\frac{1}{10}$ ,  $\frac{2}{10}$ ,  $\frac{3}{10}$ ,  $\frac{4}{10}$ ,  $\frac{5}{10}$ ,  $\frac{6}{10}$ ,  $\frac{7}{10}$ ,  $\frac{8}{10}$ , and  $\frac{9}{10}$ .

Now,



Now, by the property of the circle, we shall have

$\sqrt{1.9 \times .1} = \sqrt{.19} = .4358899 =$  the 2d ordinate, or  
the first of the aforefaid nine,

$\sqrt{1.8 \times .2} = \sqrt{.36} = .6 =$  the 3d ordinate,

$\sqrt{1.7 \times .3} = \sqrt{.51} = .7141428 =$  the 4th,

$\sqrt{1.6 \times .4} = \sqrt{.64} = .8 =$  the 5th,

$\sqrt{1.5 \times .5} = \sqrt{.75} = .8660254 =$  the 6th,

$\sqrt{1.4 \times .6} = \sqrt{.84} = .9165151 =$  the 7th,

$\sqrt{1.3 \times .7} = \sqrt{.91} = .9539392 =$  the 8th,

$\sqrt{1.2 \times .8} = \sqrt{.96} = .9797959 =$  the 9th,

$\sqrt{1.1 \times .9} = \sqrt{.99} = .9949874 =$  the 10th,

the 11th, being = 1 the radius, as was before observed.

Hence  $A = 0 + 1 = 1$ ,  $B =$  the 2d + 4th + 6th + 8th + 10th = 3.9649847, and  $C =$  the 3d + 5th + 7th + 9th = 3.296311, and  $D = .1$ .

Consequently  $\frac{A + 4B + 2C}{3} \times D = \frac{23.45256}{3} \times .1 = .78175 =$  the area pretty near the truth.

This area differs from the truth by more than in curves in general, on account of the obliquity of the curve to the ordinates near the beginning of the arc; but if by the same rule we find the arc belonging to the 9 greatest ordinates, which will be .70363, and to it add the area of the semi-segment whose altitude is .2 and base .6, considering it as a parabola, viz.  $.6 \times .2 \times \frac{2}{3} = .08$ , the sum .7836 will be nearer the true area; and if .08175 the true area of the segment had been added, we should have had .78538 the area very exact.



## EXAMPLE II.

Taking example 1 to prob. 5 of the parabola, in which the abscissa of a parabola is 2, and the base or double ordinate 12, to find the area of the parabola.

Here by taking three ordinates, of which the first and last are each nothing, and the middle one = 2, their common distance being 6; we shall have  $A = 0$ ,  $B = 2$ ,  $C = 0$ , and  $D = 6$ .

Consequently  $\frac{A + 4B + 2C}{3} \times D = \frac{8}{3} \times 6 = 16 =$  the true area as before.

## EXAMPLE III.

Given the lengths of five equidistant ordinates of an area, or sections of a solid, 10, 11, 14, 16, 16, and the length of the whole base 20; to find the area or solidity.

Here  $A = 10 + 16 = 26$ ,  $B = 11 + 16 = 27$ ,  $C = 14$ , and  $D = \frac{20}{4} = 5$ .

Wherefore  $\frac{A + 4B + 2C}{3} \times D = \frac{26 + 108 + 28}{3} \times 5 = \frac{162}{3} \times 5 = 54 \times 5 = 270 =$  the area or solidity required.

## EXAMPLE IV.

Given eleven ordinates to an hyperbola between the asymptotes, to find the area. Thus, taking the equation at the end of example 4 to corollary 5 to prop. 1 sect. 1, in which any ordinate  $y$  is =  $\frac{ab}{x+x} = \frac{1}{1+x}$ ; then supposing the first value of  $x$  to be



be nothing, and the last value equal to 1, and consequently the common distance of the ordinates  $\frac{1}{10}$  or  $\frac{1}{10}$ ; we shall have for the ordinates  $\frac{1}{1+0}, \frac{1}{1+1}, \frac{1}{1+2}, \&c.$  or  $\frac{10}{10}, \frac{10}{11}, \frac{10}{12}, \frac{10}{13}, \frac{10}{14}, \frac{10}{15}, \frac{10}{16}, \frac{10}{17}, \frac{10}{18}, \frac{10}{19}$  and  $\frac{10}{20}$ .

Here then  $A = \frac{10}{10} + \frac{10}{20} = 1 + \frac{1}{2} = 1.5$ ,  $B = \frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} = 3.4595393$ ,  $C = \frac{10}{12} + \frac{10}{14} + \frac{10}{16} + \frac{10}{18} = 2.7281745\frac{5}{8}$ , and  $D = 1$ .

Wherefore  $\frac{A + 4B + 2C}{3} \times D = .69315021 =$  the area required.

## PROPOSITION II.

*Given any equidistant ordinates of a curve, or sections of a solid, as a, b, c, d, e, f, &c. to find nearly a general expression of the area or solidity by the terms a, b, c, &c. and the base upon which they insist.*

## INVESTIGATION.

The first differences of the terms  $a, b, c, d, e, f, \&c.$  are  $b - a, c - b, d - c, e - d, f - e, \&c.$

The dif. of these are

$c - 2b + a, d - 2c + b, e - 2d + c, f - 2e + d, \&c.$

The dif. of these are

$d - 3c + 3b - a, e - 3d + 3c - b, f - 3e + 3d - c, \&c.$

The dif. of these are

$e - 4d + 6c - 4b + a, f - 4e + 6d - 4c + b, \&c.$

The dif. of these are

$f - 5e + 10d - 10c + 5b - a, \&c.$

Then by putting  $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$  for the first term of these differences respectively, and transposing, we shall have

$b =$



$$b = +a + \alpha$$

$$c = -a + 2b + \beta$$

$$d = +a - 3b + 3c + \gamma$$

$$e = -a + 4b - 6c + 4d + \delta$$

$$f = +a - 5b + 10c - 10d + 5e + \epsilon$$

$$\&c. \qquad \&c.$$

Hence, by substituting the value of  $b$  in that of  $c$ ; and the values of  $b$  and  $c$  in that of  $d$ ; and those of  $b$ ,  $c$ , and  $d$  in that of  $e$ ; and those of  $b$ ,  $c$ ,  $d$ , and  $e$  in that of  $f$ ; and so on; we shall obtain

$$a = a$$

$$b = a + \alpha$$

$$c = a + 2\alpha + \beta$$

$$d = a + 3\alpha + 3\beta + \gamma$$

$$e = a + 4\alpha + 6\beta + 4\gamma + \delta$$

$$f = a + 5\alpha + 10\beta + 10\gamma + 5\delta + \epsilon$$

$$\&c. \qquad \&c.$$

Where it is evident that the coefficients in the value of any term are the same with those of a binomial raised to the power denoted by the place or number of the term, and consequently the  $x$  term  $z$  will be  $a + x\alpha + x \cdot \frac{x-1}{2} \beta + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} \gamma + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} \cdot \frac{x-3}{4} \delta + \&c.$  Which, by actually multiplying the factors together,  $\&c.$  becomes  $z = a$

$$\begin{aligned}
 &+ \alpha x + \frac{1}{2} \beta x^2 + \frac{1}{2 \cdot 3} \gamma x^3 + \&c. \\
 &- \frac{1}{2} \beta x - \frac{1+2}{2 \cdot 3} \gamma x^2 - \frac{1+2+3}{2 \cdot 3 \cdot 4} \delta x^3 - \&c. \\
 &+ \frac{1 \cdot 2}{2 \cdot 3} \gamma x + \frac{1 \cdot 2 + 3 \cdot 1 + 2}{2 \cdot 3 \cdot 4} \delta x^2 + \frac{1 \cdot 2 + 3 \cdot 1 + 2 + 4 \cdot 1 + 2 + 3}{2 \cdot 3 \cdot 4 \cdot 5} \epsilon x^3 + \&c. \\
 &- \frac{1 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 4} \delta x - \frac{1 \cdot 2 \cdot 3 + 4 \cdot 1 \cdot 2 + 3 \cdot 1 + 2}{2 \cdot 3 \cdot 4 \cdot 5} \epsilon x^2 \qquad \&c. \\
 &+ \frac{1 \cdot 2 \cdot 3 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5} \epsilon x \qquad \&c.
 \end{aligned}$$

or



$$\begin{aligned}
 \text{or } z = & a + \alpha x + \frac{1}{2}\beta x^2 + \frac{1}{6}\gamma x^3 + \frac{1}{24}\delta x^4 + \frac{1}{120}\epsilon x^5 \&c. \\
 & - \frac{1}{2}\beta x - \frac{1}{2}\gamma x^2 - \frac{1}{4}\delta x^3 - \frac{1}{12}\epsilon x^4 \&c. \\
 & + \frac{1}{3}\gamma x + \frac{1}{24}\delta x^2 + \frac{7}{24}\epsilon x^3 \&c. \\
 & - \frac{1}{4}\delta x - \frac{5}{12}\epsilon x^2 \&c. \\
 & + \frac{1}{5}\epsilon x \&c. \\
 & \&c.
 \end{aligned}$$

And, by cor. 5 prop. 1 sect. 1, the area will be

$$\begin{aligned}
 Lx : & a + \frac{1}{2}\alpha x + \frac{1}{6}\beta x^2 + \frac{1}{24}\gamma x^3 + \frac{1}{120}\delta x^4 + \frac{1}{720}\epsilon x^5 \&c. \\
 & - \frac{1}{4}\beta x - \frac{1}{6}\gamma x^2 - \frac{1}{16}\delta x^3 - \frac{1}{80}\epsilon x^4 \&c. \\
 & + \frac{1}{6}\gamma x + \frac{1}{72}\delta x^2 + \frac{7}{96}\epsilon x^3 \&c. \\
 & - \frac{1}{8}\delta x - \frac{5}{36}\epsilon x^2 \&c. \\
 & + \frac{1}{10}\epsilon x \&c. \\
 & \&c.
 \end{aligned}$$

Where  $L$  represents the length of the base.

Which area will be had accurately true when the differences are continued till one order of them becomes equal to nothing, for then the series will break off and terminate; that is, an area is quadrable whenever one of the orders of the differences of its equidistant ordinates consists of a series of nothings or any other equals, or of a series of arithmetics; but if an order of the differences never become equal to nothing, the area will be expressed by an infinite series.

When the terms or ordinates are taken near to one another, the differences will the sooner become equal to nothing, or nearly so; and if any order of differences, and consequently its succeeding ones, be rejected as inconsiderable, we shall have an approximate value of the area, and that the nearer the truth the more of the differences  $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$  are used.

6 B

Thus,



Thus, if there be only one ordinate, or if  $\alpha, \beta, \gamma, \&c.$  be rejected, the area will be  $aL$ .

If there be two ordinates, or  $\beta, \gamma, \delta, \&c.$  be rejected,  $x$  will be  $= 1$ , and the area will be  $a + \frac{1}{2}\alpha \times L = \frac{a+b}{2} \times L$ .

If there be three ordinates, or  $\gamma, \delta, \epsilon, \&c.$  be rejected,  $x$  will be  $= 2$ , and the area will be  $a + \alpha + \frac{2}{3}\beta - \frac{1}{2}\beta \times L = a + \alpha + \frac{1}{6}\beta \times L = \frac{a+4b+c}{6} \times L$ .

If there be four ordinates, or  $\delta, \epsilon, \&c.$  be rejected,  $x$  will be  $= 3$ , and the area will be

$$\left. \begin{array}{l} a + \frac{3}{2}\alpha + \frac{3}{2}\beta + \frac{9}{8}\gamma \\ - \frac{3}{4}\beta - \frac{3}{2}\gamma \\ + \frac{1}{2}\gamma \end{array} \right\} \times L = a + \frac{3}{2}\alpha + \frac{3}{4}\beta + \frac{1}{8}\gamma \times L = \frac{a+3b+3c+d}{8} \times L.$$

And thus by putting  $x$  equal to every number successively, we shall have the following table of areas answering to the respective number of ordinates set opposite to them, of which every expression is more accurate than the preceding ones, and in which  $A$  represents the sum of the first and last terms,  $B$  the sum of the second and last but one,  $C$  the sum of the third and last but two,  $\&c.$  and the last of the letters denotes the double of the middle term when the number of terms is odd, also  $L$  denotes the length of the whole base or the distance between the first and last terms.



| N <sup>o</sup> of ordin. | Areas.                                                        |
|--------------------------|---------------------------------------------------------------|
| 1                        | $AL$                                                          |
| 2                        | $\frac{A}{2} \times L$                                        |
| 3                        | $\frac{A + 2B}{6} \times L$                                   |
| 4                        | $\frac{A + 3B}{8} \times L$                                   |
| 5                        | $\frac{7A + 32B + 6C}{90} \times L$                           |
| 6                        | $\frac{19A + 75B + 25C}{288} \times L$                        |
| 7                        | $\frac{41A + 216B + 27C + 136D}{840} \times L$                |
| 8                        | $\frac{751A + 3577B + 1323C + 2989D}{17280} \times L$         |
| 9                        | $\frac{989A + 5888B - 928C + 10496D - 2270E}{28350} \times L$ |
| &c.                      | &c.                                                           |

## EXAMPLE.

Taking the second example to the last proposition, where are given the five perpendiculars 10, 11, 14, 16, 16, and the distance between the first and last = 20; we shall have  $A = 10 + 16 = 26$ ,  $B = 11 + 16 = 27$ ,  $C = 14 \times 2 = 28$ , and  $L = 20$ .

Hence by the rule for five ordinates  $\frac{7A + 32B + 6C}{90}$   
 $\times L = \frac{182 + 864 + 168}{90} \times 20 = \frac{1214}{9} \times 2 = 269\frac{7}{9} =$  the  
 area, as before nearly.

## SCHOLIUM.

When there are many ordinates given, the case may be reduced to fewer by adding together the  
 first



first and last ordinates, the second and last but one, the third and last but two, and so on, and considering the sums as a new set of terms upon a base equal to half that of the former. Or there may be added together the two first and two last terms in one sum, then the four terms next to these, &c. or we may add three at the beginning to three at the end, then the next six, &c. always diminishing the base in proportion to the number of terms that are added into each sum. And the content will be nearly the same in each case.

## E X A M P L E.

Taking the fourth example to the last proposition, in which are given the eleven ordinates  $\frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}$ , and the distance of the first and last = 1; we shall have  $\frac{1}{10} + \frac{1}{20} = 1.5$ ,  $\frac{1}{11} + \frac{1}{19} = 1.4354069$ ,  $\frac{1}{12} + \frac{1}{18} = 1.3888889$ ,  $\frac{1}{13} + \frac{1}{17} = 1.357466$ ,  $\frac{1}{14} + \frac{1}{16} = 1.3392857$ , and  $\frac{1}{15} \times 2 = 1.3$ .

Hence  $A = 1.5 + 1.3 = 2.8$ ,  $B = 1.4354069 + 1.3392857 = 2.7746926$ ,  $C = 1.3888889 + 1.357466 = 2.7463549$ , and  $L = \frac{1}{2}$ .

Then, by the rule for six ordinates, we shall have  $\frac{19A + 75B + 50C}{288} \times L = .6931476$ , which is much nearer to the truth than the number found by the former rule, the true number being .69314718, and is the hyperbolic logarithm of 2.



## SECTION III.

*Of the Relation between the AREAS and SOLIDITIES of Figures and the Centers of Gravity of their generating PLANES and LINES.*

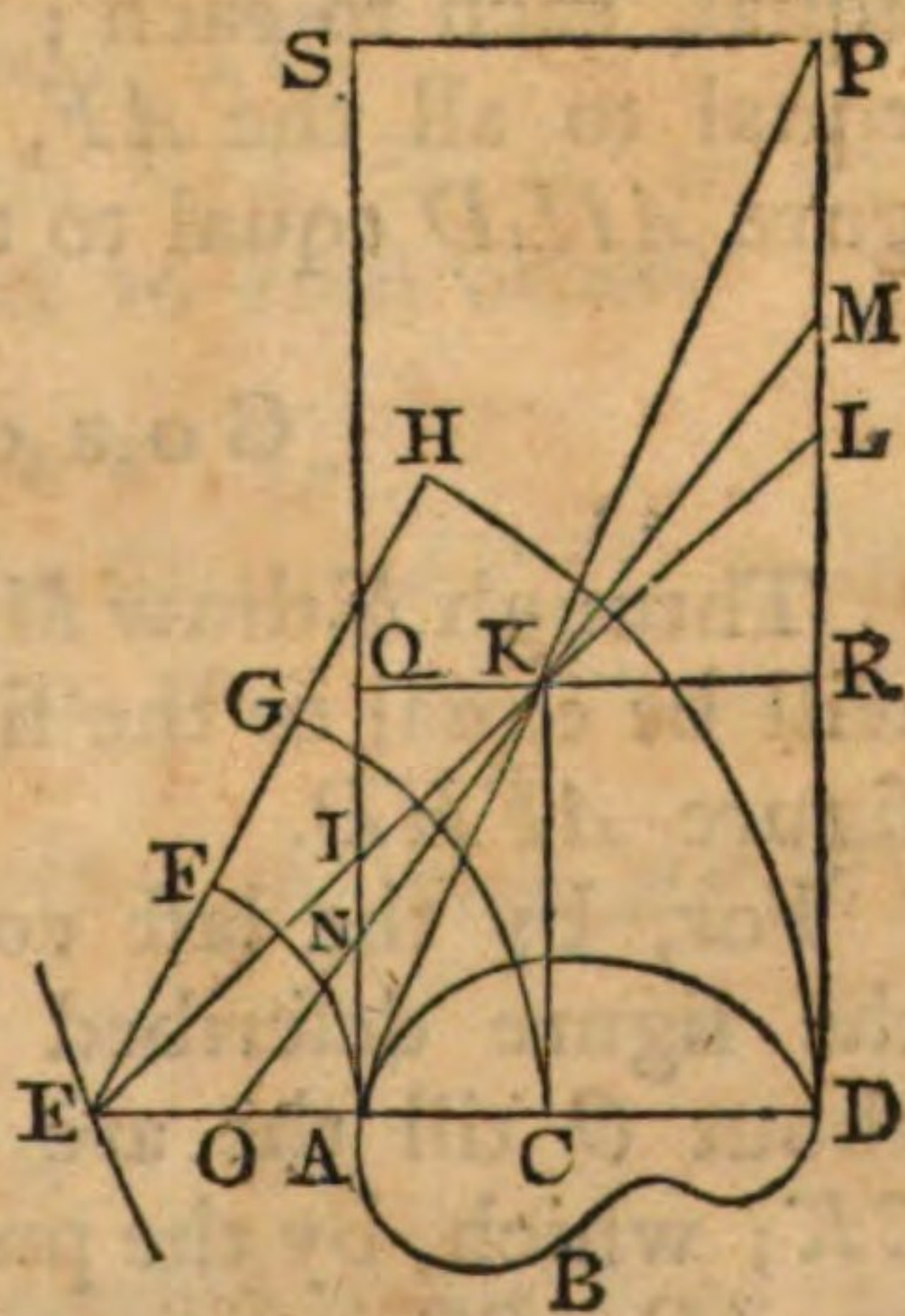
## PROPOSITION I.

**I**F any line, right or curve, or any plane figure whether it be bounded by right lines or curves, revolve about an axe in the plane of the figure; the surface or solid generated, will be respectively equal to the surface or solid whose base is the given line or figure, and height equal to the arc described by the center of gravity of the said generating line or figure; and consequently the content will be found by drawing the generating line or figure into the arc described by its center of gravity.

## DEMONSTRATION.

Let  $AFHD$  be the figure generated by the given line or plane  $ABD$ ; through  $C$  the center of gravity of which draw  $DCAE$  perpendicular to the axe of revolution, and meeting with  $HGFE$  in  $E$ ; and let every point of the base be reduced to  $AD$  by means of perpendiculars to it.

The figure  $AFHD$  generated is equal to all the  $AF$ ,  $CG$ ,  $DH$ , &c. But, by similar figures, all the





$AF, CG, DH, \&c.$  are as all the  $EA, EC, ED, \&c.$  and, by mechanics, the sum of all the  $EA, EC, ED, \&c.$  is equal to so many times  $EC$ ; therefore the sum of all the  $AF, CG, DH,$  is equal to so many times  $CG$ , or equal to  $AD \times CG$ ; that is, the figure  $AFHD$  is equal to  $ABD \times CG$  the base drawn into the line described by its center of gravity. *Q.E.D.*

## COROLLARY I.

From  $E$  draw  $EIKL$  cutting the upright prismatic figure erected upon the given base  $ABD$ , so as that any perpendicular  $AI$  may be equal to its corresponding arc  $AF$ . Then will the figure  $AILD$  be equal to the figure  $AFHD$ .

For, by similar figures, all the  $AF, CG, DH, \&c.$  are as all the  $AI, CK, DL, \&c.$  each to each; and as one of each are equal, therefore they are all equal, each to each; viz. all the  $AI, CK, DL, \&c.$  equal to all the  $AF, CG, DH, \&c.$  that is, the figure  $AILD$  equal to the figure  $AFHD$ .

## COROLLARY II.

Through  $K$  draw  $MKNO$ ; and the figure  $ANMD$  will be equal to the figure  $AIKLD$  or equal to the figure  $AFHD$ .

For, by the last corollary,  $ANMD$  is equal to the figure described by the base  $AD$  revolving about  $O$  till the arc described by  $C$  be equal to  $CK$ ; which, by the proposition, is equal to  $AD \times CK$  or  $AD \times CG$ .

Co.



## COROLLARY III.

Hence all the upright figures  $AQKRD$ ,  $AIKLD$ ,  $ANKMD$ ,  $AKPD$ , &c. of the same base, and bounded at the top by lines or planes cutting the upright sides and passing through the extremity  $K$  of the line  $CK$  erected upon the center of gravity of the base, are equal to one another; and the value of each will be equal to the base drawn into the line  $CK$ .

Hence also all figures, described by the revolution of the same line or plane about different centers or axes, will be equal to one another when the arcs described by the center of gravity are equal. But if those arcs be not equal, the figures generated will be as the arcs. And in general, the figures generated will be to one another as the revolving lines or planes drawn into the arcs described by their respective centers of gravity.

## COROLLARY IV.

Moreover, the opposite parts  $NIK$ ,  $MLK$ , of any two of these figures are equal to each other.

## COROLLARY V.

The figure  $ASPD$  is to the figure  $APD$ , as  $AS$  to  $CK$ ; or, by similar triangles, they will be as  $AD$  to  $AC$ .

For  $ASPD$  is equal to  $AD \times AS$ , and  $APD$  is equal to  $AD \times CK$ .

## COROLLARY VI.

If the line or plane be supposed to be at an infinite distance from the center about which it revolves



volves, the figure generated will be an upright surface or prism, the altitude being the line described by the center of gravity; so that the base drawn into the said line will be equal to the base drawn into the altitude, as it ought for all upright figures whose sections parallel to the base are all equal to each other.

## EXAMPLE I.

If a right line or a parallelogram revolve about a line perpendicular to the length, there will be described a ring either superficial or solid; and as the center of gravity of the describing line or parallelogram is in the middle of them, the general rule will become the same with rule 3 sect. 1 part 3, and the rule at prob. 2 sect. 7 part 3.

When the center of revolution is in the end of the line, the line will describe a circle whose radius is the said describing line, and whose circumference is double the circumference described by the center of gravity; consequently the radius drawn into half the circumference will be the area of the circle.

## EXAMPLE II.

If the right-angled triangle  $ABC$  revolve about the perpendicular  $BC$  and describe the cone  $ABD$ .

Draw  $BE$  to bisect  $AC$ , and take  $EF$  equal to one-third of  $BE$ ; and  $F$  will be the center of gravity of the triangle  $ABC$ , as is well known. Draw  $FG$  parallel to  $AC$ .





*AC.*—Then the line described by *F* will be the circumference of a circle whose radius is *FG*; and, by the general rule, the cone will be equal to  $\triangle ABC \times GF \times 8n$  (putting  $n = .785398$  &c.)  $= AC \times \frac{1}{2}CB \times \frac{2}{3}CE \times 8n = AC \times \frac{1}{2}CB \times \frac{1}{3}CA \times 8n = AC^2 \times \frac{1}{3}CB \times 4n = AD^2 \times \frac{1}{3}CB \times n =$  the base drawn into one-third of the altitude, as it ought.

Again, from *H* the middle of *AB* draw *HI* parallel to *AC*; then is *H* the center of gravity of *AB*, and consequently the surface described by *AB* will be  $AB \times$  circumference whose radius is *HI*  $= AB \times$  half the circumference whose radius is *AC*  $=$  the side drawn into half the circumference of the base  $=$  the surface of the cone, as it ought.

## EXAMPLE III.

If the semi-circle *DCA* (fig. page 14) revolve about the diameter *AD*, and describe the surface of a sphere.

If there be taken  $DC : CF :: CF : FH = \frac{CF^2}{DC} = \frac{4rr}{c}$ , putting *r* for the radius and *c* for the whole circumference; *H* will be center of gravity of the arc *DCA*, and consequently  $r : c :: (FH) \frac{4rr}{c} : 4r =$  the line or circumference described by *H* the center of gravity; and, by the general rule,  $DCA \times 4r = \frac{1}{2}c \times 4r = 2rc =$  the surface of the sphere  $=$  the circumference into the diameter, as it ought.

And for the solidity of the sphere, we shall have first  $\frac{1}{2}c : 2r :: \frac{2}{3}r : \frac{8rr}{3c} =$  the distance *FH* of the center of gravity of the semi-circle *DCAD* from  
6 D the



the diameter  $AD$ , which is two-thirds of the distance of the center of gravity of the arc  $DCA$  from the same diameter  $DA$ , in the former case; consequently the line described by the center of gravity in this case will be two-thirds of *that* in the former; but the describing line in the former case is to the describing space in *this* as 1 is to  $\frac{1}{2}r$ , wherefore  $1 : \frac{2}{3} \times \frac{1}{2}r :: \text{surface of the sphere} : \text{Solidity} = \frac{1}{3}r \times \text{surface}$ .

## COROLLARY.

The circumference of the circle whose radius is the distance of the center of gravity of the semi-circumference of any circle from its center, is equal to four times the radius of that circle.

## EXAMPLE IV.

For the solidity of the parabolic spindle, putting  $b$  = the base and  $a$  = the altitude or axe of the generating parabola, and  $n = .785398$ , &c. as before.

It is known that  $\frac{2}{3}a$  is the distance of the center of gravity from the base, and consequently  $\frac{1}{5}a$  = the line described by the center of gravity; but  $\frac{2}{3}ab$  is = the revolving area; wherefore  $\frac{1}{5}a \times \frac{2}{3}ab = \frac{2}{15}aab$  will be the content, which is  $\frac{8}{15}$  of the circumscribed cylinder.

## EXAMPLE V.

For the paraboloid. Making the notation as in the last example,  $\frac{3}{8}b$  will be the distance of the center of gravity of the semi-parabola from the axe, consequently  $\frac{3}{8}b \times 8n \times \frac{2}{3}ab = 2abbn$  = the solidity = half the circumscribed cylinder.

PART



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## PART V.

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### SECTION I.

#### Of LAND SURVEYING.

IT is supposed that land-measuring first gave rise to geometry, which hath since been gradually rising to the height at which we at present view it. Since the division of common grounds hath become so fashionable in Britain, surveying hath been universally taught and practiced throughout the nation. The mathematical instrument makers have also come in for a share of the profits, by introducing a variety of instruments, often to little and sometimes even to bad purpose. I shall here therefore give a short account of two or three of the most useful instruments.

#### CHAPTER I.

*The description and use of the most useful instruments for surveying.*

##### I. Of the Chain.

Land is measured with a chain, called *Gunter's* chain, of 4 poles or 22 yards in length, which is divided into 100 equal parts called links, each of which is in length  $\frac{22}{100}$  yards =  $\frac{66}{100}$  feet = 7.92 inches; but in turning links into inches or feet, or  
the



the contrary, we may, in many cases, without sensible error suppose each link to be 8 inches or  $\frac{2}{3}$  of a foot in length. A statute acre of land is 4840 square yards = 160 square poles or 10 square chains; or since 40 square poles or  $\frac{1}{4}$  of an acre is called a rood, the table of the divisions for land measure will stand thus,

625 square links = 1 pole or perch

40 poles = 1 rood

4 roods = 1 acre

or 100000 square links = 1 acre.

Hence, when any area is found in square links, point off five places towards the right for decimals of an acre, and the figures on the left will be so many compleat acres; then value the said five decimal places of an acre, by multiplying them by 4 for roods, and again by 40 for perches.

## II. Of the plain Table.

The plain table is so called either from its being a plain surface, or from its reducing the land, in measuring, to a plain, whether horizontal or inclined to the horizon.

This instrument consists of a plain rectangular board of any convenient size, the center of which when used is fixed by means of screws to a three-legged stand having a ball and socket at the top, by means of which when the legs are fixed on the ground the table is inclined in any direction.

To the table belongs,

I. A frame of wood made to fit round its edges, and to take off for the convenience of putting a sheet of paper upon the table. The one side of this frame



frame is divided into 360 degrees from a center which is in the middle of the table, and the other side is divided into 180 degrees to a larger radius from a center towards one side of the table; by the means of which divisions the table is to be used as a theodolite, &c.

II. A needle and compass screwed into the side of the table, to point out the directions and be a check upon the sights.

III. An index, which is a brass two-foot scale with sights erected perpendicularly upon the ends. The hairs in these sights and one edge of the index are in the same plane, and that edge is called the fiducial edge of the index.

Before you use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat upon the table, pressing down the frame upon the edges to stretch it and keep it fixed there; and when the paper is become dry, it will by contracting again stretch itself smooth or flat from any cramps or unevenness. Upon this paper is to be plotted the plan or form of the thing measured.

In using this instrument, begin at any part of the ground you think the most convenient, and make a point upon a convenient part of the paper or table to represent that point of the ground; then fix in that point one leg of the compasses, and apply thereto the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c. and from the station point draw a line with the point of the compasses along the fiducial



edge of the index; then set another object and draw its line; do the same by another, and so on till as many objects be set as may be thought necessary. Then measure from your station towards so many of the objects as may be necessary and no more, taking the requisite offsets to corners or crooks in the hedges, &c. and lay the measures down upon their respective lines upon the table. Then at any convenient place, measured to, fix the table in the same position, and set the objects which appear from thence, &c. as before; and thus continue till your work be finished, measuring such lines as are necessary, and determining so many as you can by intersecting lines of direction drawn from different stations.

And in these operations observe the following particular cautions and directions.

1. Let the lines upon which you make stations be directed towards objects as far distant as possible, and when you have set any such object, go round the table and look through the sights from the other end of the index, to see if any other remarkable object lie directly opposite; if there be not such an one, endeavour to find another forward object such as shall have a remarkable backward opposite one, and make use of it rather than the other; because the back object will be of use in fixing the table in the original position either when you have measured too near to the forward object, or when it may be hid from your sight at any necessary station by intervening hedges, &c.

2. Let the said lines, upon which the stations are taken, be pursued as far as you conveniently can;



can; for that will be the means of preserving more accuracy in the work.

3. At each station it will be necessary to prove the truth of it; that is, whether it be streight in the line towards the object, and also whether the distance be rightly measured and laid down on the paper.—To know if the table be set down streight in the line, lay the index upon the table in any manner, and move the table about till through the sights you perceive either the fore or back object; then, without moving the table, go round it and look through the sights by the other end of the index, to see if the other object can be perceived; if it be, the table is in the line; if not, it must be shifted to one side, according to your judgment, till through the sights both objects can be seen.—The aforesaid operation only informs you if the station be streight in the line; but to know if it be in the right part of the line, that is, if the distance have been rightly laid down; fix the table in the original position, by laying the index along the station line and turning the table about till the fore and back objects appear through the sights, and then also will the needle point at the same degree as at first; then lay the index over by the station point and any other point on the paper representing an object which can be seen from the station; and if the said object appear streight through the sights, the station may be depended on as right; if not, the distance should be examined and corrected till the object can be so seen. And for this very useful purpose, it is adviseable to have some high object or two, which can be seen from  
the



the greatest part of the ground, accurately laid down on the paper from the beginning of the survey, to serve continually as proof objects.

When from any station the fore and back objects cannot both be seen, the agreement of the needle with one of them may be depended on for placing the table straight on the line, and for fixing it in the original position.

*Of the shifting of the paper.*

When one paper is full, and you have occasion for more; draw a line in any manner through the farthest point, of the last station line, to which the work can be conveniently laid down; then take the sheet off the table, and fix another on, drawing a line upon it, in the part the most convenient for the rest of the work; then fold or cut the old sheet by the line drawn on it, apply the edge to the line on the new sheet, and, as they lie in that position, continue the last station line upon the new paper, placing thereon the rest of the measure, beginning at where the old sheet left it. And so on from sheet to sheet.

After the work is done, when you would fasten all the sheets together into one piece or rough plan, these lines are to be accurately joined together as when the lines were transferred from the old sheets to the new ones.

But it is to be noted that if the said joining lines, upon the old and new sheet, have not the same inclination to the side of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to respect



spect still the same degree of the compass, the easiest way of drawing the lines in the same position, is to draw them both parallel to the same sides of the table by means of the equal divisions marked on the other two sides.

*Of the Theodolite.*

The theodolite is a brazen circular ring divided into 360 degrees, having an index with sights placed upon the center, about which the index is moveable, and a compass fixed to the center, to point out courses and check the sights; the whole being fixed by the center upon a stand of a convenient height for use.

In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan upon your return home from the ground.

Begin at such part of the ground and measure in such directions as you judge most convenient, taking angles or directions to objects, and measuring such distances as appear necessary, under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and at any station placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position, after the



same manner as the plain table is fixed in the original position by laying the index along the line of the last direction.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle, quarter it, and lay upon it the several numbers of degrees cut off by the index in each direction; then, by means of a parallel ruler, draw from station to station lines parallel to lines drawn from the center to the respective points in the circumference.

#### *Of the Cross.*

The cross consists of two pair of sights set at right angles to each other, upon a staff having a sharp point to stick in the ground.

The cross is very useful to measure small and crooked pieces. The method is to measure a base or chief line, generally in the longest direction of the piece from corner to corner, taking the places of perpendiculars upon this line from the several corners and bends in the boundary of the piece, with the cross, by fixing it by trials upon such parts of the line as that through one pair of the sights both ends of the line may appear, and through the other pair you can perceive the corresponding bends or corners; and then measuring the lengths of the said perpendiculars.

#### SCHOLIUM.

Besides the fore-mentioned instruments, which are most used, there are some others, as the circumferentor much like to the theodolite in shape and



and use, and the semi-circle for the taking of angles, &c. But of all the instruments for measuring, the plain table is certainly the most excellent; not only because it may be used as a theodolite or semi-circle, according as you put uppermost the side of the frame having the 360 degrees or that which hath the semi-circle or 180; but because that it is in its own proper use by much the easiest, safest, and most accurate for the purpose; for by planning every part immediately upon the spot as soon as measured, there is not only saved a great deal of writing in the field book, but because that every thing can be planned more easily and accurately while it is in view than afterwards from a field book, in which many little things must be either neglected or mistaken; and besides, the opportunities which the plain table afford of correcting our work or proving if it be right at every station, are such advantages as can never be balanced by any other method. But although the plain table be the most generally useful instrument, it is not *always* so; there being many cases in which sometimes one instrument is the properest and sometimes another; nor is that surveyor master of his business who cannot in any case distinguish which is the fittest instrument or method, and use it accordingly: nay sometimes no instrument at all, but barely the chain itself, is the best method, particularly in regular open fields lying together; and even when you are using the plain table it is often of advantage to measure such with the chain only and from those measures lay them down upon the table; and some eminent surveyors have been so struck with the



the ease and simplicity of this method, that they have neglected and exploded the use of every instrument whatever: and indeed such practice is not so absurd as some other; for although it be often attended with much trouble in measuring, and even sometimes be impracticable, as in surveying a town, or among bushes, &c. and attended with trouble also in the planning on account of some of the lines measuring to more than they ought; yet when such lines are corrected and the parts of the plan made to fit, the true surface will then be represented by the plan: but of all the absurd or false methods *that* by the use of what is called the new-improved theodolite is of the most dangerous consequence; for it is professedly adapted to the diminishing of all land by reducing it to an *horizontal* plain, even such as is wholly upon a *regular* declivity; by that means destroying sometimes a third or fourth of the true surface; it is the surface that surveyors have to deal with, without any regard to its inclination, except indeed where a pointed hill, &c. arises out of a plain, for such all the methods would level down to the general surface; nor is it to any purpose to say that the horizontal plain will bear as many perpendiculars or piles of grass, &c. as the slope will do, for that the surveyor has no business with, let him give the true surface, and leave the owner to rate and let his land according to its quality. In short, let every gentleman, who would not have his real land reduced below its just quantity, beware of those surveyors who use *this* new-improved instrument.



## CHAPTER II.

*Of some particular methods of measuring, illustrated by examples.*

Besides the general methods of measuring, mentioned in the preceding description of instruments, may be observed the following particular cases of measuring and of computing the contents.

Many pieces of land may be very well measured by measuring any base line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by those means divided into several triangles and trapezoids all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without being taken from the sum of the whole which are both within and without, will leave the figure proposed.

In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the several perpendiculars will fall, by means of the *Cross*, and from thence measuring to the corners for the lengths of the perpendiculars.— And it will be most convenient to draw the line so as that all the perpendiculars may fall within the figure.

## EXAMPLE I.

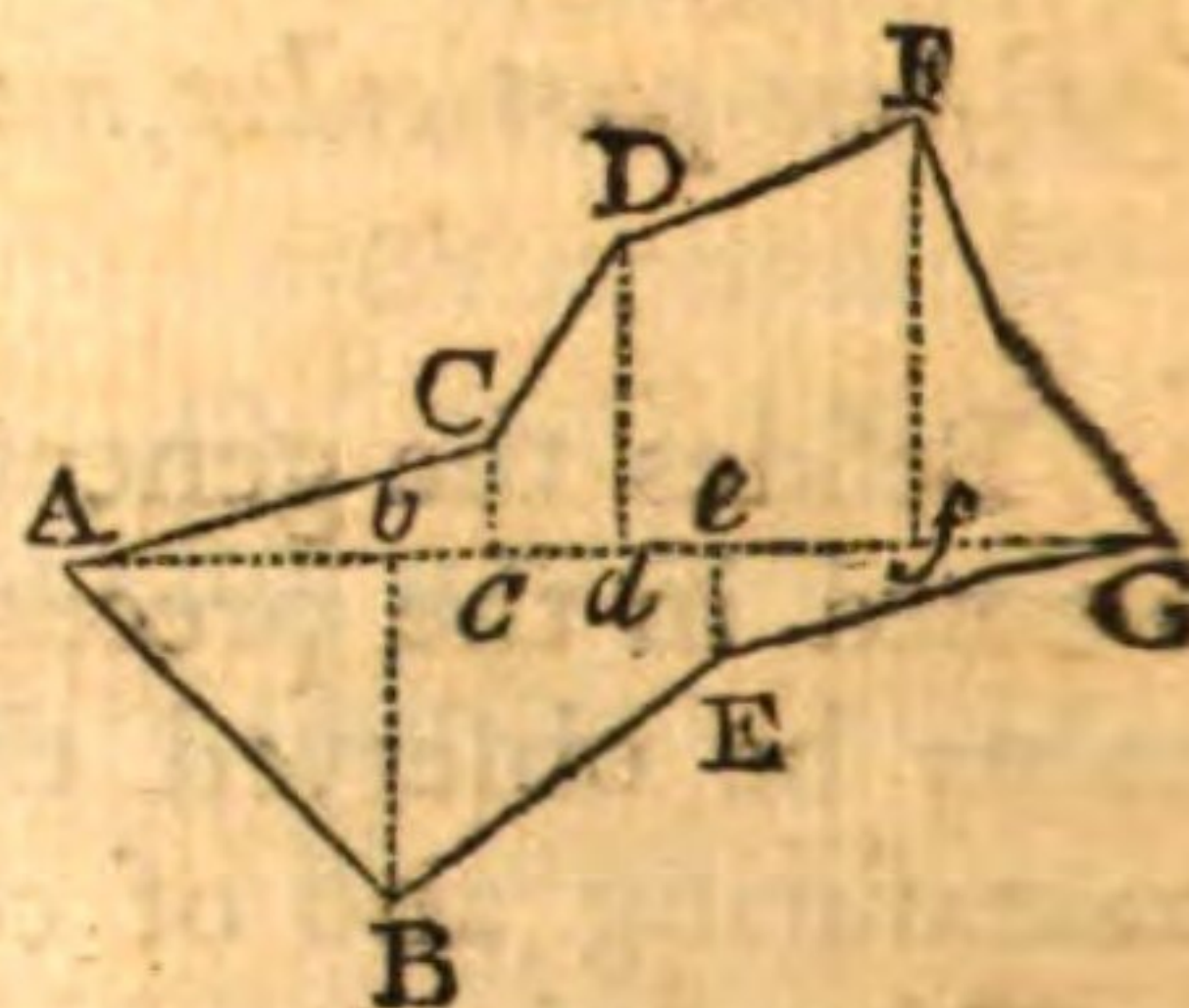
Having, in the field *ACDFGEBA*, measured the base line *AG* and perpendiculars as are expressed in the following table or field-book, in links: Required the content, and plot of the field.

6 G

Field.



| Field-Book. |                  |
|-------------|------------------|
| Base line   | Perpendiculars   |
| $Ab = 315$  | $bB = 350$ South |
| $Ac = 440$  | $cC = 70$ North  |
| $Ad = 585$  | $dD = 320$ N.    |
| $Ae = 610$  | $eE = 50$ S.     |
| $Af = 990$  | $fF = 470$ N.    |
| $AG = 1020$ | 0                |



The calculation.

$$Ab \times bB \text{ --- } = 315 \times 350 = 110250 = 2 AbB.$$

$$Ac \times cC \text{ --- } = 440 \times 70 = 30800 = 2 AcC.$$

$$cd \times cC + Dd = 145 \times 390 = 56550 = 2 cCDd.$$

$$be \times bB + Ee = 295 \times 400 = 118000 = 2 bBEe.$$

$$df \times dD + fF = 405 \times 790 = 319950 = 2 dDFf.$$

$$Ff \times FG \text{ --- } = 470 \times 30 = 14100 = 2 fFG.$$

$$Ge \times eE \text{ --- } = 50 \times 410 = 20500 = 2 GEe.$$

the sum =  $670150$  = double the whole figure; whose half =  $335075$  square links =  $3.35075$  acres =  $3$  A.  $1$  R.  $16.12$  Perches = the area required.

To plot the figure above, or any other measured in the same manner.

Draw an indeterminate line  $AG$  for the base line, and transfer upon it, from a scale of equal parts, the several distances  $Ab$ ,  $Ac$ ,  $Ad$ , &c. equal to the given measures  $315$ ,  $440$ ,  $585$ , &c. Then, at the points  $b$ ,  $c$ ,  $d$ , &c. erect the perpendiculars  $bB$ ,  $cC$ ,  $Dd$ , &c. equal to the given lengths  $350$ ,  $70$ ,  $320$ , &c. on the upper or under side of the base line accord-



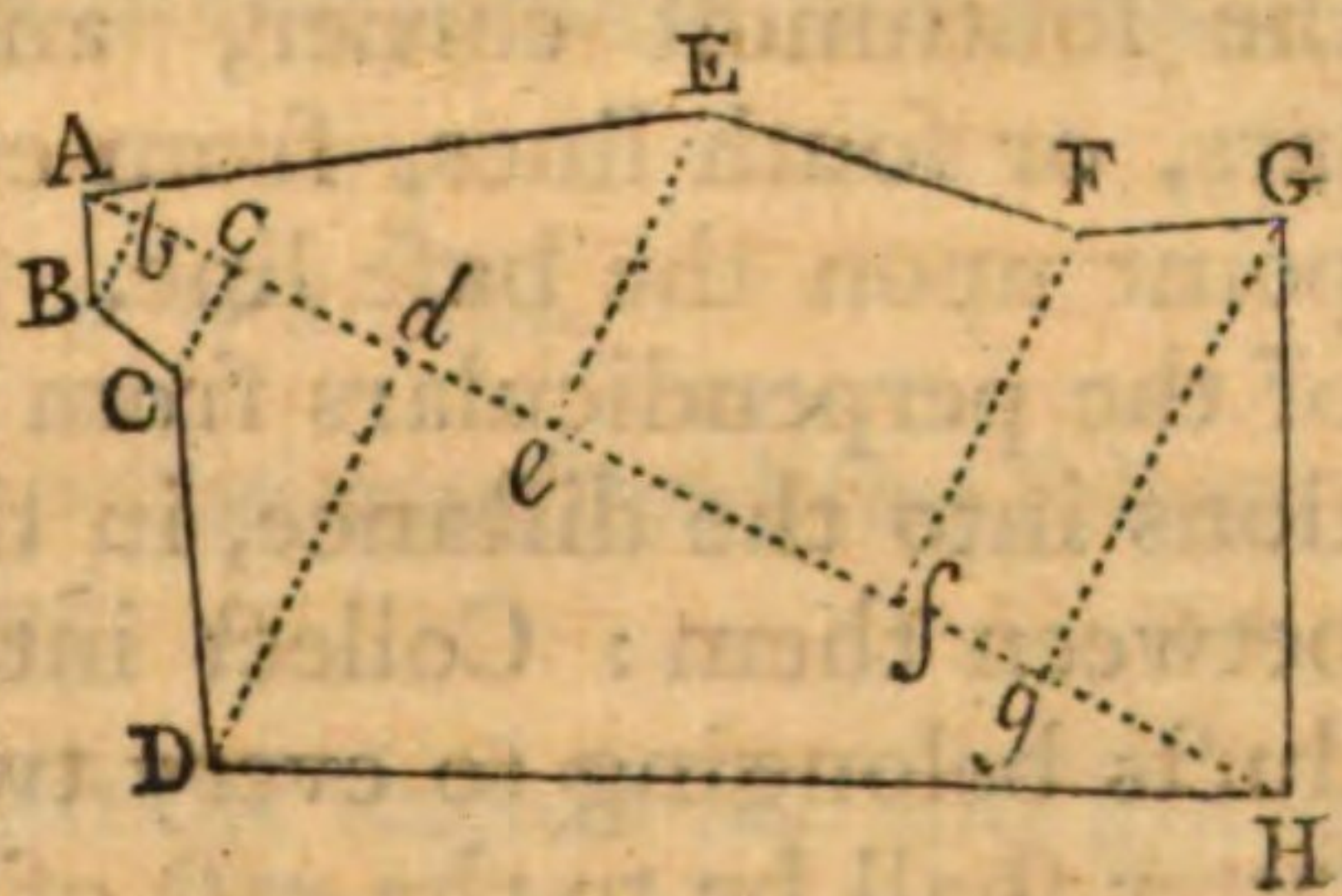
according as they are north or south. Lastly, draw the lines  $AC$ ,  $CD$ ,  $DF$ , &c. and it will be done.

## EXAMPLE II.

Required the area and plot of the figure from the following field-book.

*Note*, That a cypher in the place of a perpendicular, denotes that there the base line touches an angular point. So here the first and last perpendiculars, being cyphers, shew that the base line begins and ends at an angle.

| Field-Book. |                |
|-------------|----------------|
| Base line   | Perpendiculars |
| 0           | 0              |
| 19          | 42 S.          |
| 80          | 62 S.          |
| 184         | 235 S.         |
| 307         | 184 N.         |
| 470         | 220 N.         |
| 556         | 270 N.         |
| 697         | 0              |



## Calculation.

$$Ab \times bB \text{ --- } = 19 \times 42 = 0798 = 2ABb.$$

$$bc \times cC + Bb = 61 \times 104 = 6344 = 2BCcb.$$

$$cd \times dD + Cc = 104 \times 297 = 30888 = 2CDdc.$$

$$Dd \times dH \text{ --- } = 235 \times 513 = 120555 = 2DdH.$$

$$Ae \times eE \text{ --- } = 307 \times 184 = 56488 = 2EeA.$$

$$ef \times fF + Ee = 163 \times 404 = 65852 = 2FEef.$$

$$fg \times gG + Ff = 86 \times 490 = 42140 = 2GFfg.$$

$$Gg \times gH \text{ --- } = 270 \times 141 = 38070 = 2HGg.$$

the sum = 361135, the half of  
which = 180567.5 square links = 1.805675 acres =  
1 A. 3 R. 8.908 P. But



But in large pieces or tracts of land it is better to reduce the measures to a base line drawn in a direction exactly east-and-west, and touching the southmost angle of the figure.—And this is Mr. *Thomas Burgh's* “method to determine the areas of right-lined figures universally,” and is performed thus.

Begin at any corner, and, going round the figure, measure the length of each side and the angle formed by each and the meridian or north-and-south line; then the figure being plotted, draw the base line, in an east-and-west direction, through the southmost corner, and let fall perpendiculars, or south lines, from every station or angular point upon the base line. Then multiply the sum of the perpendiculars from every two adjacent stations into the distance, in the base line, intercepted between them: Collect into one sum all the products belonging to every two stations of which the latter shall be to the east of the former, and collect into another sum all the other products; and half the difference of the sums will be the area of the figure.\*

*Note.*

\* For, beginning at the west at *A* (see the figure to the following example) and proceeding round by *B*, *C*, &c. it is plain that  $\overline{Aa + bB} \times ab =$  double the trapezoid *AabB*,  $\overline{Bb + cC} \times bc =$  double *BbcC*,  $\overline{Cc + dD} \times cd =$  double *CcdD*,  $\overline{Ee + fF} \times ef =$  double *EefF*; and that the sum of these products will be the double of the figure *aABCDEFfa* together with the double of *EedD*, because that *EedD* is included in both *CcdD* and *EefF*.

Again  $\overline{Dd + eE} \times de =$  twice *DdeE*,  $\overline{Ff + gG} \times fg =$  twice *FfgG*,  $Gg \times gH =$  twice *GgH*,  $Hi \times iI =$  twice *HiI*,  $\overline{Ii + aA} \times ia =$  twice *IiaA*; and the sum of these products make up the doubles of the two figures *AaHIA*, *HfFGH*, together with the double of *EedD* also.

Consequently the latter sum taken from the former will leave the area of the figure *ABCDEFGHIA* required.



*Note.* The perpendiculars upon the base line from the several stations Mr. Burgh calls their absolute nothings; and the distances between the perpendiculars, the particular eastings and westings of the stations, viz. easting when the station is to the east of the preceding station, and westing when it is to west of it. Then his rule is to multiply every particular easting between two stations into the half sum of the absolute northings, collecting the products into one sum; and every particular westing between two adjacent stations into the half sum of their absolute nothings, collecting these products also: then the difference of the sums is the area.

### EXAMPLE III.

Suppose that there is to be measured the following figure by this method.

Beginning at the westmost station *A*, and going round towards the north, suppose the lengths of the lines and the angles formed by them and the meridians to be taken thus:  $AB = 1550$  links, and its direction N.  $35\frac{1}{2}^\circ$  E. that is,  $35\frac{1}{2}^\circ$  from the north towards the east; I enter these, then, in the 2d and 3d columns, of the following table, opposite to the 1st station *A*. Then proceeding to the 2d station *B*, I find the direction of *BC* to be N. E.  $72\frac{3}{4}^\circ$ , and its length = 1870 links, which, therefore, I enter in the table upon a line with the second station *B*. In the same manner I find all the other sides and their directions as expressed in order in the 2d and 3d columns of the table; the 1st column containing only the number and mark of each station.

6 H

Then



Then by constructing the figure, which is very easy to do, and measuring, by the scale, the lengths and distances of the several perpendiculars, and multiplying, &c. according to rule, we shall have the area pretty near.

But to have the area exact, the several perpendiculars and their distances must be calculated by trigonometry, or, for more expedition, by a traverse table.—Thus, in the triangle  $ABz$ , the distance  $AB$  being 1550, and the course or angle at  $B = 35\frac{1}{2}^\circ$ , I find, in the traverse table, the easting  $Az = 900$ , and northing  $zB = 1262$ ; these I enter, in my table, in the columns of northing and easting, marked N and E, upon a line with the first station  $A$ . Then, in the triangle  $BCy$ , to the distance 1870 and course  $72\frac{3}{4}^\circ$ , I find the easting  $By = 1786$ , and the northing  $yC = 555$ , which I place, in the columns  $E$  and  $N$ , opposite the 2d station. In like manner, in the triangle  $CxD$ , I find the southing  $Cx = 616$ , and easting  $xD = 1765$ , which I place accordingly in the columns of southing and easting. And in the triangle  $DEw$ , the southing  $Dw = 749$ , and westing  $wE = 994$ , which I put in the columns  $S$  and  $W$ . And so on till, by going through all the triangles, I return to the point  $A$  where I began, placing the particular northings or southings, eastings or westings in the table.—And it is evident that, being returned to the first station  $A$ , the sum of all the northings must be equal to the sum of the southings, and the sum of the eastings equal to that of the westings; and, accordingly, by summing them up in the table, I find that they are so, the sum of each of the former being



being 3982, and that of each of the latter 6844, which is a very good proof of this part of the work.

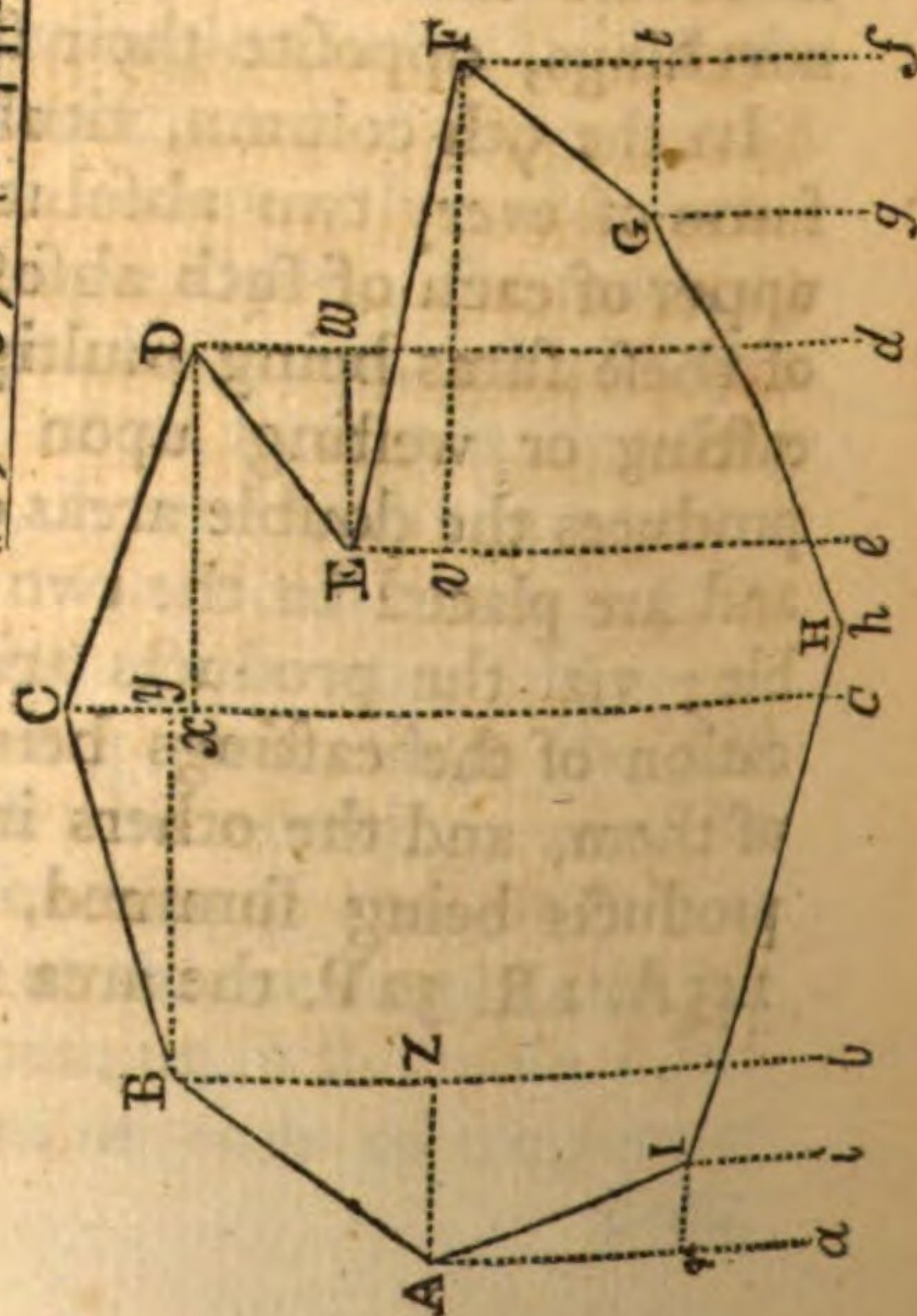
Then, for the absolute northings, it is evident that the sum of all the northings 3982 must be equal to  $cC$  the absolute northing of the northmost station  $C$ ; and this I place on the 8th column of the table, marked Absol. N. being the column of absolute northings, opposite the station  $C$ . Then  $Cx$ , the particular southing of  $C$ , taken from its absolute northing, gives the absolute northing  $dD$  of the next station  $D$ : again, the particular southing of  $D$  taken from its absolute northing, gives the absolute northing of the next station  $E$ : and so on, to or from the absolute northing of every station, adding its particular northing or subtracting its particular southing, for the absolute northing of the next succeeding station. And these are all entered, in the column of absolute northings, opposite their stations.

In the 9th column, titled *Sums*, are placed the sums of every two absolute northings opposite the upper of each of such absolute northings.—Each of these sums being multiplied into the particular easting or westing upon the same line with it, produces the double areas of the several trapezoids, and are placed in the two last columns of the table; viz. the products arising from the multiplication of the eastings being written in the former of them, and the others in the latter. Then these products being summed, half their difference is 145 A. 1 R. 32 P. the area sought.

Stations



| Stations | Angles                     | Sides | N    | S    | E    | W    | Abfol. Ns. | Sums | E. Prod. | W. Prod. |
|----------|----------------------------|-------|------|------|------|------|------------|------|----------|----------|
| 1. A     | NE $35\frac{1}{2}^{\circ}$ | 1550  | 1262 |      | 900  |      | 2165 = aA  | 5592 | 5032800  |          |
| 2. B     | NE $72\frac{3}{4}$         | 1870  | 555  |      | 1786 |      | 3427 = bB  | 7409 | 13232474 |          |
| 3. C     | SE $70\frac{3}{4}$         | 1870  |      | 616  | 1765 |      | 3982 = cC  | 7348 | 12969220 |          |
| 4. D     | SW 53                      | 1245  |      | 749  |      | 994  | 3366 = dD  | 5983 |          | 5947108  |
| 5. E     | SE $83\frac{1}{4}$         | 2410  |      | 283  | 2393 |      | 2617 = eE  | 4951 | 11847743 |          |
| 6. F     | SW $31\frac{1}{4}$         | 1520  |      | 1299 |      | 789  | 2334 = fF  | 3369 |          | 2658141  |
| 7. G     | SW $62\frac{3}{4}$         | 2260  |      | 1035 |      | 2008 | 1035 = gG  | 1035 |          | 2078280  |
| 8. H     | NW $73\frac{1}{2}$         | 2730  | 775  |      |      | 2617 | 0 = hH     | 775  |          | 2026635  |
| 9. I     | NW $17\frac{5}{2}$         | 1456  | 1399 |      |      | 436  | 775 = iI   | 2940 |          | 1281840  |
|          |                            |       |      |      |      |      | 3982       | 3982 | 6844     | 6844     |
|          |                            |       |      |      |      |      |            |      |          | 43082237 |
|          |                            |       |      |      |      |      |            |      |          | 13992004 |

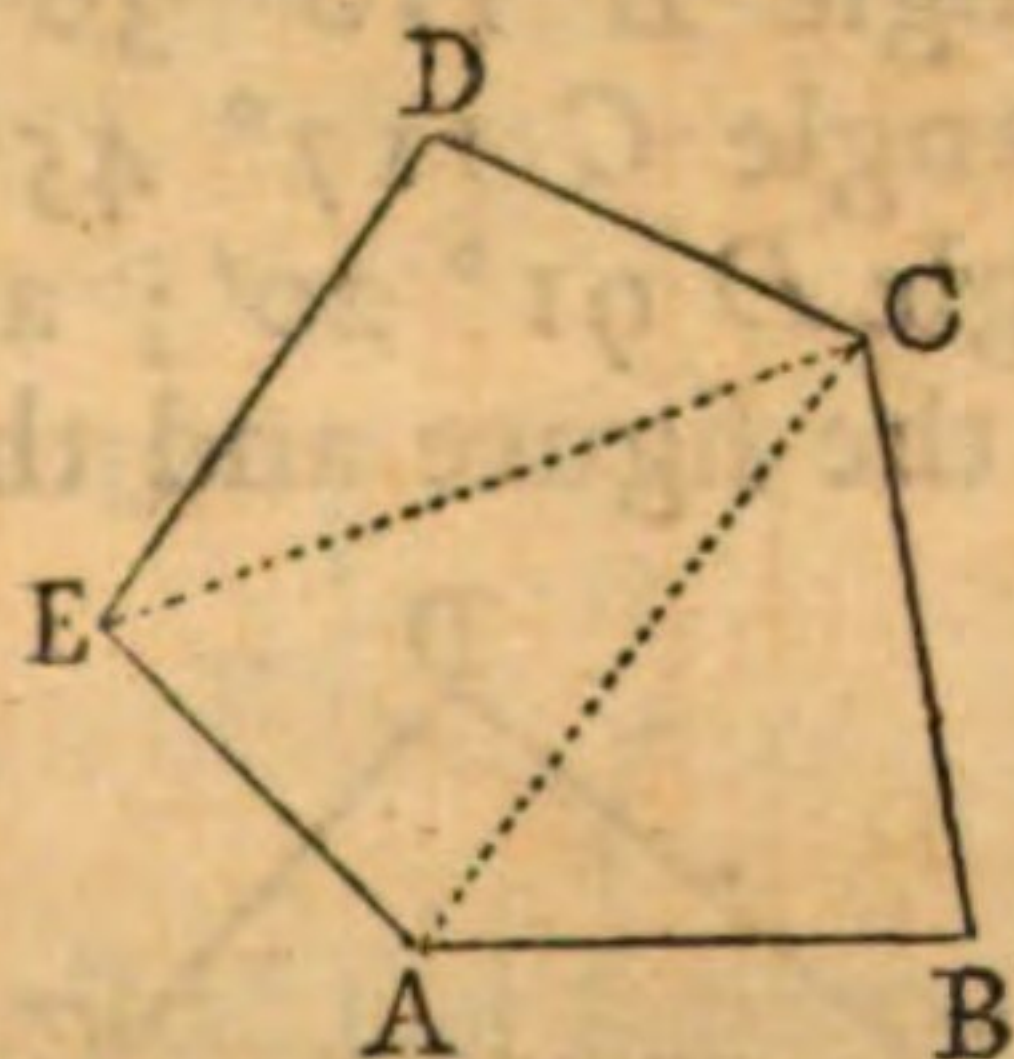




*Some other examples, promiscuously set down.*

EXAMPLE IV.

In a pentangular field, beginning with the south side, and measuring round towards the east first, the 1st or south side was = 2735 links, the 2d = 3115, the 3d = 2370, the 4th = 2925, and the 5th = 2220; also the diagonal from the 1st angle to the 3d was 3800, and that from the 3d to the 5th was 4010; Required the figure and area.



Having constructed the figure, I first take the distance of the angle *B* from the diagonal *AC* between the compasses, and, by applying them to the scale, find it to be 2210, which is the length of the perpendicular from *B* upon *AC*. Wherefore  $3800 \times 2210 = 8398000$  double the triangle *ABC*.

Then in the same manner I find the distances of *A* and *D* from the diagonal *CE* to be 2065 and 1710. Wherefore  $4010 \times 2065 + 1710 = 4010 \times 3775 = 15137750 =$  double the trapezium *ACDE*.

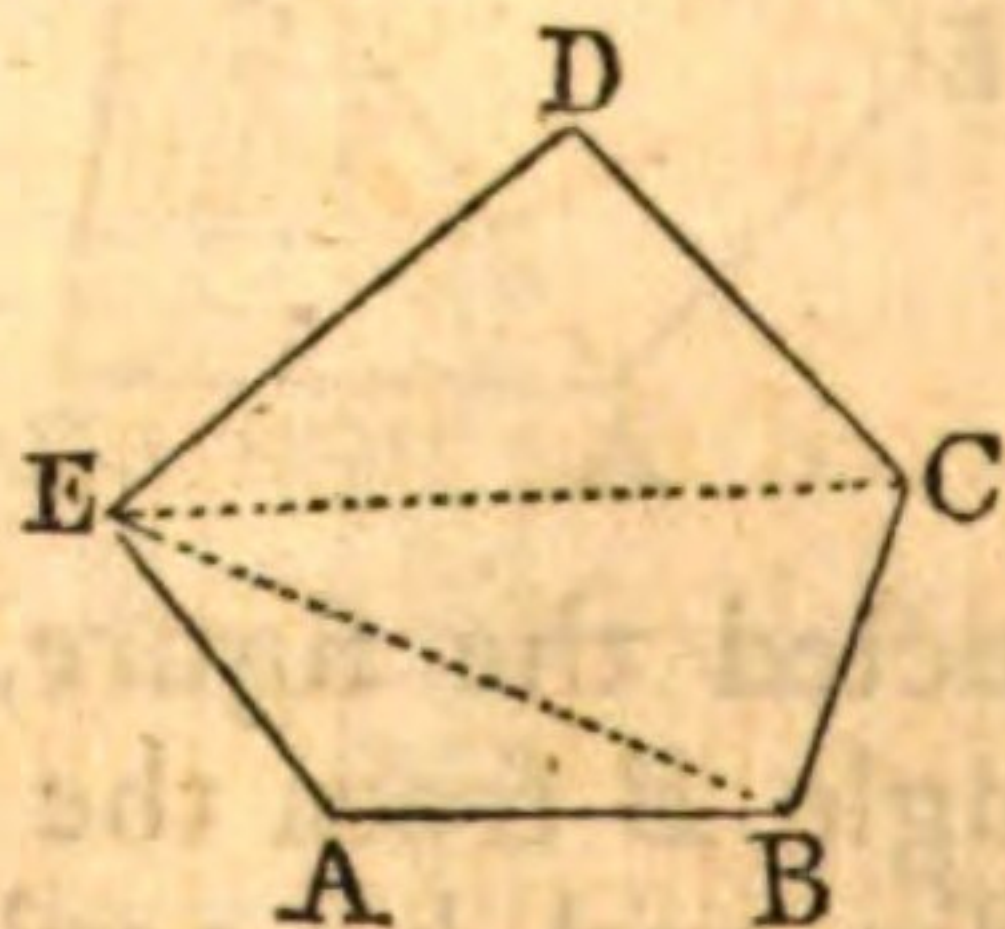
Whence the sum of the two products is 23535750, the half of which is 11767875 square links, equal 117 A. 2 R. 28.6 P. the area sought.



*Note.* As this figure consists of three triangles all whose sides are given, by calculating their areas by rule 2 of problem 2, their sum will be that of the whole figure accurately, and you will then find how much the area above assigned differs from the truth, which, I believe, will be but about the 30th part of an acre.

## EXAMPLE V.

In going round a five-sided field  $ABCDE$ , the sides and angles were thus: The side  $AB = 1940$  links, and the angle  $B = 110^\circ 30'$ ; the side  $BC = 1555$ , and the angle  $C = 117^\circ 45'$ ; the side  $CD = 2125$ , and the angle  $D = 91^\circ 20'$ ; and the side  $DE = 2741$ : Required the figure and the content.



Having constructed the figure and drawn the two diagonals  $EB$ ,  $EC$ , which measure 3180 and 3500, I measure the distance between  $A$  and the diagonal  $EB$ , and find it = 780, and the distances of  $B$  and  $D$  from the diagonal  $EC$  are 1430 and 1670.

Then  $3180 \times 780 = 2480400 = \text{double the } \triangle ABC$ ,  
And  $3500 \times 1430 + 1670 = 3500 \times 3100 = 10850000$   
= double the trapezium  $EBCD$ .

Wherefore  $\frac{10850000 + 2480400}{2} = 6665200$  square links = 66 A. 2 R. 24' 32 P. the area sought.

Ex-

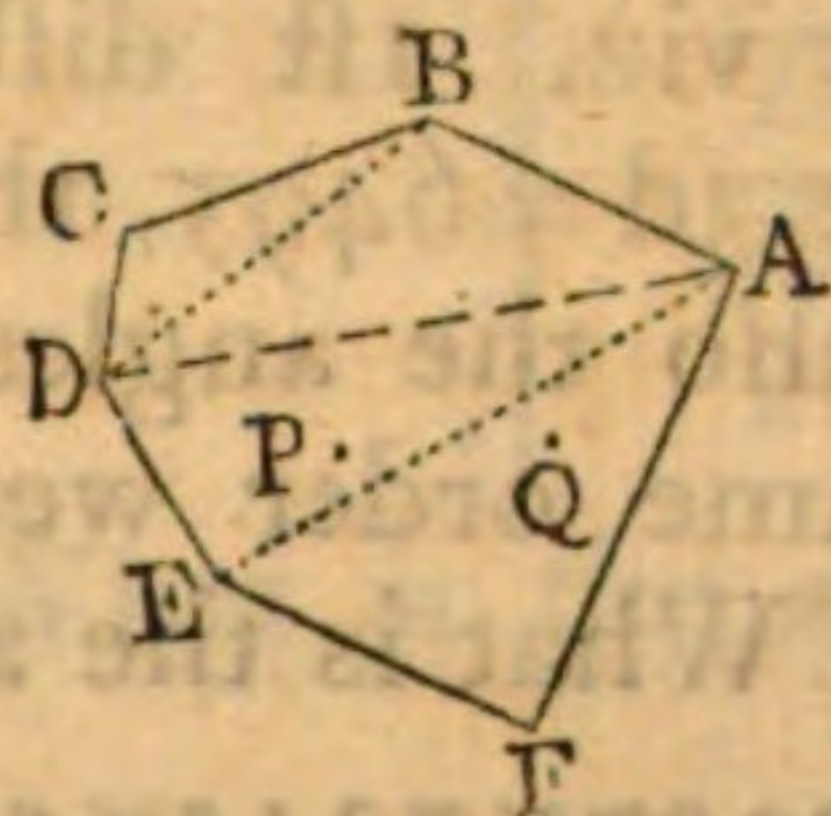


## EXAMPLE VI.

In a field I took two stations  $P$  and  $Q$ , at the distance of 10 chains from each other:

|                               |   |                               |   |                            |   |                          |
|-------------------------------|---|-------------------------------|---|----------------------------|---|--------------------------|
| From $P$ I took<br>the angles | { | $\angle P A = 21^{\circ} 20'$ | } | And from $Q$ the<br>angles | { | $P Q D = 10^{\circ} 40'$ |
|                               |   | $\angle A P B = 49 \quad 10$  |   |                            |   | $D Q C = 18 \quad 30$    |
|                               |   | $\angle B P C = 57 \quad 12$  |   |                            |   | $C Q B = 42 \quad 00$    |
|                               |   | $\angle C P D = 29 \quad 40$  |   |                            |   | $B Q A = 67 \quad 05$    |
|                               |   | $\angle D P E = 64 \quad 25$  |   |                            |   | $A Q F = 137 \quad 00$   |
|                               |   | $\angle E P F = 79 \quad 16$  |   |                            |   | $F Q E = 62 \quad 52$    |

Required the figure and area.



Having constructed the figure, by the line  $AD$ , I divide it into two trapeziums  $ABCD$ ,  $ADEF$ : In the former of which the diagonal  $BD$  measures 1860, and the perpendiculars 1420 and 440; and in the latter, the diagonal  $AE = 2940$ , and the perpendiculars 1050 and 1350.

Whence  $1860 \times 1420 + 440 = 1860 \times 1860 = 3459600$

and  $2940 \times 1050 + 1350 = 2940 \times 2400 = 7056000$

$2) 10515600$

Acres 52.578

A. R. P.

Anf. 52 2 12

Roods 2.312

Perches 12.48

If



If from a station within a field, the distances to the several angles be measured, as also the angles formed by these distances.

Then multiply the product of every two adjacent distances by the natural sine of the angle between them, and half the sum of these products will be the *exact* area.

#### EXAMPLE VII.

From a station within a field of five sides I measured the distances to the several corners, beginning with that on the west, and measuring round towards the north, viz. 1st distance 7345 links, the 2d = 5980, the 3d = 6495, the 4th = 6015, and the 5th = 7050; also the angles formed by these distances in the same order were  $71\frac{1}{2}$ ,  $55\frac{3}{4}$ ,  $49\frac{1}{4}$ , and  $81\frac{1}{2}$  degrees. What is the area.

$$\begin{aligned} & (\text{Sine of } 71\frac{1}{2}^\circ) \cdot 9483237 \times 7345 \times 5980 = 41653316.25 \\ & \left( \begin{array}{l} \text{---} \\ \text{---} \end{array} - 55\frac{3}{4} \right) \cdot 8265897 \times 5980 \times 6495 = 32104826.57 \\ & \left( \begin{array}{l} \text{---} \\ \text{---} \end{array} - 49\frac{1}{4} \right) \cdot 7575650 \times 6495 \times 6015 = 29596113.97 \\ & \left( \begin{array}{l} \text{---} \\ \text{---} \end{array} - 81\frac{1}{2} \right) \cdot 9890159 \times 6015 \times 7050 = 41939961.01 \\ & \left( \begin{array}{l} \text{---} \\ \text{---} \end{array} - 78 \right) \cdot 9781476 \times 7050 \times 7345 = 50650683.55 \end{aligned}$$

$$2) 195944901.35$$

$$\text{Acres } 979.72450675$$

4

$$\text{Roods } 2.89802700$$

40

$$\text{Perches } 35.92108$$

*Note.* The last angle ( $78^\circ$ ) is found by subtracting the sum of all the other angles from  $360^\circ$ ; for all the angles formed by lines about a point make up four right angles or  $360^\circ$ .

Ex-



## EXAMPLE VIII.

From a place  $O$ , near the middle of a field  $ABCDEF$ , from which I could see all the angles, I measured the distances to the several angles, and observed the quantities of the angles formed at  $O$  by those distances as below.

| Distances         | Angles               |
|-------------------|----------------------|
| $OA = 4315$ links | $AOB = 60^\circ 30'$ |
| $OB = 2982$       | $BOC = 47 40$        |
| $OC = 3561$       | $COD = 49 50$        |
| $OD = 5010$       | $DOE = 57 10$        |
| $OE = 4618$       | $EOF = 64 15$        |
| $OF = 3606$       | $FOA = 80 35$        |

What is the area.

$$\begin{aligned}
 &(\text{Sine of } 60^\circ 30') \cdot 8703557 \times 4315 \times 2982 = 11199154 \cdot 17 \\
 &\left\{ \begin{array}{l} - - 47 40 \end{array} \right\} \cdot 7392394 \times 2982 \times 3561 = 7849910 \cdot 77 \\
 &\left\{ \begin{array}{l} - - 49 50 \end{array} \right\} \cdot 7641714 \times 3561 \times 5010 = 13633284 \cdot 15 \\
 &\left\{ \begin{array}{l} - - 57 10 \end{array} \right\} \cdot 8402513 \times 5010 \times 4618 = 19440205 \cdot 26 \\
 &\left\{ \begin{array}{l} - - 64 15 \end{array} \right\} \cdot 9006982 \times 4618 \times 3606 = 14998884 \cdot 03 \\
 &\left\{ \begin{array}{l} - - 80 35 \end{array} \right\} \cdot 9865246 \times 3606 \times 4315 = 15350214 \cdot 66
 \end{aligned}$$

$$2) 82471653 \cdot 04$$

$$\text{Acres } 412 \cdot 3582652$$

4

$$\text{Rood } 1 \cdot 4330608$$

40

$$\text{Perches } 17 \cdot 322432$$



## EXAMPLE IX.

Having made choice of a station within a piece of land, I measured from thence the several bearings and distances of the corners as below: Required the area.

|     | Bearings                          | Distances |
|-----|-----------------------------------|-----------|
| 1st | N.E.                              | 3540      |
| 2d  | N. $\frac{1}{2}$ E.               | 4785      |
| 3d  | N.W. b.W.                         | 3915      |
| 4th | S.W. b.S.                         | 4125      |
| 5th | S.S.E. $7^{\circ}$ E.             | 2030      |
| 6th | E. b.S. $3\frac{1}{2}^{\circ}$ E. | 2945      |

The  $\angle$  between the 1st & 2d dist. is  $3\frac{1}{2}$  points or  $39^{\circ} 22\frac{1}{2}'$

2d & 3d —  $5\frac{1}{2}$  — or  $61 52\frac{1}{2}'$

3d & 4th — 8 — or  $90 00$

4th & 5th — 5 pts.  $7^{\circ}$  or  $63 15$

5th & 6th — 4 pts.  $7\frac{3}{4}^{\circ}$  or  $52 45$

6th & 1st — 4 pts.  $7\frac{3}{4}^{\circ}$  or  $52 45$

Then . /

Sum 32 pts. or 360 00

$$\begin{aligned}
 & \left( \begin{array}{l} \text{fine of } 39 22\frac{1}{2}' \\ \text{--- } 61 52\frac{1}{2}' \\ \text{--- } 90 00 \\ \text{--- } 63 15 \\ \text{--- } 52 45 \\ \text{--- } 52 45 \end{array} \right) \begin{array}{l} \cdot 6343932 \times 3540 \times 4785 = 10745923 \cdot 32 \\ \cdot 8819212 \times 4785 \times 3915 = 16521272 \cdot 60 \\ 1 \cdot 0000000 \times 3915 \times 4125 = 16149375 \cdot 00 \\ \cdot 8929789 \times 4125 \times 2030 = 7477582 \cdot 14 \\ \cdot 7960020 \times 2030 \times 2945 = 47960020 \cdot 26 \\ \cdot 7960020 \times 2945 \times 3540 = 8257338 \cdot 26 \end{array}
 \end{aligned}$$

2) 63951491 32

Acres 319 75745 66

4

Roods 3 0298264

40

Perches 1 193056

CHAP.



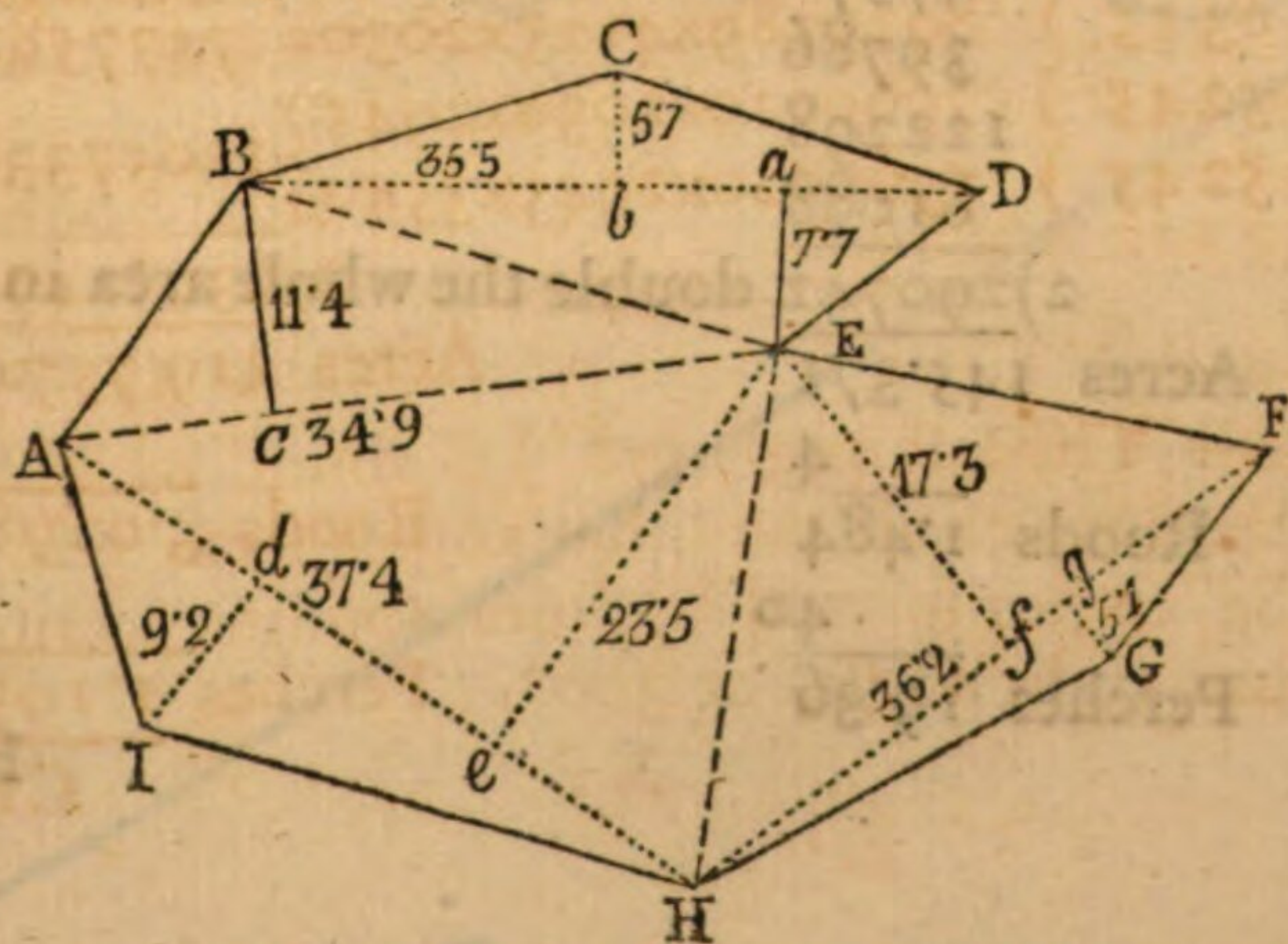
## CHAPTER III.

*Of the general method of casting up the contents of surveys.*

After the survey is finished and the whole put together into one rough plan, the general method of casting up the contents, is to divide the whole into trapeziums and triangles, by drawing lines from angle to angle, and measuring the necessary parts of these figures, by taking their lengths between the compasses and applying them to the scale from which the whole figure was laid down; and then computing their contents by their proper rules laid down in former parts. By these means the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation; and, among a number of parts thus taken and applied to a scale, it is likely that some will be taken too little and some too great, so that they will, upon the whole, in all probability, nearly balance one another.

## EXAMPLE I.

Suppose that, after having planned the figure in the 3d example, I would calculate it in this manner.



First,



First, then, from the angle  $E$  I draw lines to the angles  $B, A, H$ , dividing the figure into three trapeziums  $EDCB, EAIH, EHGF$ , and a triangle  $EBA$ . Then I draw the diagonals  $DB, AH, HF$ , and let fall the perpendiculars  $Ea, Cb, Bc, Id, Ee, Ef, Gg$ .

Then in the trapezium  $EDCB$ , the diagonal measures 3550 links and the perpendiculars 570 and 770. Wherefore  $3550 \times 570 + 770 = 3550 \times 1340 = 47570 =$  double the trapezium  $EDCB$ .

And, in the triangle  $EBA$ ,  $AE$  measures 3490 and the perpendicular 1140, their product is 39786 equal double the triangle.

Also, in the trapezium  $EAIH$ , the diagonal measures 3740, and the perpendiculars 920 and 2350. Wherefore  $3740 \times 920 + 2350 = 3740 \times 3270 = 122298$  equal double this trapezium.

Lastly, in the trapezium  $EHGF$ , the diagonal measures 3620, and the perpendiculars 1730 and 510. Wherefore  $3620 \times 1730 + 510 = 3620 \times 2240 = 81088$  equal double that trapezium.

$$\begin{array}{r}
 47570 \\
 39786 \\
 122298 \\
 81088 \\
 \hline
 2)290742 \text{ double the whole area in links} \\
 \hline
 \text{Acres } 145 \cdot 371 \\
 \quad \quad 4 \\
 \text{Roods } 1 \cdot 484 \\
 \quad \quad 40 \\
 \text{Perches } 19 \cdot 36
 \end{array}$$

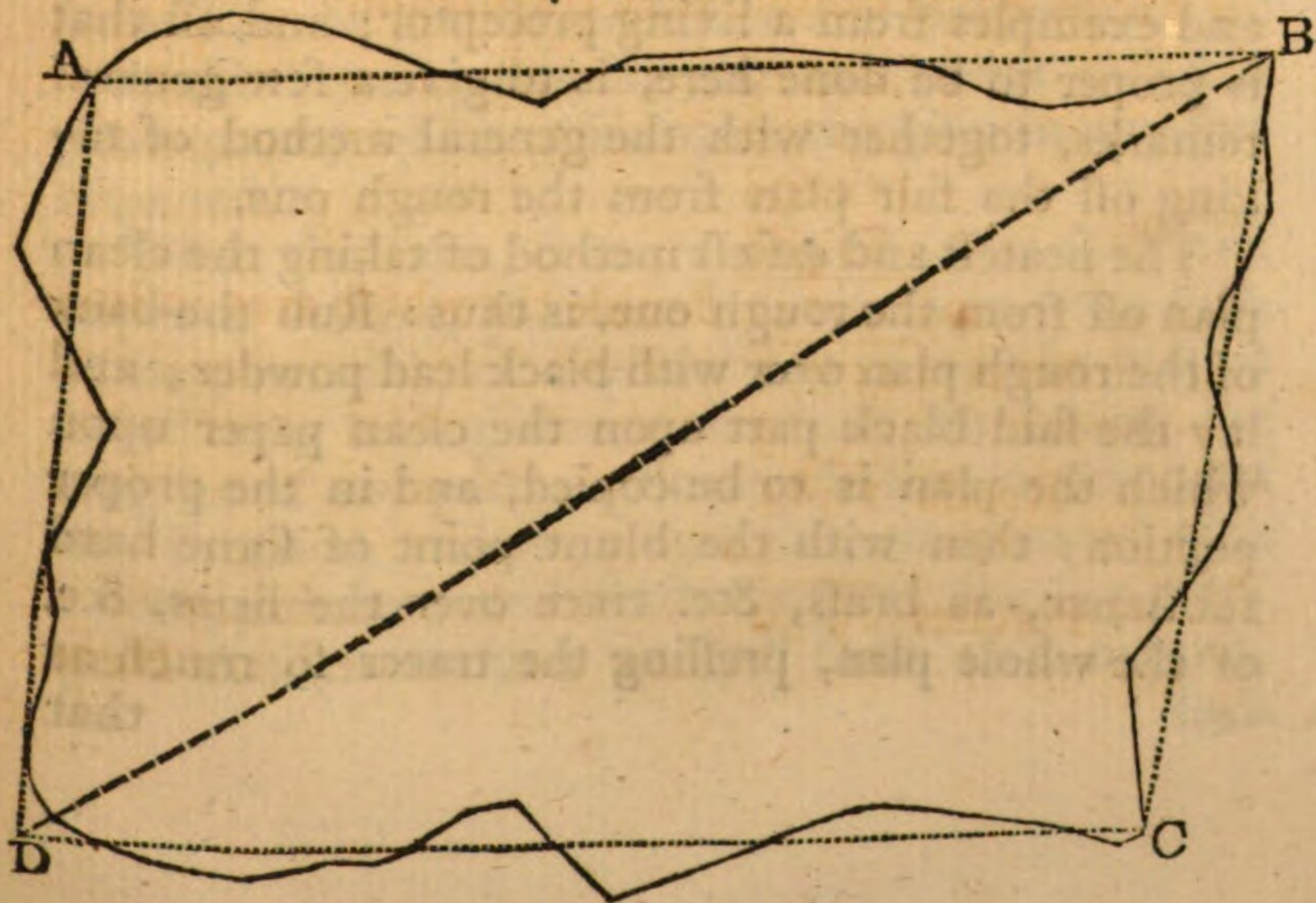
Hence



Hence the area by this method is 145A. 1R. 19P. and is less than the former by 13 perches only, which is less than  $\frac{1}{12}$  part of an acre; a very inconsiderable quantity in so large a figure.

But the chief secret of casting up consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing those lines to straight ones, and is easiest and very accurately performed thus: Stretch a horse hair over the crooked sides so as that the small parts cut off the figure by it may be equal to those which are taken in, which equality of the parts you will presently be able to judge of by a little practice; then mark the ends of the hair with two points, and with a pencil draw a line through those two points: Do the same by the next adjoining side, &c. till the whole be reduced to a straight-sided figure; then find its content as above.

Thus if it were required to find the content of the following figure bounded by the irregular black line, from a scale of 4 chains to the inch. 6 L





I draw the four lines  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  cutting off equal quantities on each side of them, by which means the irregular figure is reduced to the right-lined trapezium  $ABCD$  as nearly equal to it as can be judged. Then, having drawn the diagonal  $DB$ , by measuring I find the distance of  $A$  from  $DB$  to be 790, and the distance of  $C$  from  $DB = 760$ ; whose sum 1550 drawn into  $DB = 1916$  produces 2969800, the half of which or 1484900 square links = 14A. 3R. 15.84P. = the content required.

NOTE that the easiest method of using the hair is to string a small bow of wire, cane, or whale-bone with it; for, *this* keeping it always stretched, it can be used with one hand, while the other is at liberty to make the marks.

#### CHAPTER IV.

##### *Of Planning.*

The ornamenting of plans, as well as the chief part of the subject of surveying, must be got by much practice together with proper instructions and examples from a living preceptor; and all that is proper to be done here, is to give a few general remarks, together with the general method of taking off the fair plan from the rough one.

The neatest and easiest method of taking the clean plan off from the rough one, is thus: Rub the back of the rough plan over with black lead powder; and lay the said black part upon the clean paper upon which the plan is to be copied, and in the proper position; then with the blunt point of some hard substance, as brass, &c. trace over the lines, &c. of the whole plan, pressing the tracer so much as  
that



that the black lead under the lines may be transferred to the clean paper; after which take off the rough plan, and trace over the leaden marks with common or with india ink, &c.—Or, instead of blacking the rough plan, you may keep constantly a blacked paper to lay between the plans.

Another method of copying plans is by means of squares; which is performed by dividing both ends and sides of the plan to be copied into any convenient number of equal parts, and connecting the corresponding points of division with lines, which will divide the plan into a number of little squares; divide the paper upon which the plan is to be copied into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others in the proportion in which the plan is to be increased or diminished, when of a different size; then copy into the clean squares the parts contained in the corresponding squares of the old plan, and you will have the plan copied either of the same size, or greater or less in any proportion.

A third method is by the instrument called a pentagraph, which also copies the plan in any size required.

When the lines, &c. are copied upon the clean paper or vellum, the next business is to write such names, remarks, or explanations as may be judged necessary; laying down the scale for taking the lengths of any parts, a flower-de-luce to point out the direction, and the proper title ornamented with a compartment; and illustrating or colouring every part in such manner as shall seem most natural, such



such as shading rivers or brooks with crooked lines, drawing the representations of trees, bushes, hills, woods, hedges, houses, gates, roads, &c. in their proper places; running a single dotted line for a foot path, and a double one for a carriage road; and either representing the bases or the elevations of buildings, &c.—But to do all these things neatly, you ought to be properly instructed in the art of drawing.

## CHAPTER V.

*Of the Division of Lands.*

In the division of commons, after the whole is surveyed and cast up, and the proper quantities to be allowed for roads, &c. deducted, divide the neat quantity remaining among the several proprietors, by the following problem, in proportion to the real value of their estates, and you will thereby obtain their proportional quantities of the land. But as this division supposes the land, which is to be divided, to be all of an equal goodness, you must observe that if the part in which any one's share is to be marked off, be better or worse than the general mean quality of the land, then you must diminish or augment the quantity of his share in the same proportion as near as you can judge.

## PROBLEM I.

*It is required to divide any given quantity of ground into any given number of parts, and in proportion as any given numbers.*

## RULE.

Divide the given piece after the rule of fellowship, by dividing the whole content by the sum  
of



of the numbers expressing the proportions of the several shares, and multiplying the quotient severally by the said proportional numbers for the respective shares required.

## E X A M P L E.

It is required to divide 300 acres of land among *A, B, C, D, E, F, G,* and *H*, whose claims upon it are respectively in proportion as the numbers 1, 2, 3, 5, 8, 10, 15, 20.

The sum of these proportional numbers is 64, by which dividing 300, the quotient is 4.6875, which being multiplied by each of the numbers 1, 2, 3, 5, &c. we obtain for the several shares as below :

|            |            |      |      |       |
|------------|------------|------|------|-------|
| <i>A</i> = | 4.6875 =   | 4 A. | 2 R. | 30 P. |
| <i>B</i> = | 9.3750 =   | 9    | 1    | 20    |
| <i>C</i> = | 14.0625 =  | 14   | 0    | 10    |
| <i>D</i> = | 23.4375 =  | 23   | 1    | 30    |
| <i>E</i> = | 37.5000 =  | 37   | 2    | 00    |
| <i>F</i> = | 46.8750 =  | 46   | 3    | 20    |
| <i>G</i> = | 70.3125 =  | 70   | 1    | 10    |
| <i>H</i> = | 93.7500 =  | 93   | 3    | 00    |
| Sum =      | 300.0000 = | 300  | 0    | 00    |

## P R O B L E M II.

*To cut off from a plan a given number of acres, &c. by a line drawn from any point in the side of it.*

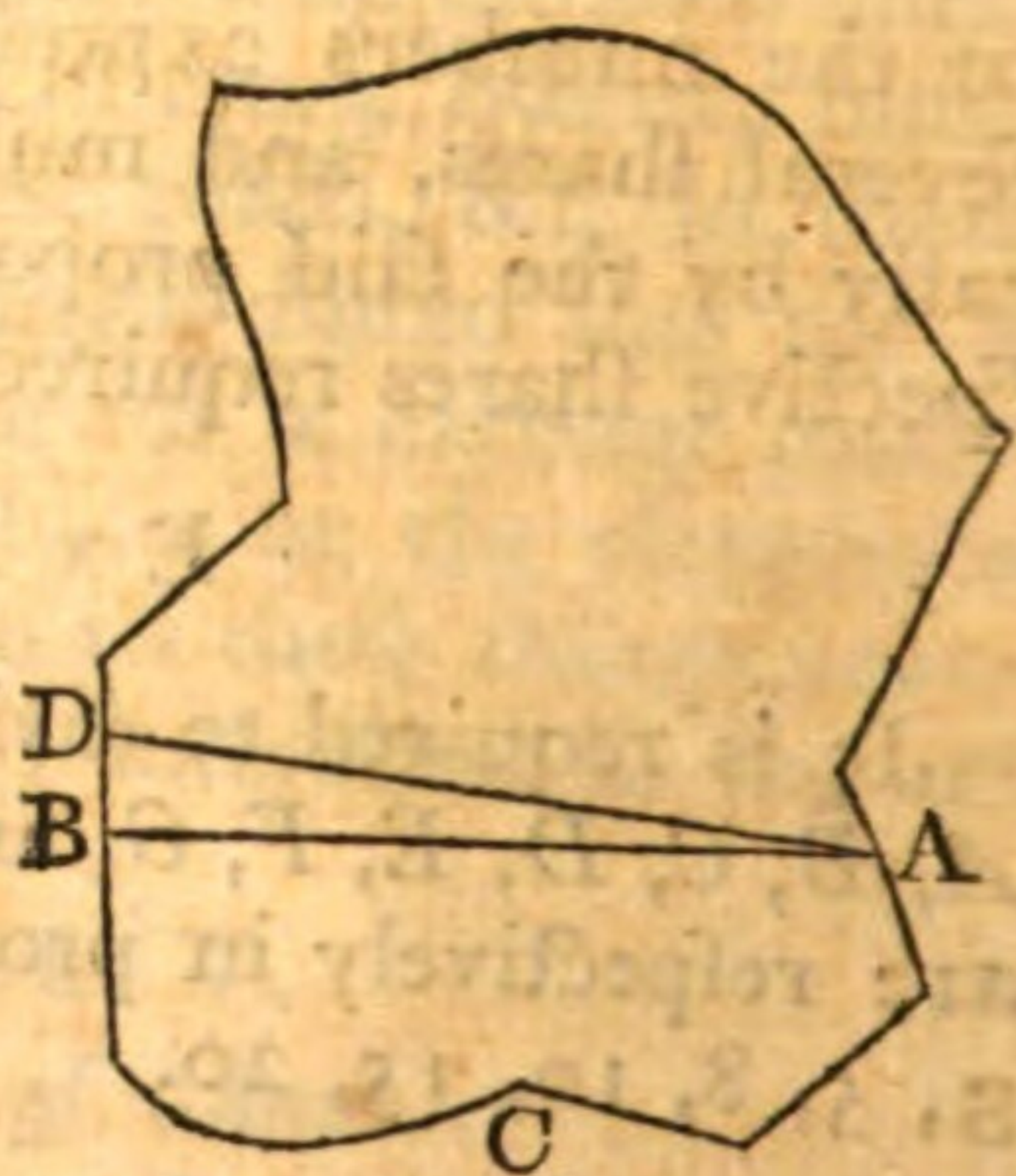
Let *A* be the given point in the annexed plan, from which a line is to be drawn cutting off 5 A. 2 R. 14 P.

6 M

Draw



Draw  $AB$  cutting off the part  $ABC$  as near as can be judged equal to the quantity proposed, and let the true quantity of  $ABC$ , when calculated, be only 4A. 3R. 20P. which is less than 5A. 2R. 14P. the true quantity, by 0A. 2R. 34P. or 71250 square links; then measure  $AB$ , which suppose = 1234 links, and by 617 the half of which divide 71250, the quantity to be added, and the quotient 115 links will be the altitude of the triangle to be added and whose base is  $AB$ ; wherefore if upon the center  $B$  with the radius 115 an arc be described, and  $AD$  be drawn to touch that arc,  $AD$  will be the line cutting off the required quantity  $ADCA$ .



NOTE that if the first piece had been too big, then  $D$  must have been set below  $B$ .

In this manner the several shares of commons to be divided may be laid down upon the plan, and transferred from thence to the ground itself.

Also for the greater ease and perfection in this business, I shall subjoin the following problems.

### PROBLEM III.

*From an angle in a given triangle to draw lines to the opposite side, dividing the triangle into any number of parts, which shall be in any assigned proportion to each other.*

### RULE.

Divide the base into the same number of parts and in the same proportion, by problem 1; then from

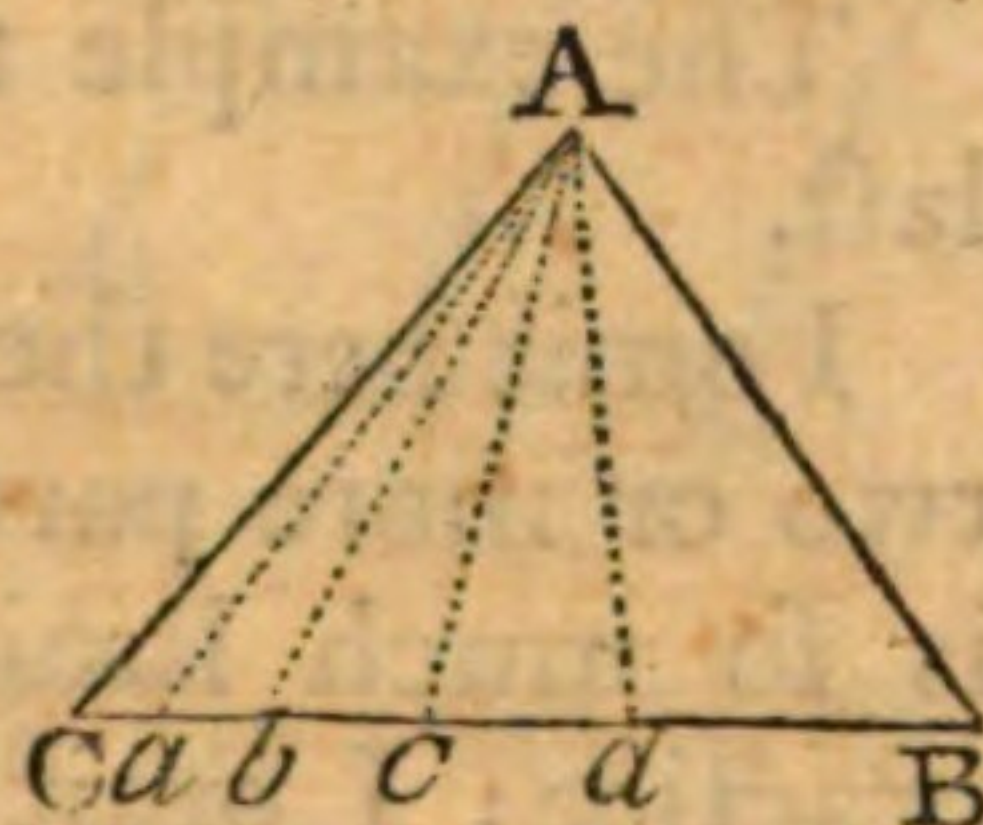


from the several points of division draw lines to the proposed angle, and they will divide the triangle as required.\*

## EXAMPLE.

Let the triangle  $ABC$  of 20 acres be divided into five parts which shall be in proportion to the numbers 1, 2, 3, 5, 9, the lines of division being to be drawn from  $A$  to  $BD$  whose length is 1600 links.

Here  $1 + 2 + 3 + 5 + 9 = 20$ , and  $1600 \div 20 = 80$ , which being multiplied by each of the proportional numbers, we have 80, 160, 240, 400, and 720; wherefore I make  $Ca = 80$ ,  $ab = 160$ ,  $bc = 240$ ,  $cd = 400$ , and  $dB = 720$ ; then by drawing the lines  $Aa$ ,  $Ab$ ,  $Ac$ ,  $Ad$ , the triangle is divided as required.

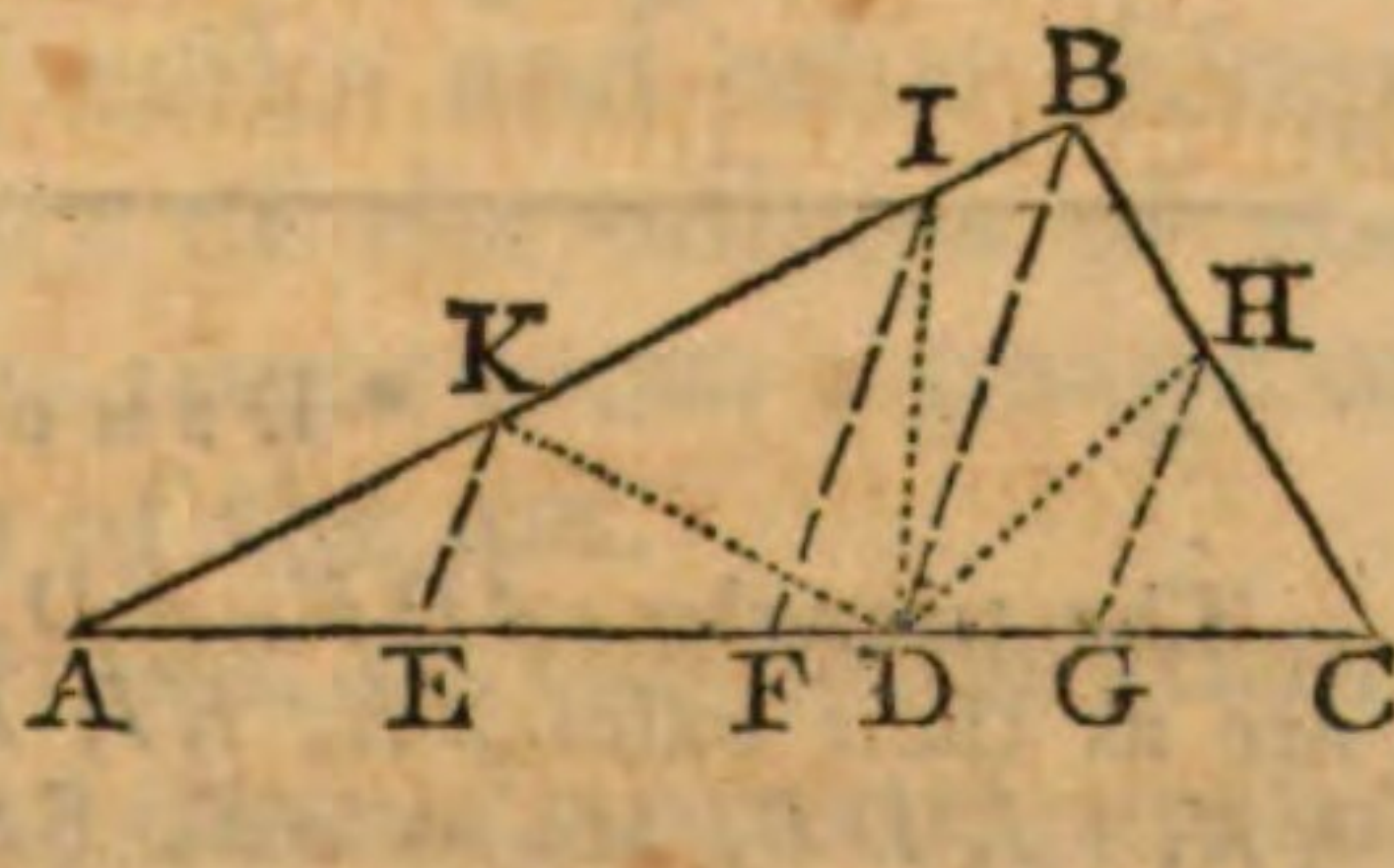


## PROBLEM IV.

*From any point in one side of a given triangle, to draw lines to the other two sides, dividing the triangle into any number of parts which shall be in any assigned ratio.*

## RULE.

From the given point  $D$  draw  $DB$  to the angle opposite to the side  $AC$  in which the point is taken; then divide the same side  $AC$



## \* DEMONSTRATION.

For the several parts are triangles of the same altitude, and which therefore are as their bases, which bases are taken in the assigned proportion.



$AC$  into so many parts  $AE, EF, FG, GC$ , and in the same proportion with the required parts of the triangle, like as was done in the last problem; and from the points of division draw lines  $EK, FI, GH$ , parallel to the line  $BD$ , and meeting with the other sides of the triangle in  $K, I, H$ ; lastly, draw  $KD, ID, HD$ , so shall  $ADK, KDI, IDHB, HDC$  be the parts required.\*

The example to this will be done exactly as the last.

I omit here the division of other figures into either two or more parts, because the method of doing it is so much limited as to render it of no use in practice; I omit also their reduction to triangles, as that is done only for the sake of their division; and shall add here only one problem more, shewing how readily to increase or diminish a figure by changing it from one scale to another; but when large or complex surveys are to be changed to a different scale, a pentagraph ought to be used for the purpose, or else the method of squares as described in chapter 4.

PRO-

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\* DEMONSTRATION.

The triangles  $ADK, KDI, IDB$ , being of the same height, are as their bases  $AK, KI, IB$ , which, by means of the parallels  $EK, FI, DB$ , are as  $AE, EF, FD$ ; in like manner, the triangles  $CDH, HDB$  are to each other as  $CG, GD$ : but the two triangles  $IDB, BDH$ , having the same base  $BD$ , are to each other as the distances of  $I$  and  $H$  from  $BD$ , or as  $FD$  to  $DG$ ; consequently the parts  $DAK, DK I, DIBH, DHC$  are to each other as  $AE, EF, FG, GC$ .

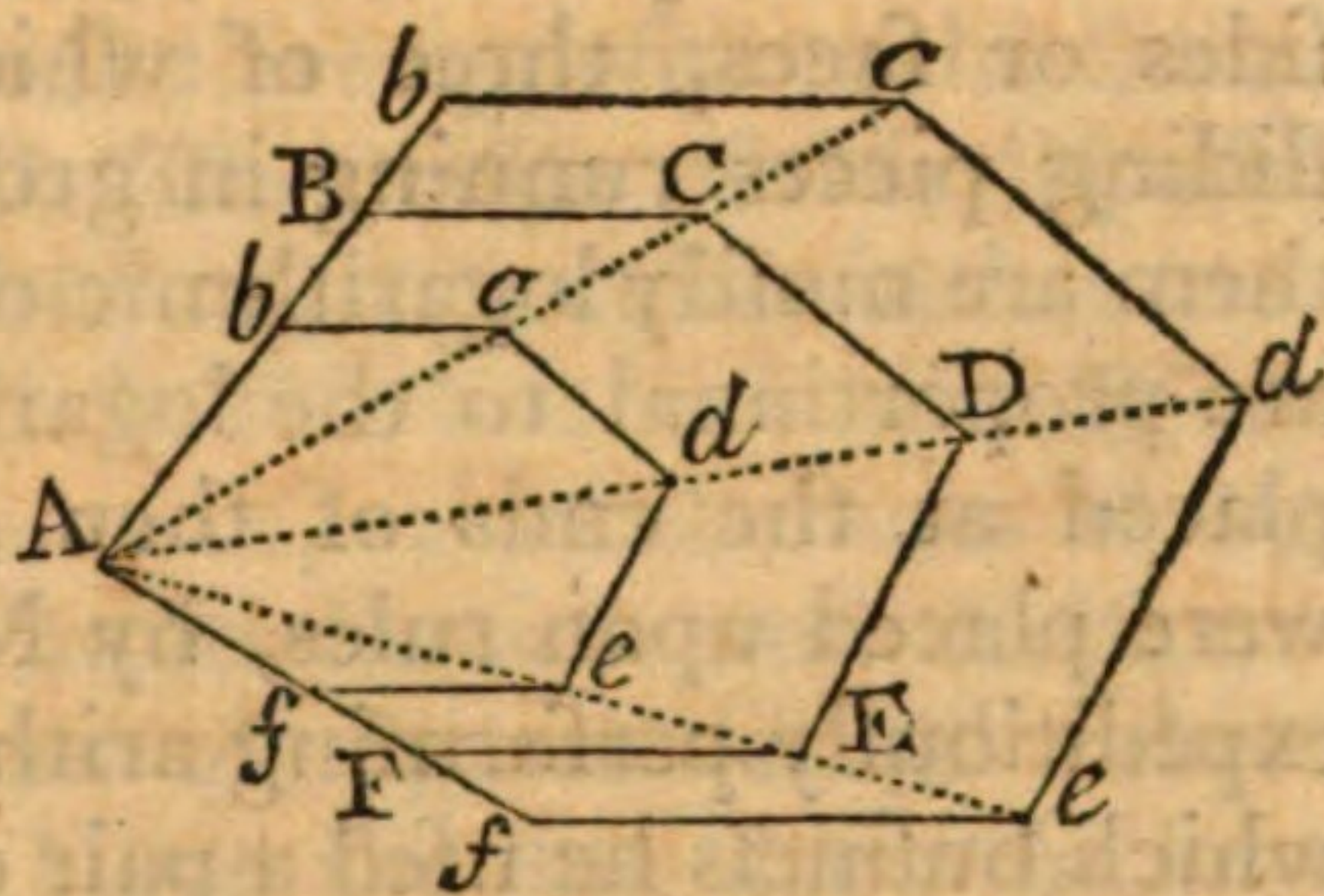


## PROBLEM V.

*To change a figure from one scale to another.*

## R U L E.

From one angle *A* draw lines *AC*, *AD*, *AE*, &c. to all the other angles of the figure given; then augment or diminish one side *AB* till *Ab* be to *AB* in the given proportion of the scales; and, by means of a parallel ruler, draw *bc* parallel to *BC* and meeting *AC* in *c*, and in the same manner *cd* parallel to *CD*, *de* parallel to *DE*, *ef* parallel to *EF*, so shall *AbcdefA* be the figure required.



## SECTION II.

*Of Cask Gauging.*

THE meaning of the word *Gauging* is restricted to the measuring of casks and other things falling under the cognizance of the officers of the excise; and hath received its name from a guage or rod used by the practitioners of the art.

The business being performed, or the calculations made, generally, by means of the instruments, called the gauging or diagonal rod, and the sliding rule or gauging rule, it will be necessary to treat of these instruments, which I shall do as below.



## CHAPTER I.

*The Description and Use of the Sliding Rule.*

This is a square rule having consequently four sides or faces, three of which are furnished with sliding pieces running in grooves. The lines upon them are mostly logarithmic ones, or distances which are proportional to the logarithms of the numbers placed at the ends of them; which kind of lines were placed upon rules, by Mr *Edmund Gunter*, for expeditiously performing arithmetical operations; in which business he used a pair of compasses for taking the several logarithmic distances: but instead of the compasses, sliding pieces were added, by Mr *Thomas Everard*, as being more convenient and certain in practice.

Upon the first face are three lines, marked *A*, *B*, and *MD*; *B* being upon the slider, and is a kind of double line, being marked at both the edges of the slider, for the convenience of application to both the lines *A*, and *MD*. These three are such logarithmic lines as are described above, and are all of the same radius or distance from 1 to 10, containing each twice the length of the radius; *A* and *B* are placed and numbered exactly alike, the numbering beginning at 1, by which means the middle division will be 10 times and the last 100 times the value assigned to the first division, which, both here and in all logarithmic lines, may have any value put upon it at pleasure, either 1, 10, 100, 1000, &c. or .0, .01, .001, .0001, &c. But the number 1 upon the line *MD* is set opposite to the division 215 or more exactly



actly 215042 on the other lines, and then numbered in the order retrograde to that of the lines *A* and *B*; by which means it happens that not only both the divisions, on *MD*, marked 1 are opposite to 215042 on the lines *A* and *B*, but also that all the three divisions, on *A* and *B*, marked 1 are opposite to the same division 215042 on *MD*. The reason of this disposition of the line *MD* is that 2150.42 cubic inches are contained in a malt bushel; this line in conjunction with *B* serving to gauge malt, while the lines *A*, *B*, serve for operations in multiplication and division. Moreover upon the lines *A*, *B*, are several other marks and letters: viz. Upon the line *A* are the letters *MB*, for malt bushel, at the aforesaid number 2150.42; and at 282, the inches in an ale gallon, are a mark and the letter *A*, denoting ale: and upon the line *B* are *si*, for square inscribed, and a mark at .707 &c. the side of a square inscribed in a circle whose diameter is 1; and *se*, for square equal, and a mark at .886 &c. the side of a square equal to the same circle; also *C*, for circumference, and a mark at 3.14159 &c. the circumference of the same circle; and moreover *W*, for wine, and a mark at 231, the inches in a wine gallon.

The second face, being that opposite to the first, is marked *D*, *C*, *D*, *E*, at one end, and Root, Square, Root, Cube, at the other, the lines *C* and *E* containing respectively the squares and the cubes of the divisions set opposite to them on the lines *D*; the two middle lines *C*, *D*, being on a slider. These, also, are logarithmic lines, but the radius of *D* is double to that of *A*, *B*, and *C*, and triple to that of *E*. And whatever the first 1 on *D* denotes,  
the



the first on *C* is the square of it, and the first on *E* the cube of it; so if *D* begin with 1, *C* will begin with 1 and *E* with 1; but if *D* begin with 10, *C* will begin with 100 and *E* with 1000. Upon the line *D* are several points marked, viz. at 17.15, the wine gauge point for circles, is placed *WG* for wine gauge; at 18.95 is marked *AG* for ale gauge; at 46.37, the malt gauge point for square measure, is set *MS* for malt square; and at 52.32 is put *MR*, in like manner, for malt round or circular measure; also at 6.23, the neat tallow gauge point, are placed the letters *TP* for tallow pounds. Moreover upon the line *C* is marked *oc* at .07957, and *od* at .7854, the former being the area of a circle whose circumference is 1, and the latter that of a circle whose diameter is 1.

Upon the third face are three lines; one, upon a slider, marked *N*, which is a logarithmic line of the same radius or size with *A*, *B*, *C*, and *MD*; and two others, marked *S.S.* and *S.L.* denoting segment standing and segment lying, this face being for ullaging standing and lying casks.

And upon the fourth or opposite face are a scale of inches with three other scales, marked spheroid or 1st variety, 2d variety, 3d variety; the line for the fourth or conic variety being upon the inside of the slider in the third face. The use of these lines is to find the mean diameters of casks.

Besides the above lines, are two others upon the insides of the two first sliders, being continued from the one slider upon the other; the one is a scale of inches from  $12\frac{1}{2}$  to 36, and the other a scale of ale gallons from the corresponding numbers .435 to  $3\frac{61}{100}$   
ale



ale gallons; which form a table to shew, in ale gallons, the contents of all cylinders whose diameters are from  $12\frac{1}{2}$  to 36 inches, the common height of them being 1 inch.

As the sliding rule is for performing, very expeditiously, any operations of multiplication, division, and extraction of roots, required by any rule proposed in words &c. so the manner of making these operations will appear in the following problems.

#### PROBLEM I.

*To find the product of two given numbers, by the sliding rule.*

**RULE.** To either of the given numbers on *A* set 1 on *B*, then against the other number on *B* is the product on *A*.

**EXAMPLE I.** Required the product of 12 and 25.  
By placing 1 on *B* under 12 on *A*, above 25 on *B* stands 300 on *A*, which is the product required.

**NOTE.** When the 1 on *B* hath been set to the one factor on *A*, if it happen that the other factor on *B* fall beyond the division on either *A* or *B*, divide it by 10, or 100, &c. till the quotient found on *B* fall under some division on the line *A*, and multiply this said division by the same 10, or 100, &c. for the product required.

**EXAMPLE II.** So when 250 is to be multiplied by 56: Having set 1 on *B* to 250 on *A*, although 56 be found on *B*, it is beyond the end of *A*; therefore dividing it by 10, I find that opposite to the quotient 5.6 on *B* is the division 1400 on *A*, which being multiplied by 10, we obtain 14000 for the product required.



EXAMPLE III. But if 250 were to be multiplied by 1120: Having set 1 to 250 as before, 1120 is beyond the end of *B*, but being divided by 100, opposite to the quotient 11.2 on *B* I find 2800 on *A*, which being multiplied by 100 we have 280000 for the product required.

#### PROBLEM II.

*To find the quotient of two numbers.*

RULE. Set 1 on *B* to the divisor on *A*, then against the dividend on *A* is the quotient on *B*.

EXAMPLE I. To divide 300 by 25. Having set 1 on *B* to 25 on *A*, opposite to 300 on *A* I find 12 on *B*, the quotient required.

NOTE. When the dividend falls beyond the end of the line *A*, let it be divided by 10, 100, or some other power of 10 till it fall within the line, and use the quotient instead of it, multiplying the result by the same power of 10 as before.

EXAMPLE II. So if 14000 must be divided by 56. Having set 1 to 56, the dividend cannot be found on *A* till it be divided by 100, the quotient being 140, opposite to which I find 2.5 on *B*, which being multiplied by 100 we obtain 250 for the quotient required.

#### PROBLEM III.

*To work the rule-of-three on the sliding rule; or having three numbers given, to find a fourth which shall be to the third as the second is to the first.*

RULE. Set the first term on *B* to either the second or third on *A*, then against the remaining term on *B* stands the fourth term required on *A*.

Ex-



EXAMPLE. If 8 yards of cloth cost 24 shillings, what will 96 yards cost at the same rate.

Having set 8 on *B* to 24 on *A*, opposite to 96 on *B*, I find, on *A*, 288 shillings or 14l. 8s. which is the answer.

## PROBLEM IV.

*To extract the square root by the sliding rule.*

RULE. The first 1 on *C* standing against the first 1 on *D* (on the stock), opposite to the given number on *C* is its root on *D*.

EXAMPLE. To find the side of a square, which shall be equal to a triangle, or circle, &c. whose area is 225; or, to extract the root of 225.—Here opposite to 225 on *C* stands 15 on *D*, which is the answer required.

## PROBLEM V.

*To extract the cube root by the sliding rule.*

RULE. The line *D* upon the slide being set streight with *E*; find the given number on *E*, and opposite to it will be its cube root on *D*.

EXAMPLE. To find the side of a cube equal to any other solid whose content is 3375; or to find the cube root of 3375.—Here opposite to 3375 on *E* stands 15 on *D*, which is the answer required.

NOTE. It is evident that the same lines, as are used in these two last problems, will serve to find the square or the cube of any given number, by taking the given number on the contrary lines.

PRO-



## PROBLEM VI.

*To find a mean proportional between two given numbers.*

RULE. Set one of the given numbers on *C* to the like or same number on *D*, then against the other given number on *C* is the number required on *D*.

EXAMPLE. To find the side of a square whose area shall be equal to that of a parallelogram whose length is 9 and its breadth 4 feet; or, to find a mean proportional between 4 and 9.

Having set 4 on *C* to 4 on *D*, against 9 on *C* stands 6 on *D*, which is the number sought.

## PROBLEM VII.

*To find a number which shall be to a given number in a given duplicate proportion; or, having given three numbers, to find a fourth, which shall be to the third as the square of the second is to the square of the first.*

RULE. Set the third number on *C* to the first on *D*, then against the second on *D* will be found, on *C*, the fourth required.

EXAMPLE. If the area of a parallelogram or any other figure be 120; it is required to find the area of a similar figure, their like dimensions or sides being as 2 to 3.

Similar figures being as the squares of their like dimensions, by setting 120 on *C* to 2 on *D*, against 3 on *D* stands 270 on *C*, for the number sought.

PRO-



## PROBLEM VIII.

*To find a number which shall be to a given number in a given subduplicate proportion; or, having given three numbers, to find a fourth, which shall be to the third as the root of the second is to the root of the first.*

RULE. Set the first number on *C* to the third on *D*, then against the second on *C* will be found the fourth on *D*.

EXAMPLE. The side of a regular figure is 2, and its area 120; it is required to find the side of a similar figure whose area is 270.

The roots of the areas of similar figures being as their sides, we must find a number which shall be to 2 as the root of 270 is to the root of 120. Wherefore, having set 120 on *C* to 2 on *D*, against 270 on *C* will be found 3 on *D*, which is the number sought.

## PROBLEM IX.

*To find a number in a given triplicate proportion to a number given; or, having three numbers given, to find a fourth, which shall be to the third as the cube of the second is to the cube of the first.*

RULE. Set the first number on the slide *D* to the third number on *E*, then against the second on *D* is the fourth required on *E*.

EXAMPLE. If a cask, whose length is 40 inches, contain 100 gallons, what will be the content of a similar cask whose length is 36 inches.

6 P

Similar



Similar solids being as the cubes of their like sides, the content required must be to 100 gallons as  $40^3$  is to  $36^3$ . Wherefore setting 40 on *D* to 100 on *E*, against 36 on *D* will be found 72.9 gallons on *E*, which is the content required.

#### PROBLEM X.

*To find a number in a given subtriplicate proportion to a given number; or, having three numbers given, to find a fourth, which shall be to the third as the cube root of the second is to the cube root of the first.*

RULE. Set the third number on *D* to the first on *E*, then against the second on *E* will stand the fourth on *D*.

EXAMPLE. What is the length of a cask whose content is 72.9 gallons, supposing the length of a similar cask to be 40 inches and its content 100 gallons.

Since the dimensions of similar solids are as the cube roots of their contents, we must find a number which shall be to 40 as the cube root of 72.9 is to the cube root of 100. Wherefore, having set 40 on *D* to 100 on *E*, against 72.9 on *E* will be found 36 on *D*, which is the length required.

#### PROBLEM XI.

*The length and breadth of a parallelogram being given, to find its area in malt bushels by the line MD.*

RULE. Set either of the given dimensions on *B* to the other on *MD*, then against 1 on *A* is the required area on *B*.

Ex-



EXAMPLE. How many malt bushels can be contained on every inch of the depth of a cistern whose length is 180 and breadth 72 inches.

By setting 72 on *B* to 180 on *MD*, against 1 on *A* will appear nearly 6 bushels on *B*, which is the quantity required.

#### PROBLEM XII.

*To find, by the line MD, the malt bushels which may be contained in a couch, floor, or cistern, whose length, breadth, and depth are given.*

RULE. Set one of the dimensions on *B* to another on *MD*, then against the third on *A* will appear the content on *B*.

EXAMPLE. Required the number of bushels in the cistern whose length is 230, breadth 58.2, and depth 5.4 inches.

Having set 230 on *B* to 5.4 on *MD*, against 58.2 on *A* I find 33.6 bushels on *B*, which is the content nearly.

The use of the other parts or marks on the rule will appear in the examples farther on.

#### CHAPTER II.

##### *Of the Gauging or Diagonal Rod.*

The diagonal rod is a square rule having four sides or faces, being generally four feet long, and folding together by means of joints.

It takes its name from its use in measuring the diagonals of casks, and computing the contents from



from the said diagonal only; where it may be noted, that by the diagonal of a cask is meant the line from the bung to the intersection of the head with the stave opposite to it, and is generally the longest line that can be drawn from the bung to any part within the cask.

And, accordingly, upon one face of the rule is a scale of inches for taking the measure of the diagonal; to which is adapted the areas, in ale gallons, of circles to the corresponding diameters, like to the lines on the under sides of the three slides in the sliding rule.

Upon the opposite face are two scales of ale and wine gallons, expressing the contents of casks having the corresponding diagonals; and these are the lines which chiefly constitute the difference between this instrument and the sliding rule; for all the other lines upon it are the same with those on that instrument, and are to be used in the same manner.

#### EXAMPLE.

Let it be required to find the content of a cask whose diagonal measures 34.4 inches, which agrees with the cask in the following chapter, whose head and bung diameters are 32 and 24, and length 40 inches; for if to the square of 20, half the length, be added the square of 28, half the sum of the diameters, the square root of the sum will be 34.4 nearly.

Now to this diagonal 34.4, corresponds, upon the rule, the content  $90\frac{3}{4}$  ale or 111 wine gallons; which differs from all the contents, in the next chapter, obtained by considering the cask as belonging



longing to each of the four proposed varieties. So that it is evident there are many cases in which the diagonal line cannot exhibit the true contents; however in most cases I do not doubt but that it may be fully as accurate as guessing at the form or variety of the cask, and then computing the content by the rule proper to that variety.

## CHAPTER III.

*Of Casks considered as divided into several Varieties.*

According to the custom of most writers on this subject, I shall distinguish casks into four forms or varieties, viz.

1. The middle frustum of a spheroid,
2. The middle frustum of a parabolic spindle,
3. The two equal frustums of a paraboloid,
4. The two equal frustums of a cone.

The middle frustums of circular, elliptic, and hyperbolic spindles, I omit here on account of the difficulty of their rules, which renders them unfit for the purpose of practical gauging. And indeed some of the above four forms are of very little real use; for very few, if any, casks are to be met with which will hold so much as the first form, or so little as the third or fourth; so that the second is the most generally, if not the only useful one, of the four varieties.

*Note.* 282 cubic inches make one ale gallon,

231 — — — wine —,

2150.42 — — — a malt bushel.

It is also to be noted that the dimensions are supposed to be inches, in the following rules.

6 Q

PRO-



## PROBLEM I.

*To find the content of a cask of the first or spheroidal variety.*

## RULE.\*

To the square of the head diameter add twice the square of the bung diameter, and multiply the sum by the length of the cask; then if the product be mult.  $\{.00092837\}$  or div.  $\{1077.157\}$  for  $\{ \text{ale} \}$ , by  $\{.00113333\}$  by  $\{882.355\}$  for  $\{ \text{wine} \}$ , the product or quotient will be the measure in gallons.

## BY THE SLIDING RULE.

Set the length of the cask  $\{32.82\}$  for  $\{ \text{ale} \}$  on  $C$  to  $\{29.7\}$  for  $\{ \text{wine} \}$  on  $D$ , and on  $D$  find the bung and head diameters, noting the numbers opposite to them on  $C$ ; then if the latter of these two numbers be added to the double of the former, the sum will be the measure in gallons.

## EXAMPLE.

Required the content of a spheroidal cask whose bung and head diameters are 32 and 24, and length 40 inches.

By

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\* For, by prob. 12 sect. 3 part 3, the content in inches is  $\frac{2B^2 + H^2}{3} \times L$ , which being divided by 282 and 231, becomes  $\frac{2B^2 + H^2}{1077.157} \times L$  or  $\frac{2B^2 + H^2}{882.355} \times L$  or  $\frac{2B^2 + H^2}{1077.157} \times L$  in the one case, and  $\frac{2B^2 + H^2}{882.355} \times L$  in the other;  $B$  being the bung and  $H$  the head diameter, and  $L$  the length of the cask.—And in working by the sliding rule, it need only be remarked that 32.82 and 29.7 are the roots of 1077.157 and 882.355.



*By the Pen.* Here  $2 \times 32^2 + 24^2 \times 40 \times$   
 $\left\{ \begin{smallmatrix} .00092837 \\ .00113333 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 97.4417 \text{ ale} \\ 118.9543 \text{ wine} \end{smallmatrix} \right\}$  gallons = the con-  
 tent required.

*By the Sliding Rule.* Having set 40 on *C* to 32.82 on *D*, against 32 and 24 on *D* stand 38 and 21.3, as near as can be judged, on *C*; then  $2 \times 38 + 21.3 = 76 + 21.3 = 97.3$  ale gallons.

And having set 40 on *C* to 29.7 on *D*, against 32 and 24 on *D* stand 46.5 and 26.1 on *C*; then  $2 \times 46.5 + 26.1 = 93 + 26.1 = 119.1$  wine gallons.

### PROBLEM II.

*To find the content of a cask of the second or parabolic spindle form.*

#### RULE.\*

To the square of the head diameter add twice that of the bung diameter, and from the sum take two-fifths of the square of the difference of the said diameters; then multiply the remainder by the length, and the product multiplied or divided by the same numbers as in the rule to the last problem will give the content.

#### BY THE SLIDING RULE.

As in the last problem, set the length on *C* to 32.82 or 29.7 on *D*, and on *D* find both the bung and head diameters, and also their difference, taking out

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\* For, by prob. 18 sect. 4 part 3, the content in inches is  

$$\frac{8B^2 + 4BH + 3H^2}{15} \times L = \frac{Ln}{3} \times \frac{10B^2 + 4BH + 5H^2}{5} - \frac{2B^2 + 2H^2}{5}$$

$$= \frac{Ln}{3} \times 2B^2 + H^2 - \frac{2}{3} \times B - H^2, \text{ and } \frac{n}{3} \text{ will give the same}$$
 numbers as in the last problem.



out the three numbers opposite to them on *C*; then if to twice the first be added the second, and two-fifths of the third be taken from the sum, the remainder will be the content.

## E X A M P L E.

Required the content of a cask of the second variety, whose bung and head diameters are 32 and 24, and length 40 inches.

*By the Pen.* Here  $2 \times 32^2 + 24^2 - \frac{2}{5} \times 8^2 \times 40 \times$   
 $\left\{ \begin{smallmatrix} .00092837 \\ .00113333 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 96.491 \text{ ale} \\ 117.793 \text{ wine} \end{smallmatrix} \right\} \text{ gallons} = \text{the con-}$   
 tent required.

*By the Sliding Rule.* Having set 40 on *C* to 32.82 on *D*, against 32, 24, and 8 on *D* stand 38, 21.3, and 2.4; then  $2 \times 38 + 21.3 - \frac{2}{5} \times 2.4 = 76 + 21.3 - 0.9 = 96.4$  ale gallons.

And, having set 40 on *C* to 29.7 on *D*, against 32, 24, and 8 on *D* stand 46.5, 26.1, and 2.9; then  $2 \times 46.5 + 26.1 - \frac{2}{5} \times 2.9 = 93 + 26.1 - 1.2 = 117.9$  wine gallons.

## P R O B L E M III.

*To find the content of a cask of the third or paraboloidal variety.*

## R U L E.

To the square of the bung diameter add that of the head diameter, and multiply the sum by the length; then if the sum be

mult.  $\left\{ \begin{smallmatrix} .00139255 \\ .0017 \end{smallmatrix} \right\}$  or div.  $\left\{ \begin{smallmatrix} 718.105 \\ 588.233 \end{smallmatrix} \right\}$  for  $\left\{ \begin{smallmatrix} \text{ale} \\ \text{wine} \end{smallmatrix} \right\}$ ,  
 by the product or quotient will be the content required.

By



## BY THE SLIDING RULE.

Set the length on *C* to  $\left\{ \begin{smallmatrix} 26.8 \\ 24.25 \end{smallmatrix} \right\}$  on *D*  $\left\{ \begin{smallmatrix} \text{for ale} \\ \text{for wine} \end{smallmatrix} \right\}$ ; then find the bung and head diameters on *D*, noting the two opposite numbers on *C*, whose sum will be the content required.\*

## EXAMPLE.

Required the content of a cask of the third variety, whose bung and head diameters are 32 and 24, and length 40 inches.

*By the Pen.* Here  $32^2 + 24^2 \times 40 \times \left\{ \begin{smallmatrix} .00139255 \\ .0017 \end{smallmatrix} \right\}$   
 $= \left\{ \begin{smallmatrix} 89.1232 \text{ ale} \\ 108.8 \text{ wine} \end{smallmatrix} \right\} \text{ gallons} = \text{the content required.}$

*By the Sliding Rule.* Having set 40 on *C* to 26.8 on *D*, against 32 and 24 on *D* stand 57.3 and 32; whose sum is 89.3 ale gallons.

And having set 40 on *C* to 24.25 on *D*, against 32 and 24 on *D* stand 69.8 and 39.1, whose sum is 108.9 wine gallons.

## PROBLEM IV.

*To find the content of a cask of the fourth or conical variety.*

## RULE.

To three times the square of the sum of the diameters add the square of the difference of the diameters;  
                                                 6 R                                                  ameters;

\* For, by prob. 12 sect. 4 part 3, the content in inches is  $\frac{B^2 + H^2}{2} \times L$ ; and  $\frac{n}{2 \times 282} = \frac{1}{718.105} = .00139255$ , and  $\frac{n}{2 \times 231} = \frac{1}{588.233} = .0017$ .

Also the numbers 26.8 and 24.25 are the roots of the numbers 718.105 and 588.233.



ameters; then if the sum be  
 mult.  $\{.00023209\}$  or div.  $\{4308.628\}$  for  $\{ale\}$   
 by  $\{.00028333\}$  by  $\{3529.42\}$  for  $\{wine\}$   
 the product or quotient will be the content required.

BY THE SLIDING RULE.

Set the length on  $C$  to  $\{65.64\}$  for  $\{ale\}$  on  $D$ ,  
 and on  $D$  find the sum and the difference of the  
 diameters, noting their opposite numbers on  $C$ ; then  
 if the second be added to three times the first, the  
 sum will be the content.\*

#### EXAMPLE.

Required the content of a cask of the fourth variety, whose bung and head diameters are 32 and 24, and length 40 inches.

*By the Pen.* Here  $3 \times 56^2 + 8^2 \times 40 \times \{.00023209\}$   
 $= \{87.9342 ale\}$   
 $\{107.348 wine\}$  gallons = the content required.

*By the Sliding Rule.* Having set 40 on  $C$  to 65.64 on  $D$ , against 56 and 8 on  $D$  stand 29.1 and 0.6; then  $3 \times 29.1 + 0.6 = 87.3 + 0.6 = 87.9$  ale gallons.

And, having set 40 on  $C$  to 59.41 on  $D$ , against 56 and 8 on  $D$  stand 35.6 and 0.7; then  $3 \times 35.6 + 0.7 = 107.5$  wine gallons.

CHAP-

\* For, by prob. 8 sect. 1 part 3, the content in inches is  
 $\frac{B^2 + BH + H^2}{3} \times L n = \frac{Ln}{12} \times 3 \times \frac{B + H}{2} \times \frac{B + H}{2} + B - H$ ; where

$\frac{n}{12 \times 282} = \frac{1}{4308.628} = .00023209$ , and  $\frac{n}{12 \times 231} = \frac{1}{3529.42} = .00028333$ . Also 65.64 and 59.41 are the roots of the numbers 4308.628 and 3529.42.



## CHAPTER IV.

*Of gauging Casks by their mean diameters.*

## PROBLEM I.

*To find the mean diameter of a cask of any of the four varieties, having given the bung and head diameters.*

## RULE.\*

Divide the head by the bung diameter, and find the quotient in the first column of the following table, marked Qu. then if the bung diameter be multiplied by the number upon the same line with it, and in the column answering to the proper variety, the product will be the true mean diameter required, or the diameter of a cylinder of the same content with the cask proposed.

Qu.

\* *The Investigation of the numbers of this table.*

By the 3d part it appears that if the bung diameter be represented by 1, and the head diameter by  $h$ , then the content for the 1st, 2d, 3d, and 4th varieties will be respectively  $ln$  drawn into

$$\left. \begin{array}{l} \frac{2+hh}{3}, \\ \frac{8+4h+3hh}{15}, \\ \frac{1+hh}{2}, \\ \frac{1+h+hh}{3}, \end{array} \right\} \begin{array}{l} \text{but the content must also be} \\ \text{equal to } ln \text{ drawn into the} \\ \text{square of the mean diamete-} \\ \text{ter; and consequently the} \\ \text{said mean diameter, or mul-} \\ \text{tiplier in the table, will be} \\ \text{respectively} \end{array} \left\{ \begin{array}{l} \sqrt{\frac{2+hh}{3}}, \\ \sqrt{\frac{8+4h+3hh}{15}}, \\ \sqrt{\frac{1+hh}{2}}, \\ \sqrt{\frac{1+h+hh}{3}}, \end{array} \right.$$

Then by writing, in these forms, the several values of  $h$ , viz. .50, .51, .52, &c. there will result the corresponding numbers of the table.—A farther dissertation upon these mean diameters may be seen in *Moss's gauging*.



| Qu. | 1 Var. | 2 Var. | 3 Var. | 4 Var. | Qu. | 1 Var. | 2 Var. | 3 Var. | 4 Var. |
|-----|--------|--------|--------|--------|-----|--------|--------|--------|--------|
| 50  | 8660   | 8465   | 7905   | 7637   | 76  | 9270   | 9227   | 8881   | 8827   |
| 51  | 8680   | 8493   | 7937   | 7681   | 77  | 9296   | 9258   | 8924   | 8874   |
| 52  | 8700   | 8520   | 7970   | 7725   | 78  | 9324   | 9290   | 8967   | 8922   |
| 53  | 8720   | 8548   | 8002   | 7769   | 79  | 9352   | 9320   | 9011   | 8970   |
| 54  | 8740   | 8576   | 8036   | 7813   | 80  | 9380   | 9352   | 9055   | 9018   |
| 55  | 8760   | 8605   | 8070   | 7858   | 81  | 9409   | 9383   | 9100   | 9066   |
| 56  | 8781   | 8633   | 8104   | 7902   | 82  | 9438   | 9415   | 9144   | 9114   |
| 57  | 8802   | 8662   | 8140   | 7947   | 83  | 9467   | 9446   | 9189   | 9163   |
| 58  | 8824   | 8690   | 8174   | 7992   | 84  | 9496   | 9478   | 9234   | 9211   |
| 59  | 8846   | 8720   | 8210   | 8037   | 85  | 9526   | 9510   | 9280   | 9260   |
| 60  | 8869   | 8748   | 8246   | 8082   | 86  | 9556   | 9542   | 9326   | 9308   |
| 61  | 8892   | 8777   | 8282   | 8128   | 87  | 9586   | 9574   | 9372   | 9357   |
| 62  | 8915   | 8806   | 8320   | 8173   | 88  | 9616   | 9606   | 9419   | 9406   |
| 63  | 8938   | 8835   | 8357   | 8220   | 89  | 9647   | 9638   | 9466   | 9455   |
| 64  | 8962   | 8865   | 8395   | 8265   | 90  | 9678   | 9671   | 9513   | 9504   |
| 65  | 8986   | 8894   | 8433   | 8311   | 91  | 9710   | 9703   | 9560   | 9553   |
| 66  | 9010   | 8924   | 8472   | 8357   | 92  | 9740   | 9736   | 9608   | 9602   |
| 67  | 9034   | 8954   | 8511   | 8404   | 93  | 9772   | 9768   | 9656   | 9652   |
| 68  | 9060   | 8983   | 8551   | 8450   | 94  | 9804   | 9801   | 9704   | 9701   |
| 69  | 9084   | 9013   | 8590   | 8497   | 95  | 9836   | 9834   | 9753   | 9751   |
| 70  | 9110   | 9044   | 8631   | 8544   | 96  | 9868   | 9867   | 9802   | 9800   |
| 71  | 9136   | 9074   | 8672   | 8590   | 97  | 9901   | 9900   | 9851   | 9850   |
| 72  | 9162   | 9104   | 8713   | 8637   | 98  | 9933   | 9933   | 9900   | 9900   |
| 73  | 9188   | 9135   | 8754   | 8685   | 99  | 9966   | 9966   | 9950   | 9950   |
| 74  | 9215   | 9166   | 8796   | 8732   | 100 | 10000  | 10000  | 10000  | 10000  |
| 75  | 9242   | 9196   | 8838   | 8780   |     |        |        |        |        |

## EXAMPLE.

Supposing the diameters to be 32 and 24, it is required to find the mean diameter for each variety.

Dividing 24 by 32 we obtain .75, which being found in the column of quotients, opposite thereto stand the numbers

$\left\{ \begin{array}{l} .9242 \\ .9196 \\ .8838 \\ .8780 \end{array} \right\}$  which being each multiplied by 32, produce respectively
  $\left\{ \begin{array}{l} 29.5744 \\ 29.4272 \\ 28.2816 \\ 28.0960 \end{array} \right\}$  the corresponding mean diameters required.

By



## BY THE SLIDING RULE.

Find the difference between the bung and head diameters upon the fourth face of the rule, or inside of the third slider, and opposite thereto is, for each variety, a number to be added to the head diameter for the mean diameter required.

So in the above example, against 8, the difference of the diameters, are found the numbers

$\left. \begin{array}{l} 5.60 \\ 5.10 \\ 4.56 \\ 4.12 \end{array} \right\} \begin{array}{l} \text{which be-} \\ \text{ing added} \\ \text{to 24, there} \\ \text{result} \end{array} \left\{ \begin{array}{l} 29.60 \\ 29.10 \\ 28.56 \\ 28.12 \end{array} \right\} \begin{array}{l} \text{for the respective mean} \\ \text{diameters, all of which} \\ \text{are too great except the} \\ \text{2d, which is too little.} \end{array}$

So that this method does not give the true mean diameter.

## PROBLEM II.

*To find the content of a cask by the mean diameter on the sliding rule.*

## RULE.

Set the length on *C* to the \* gage point, 18.95 for ale, or 17.15 for wine, on *D*; then against the mean diameter on *D*, is the content on *C*.

## EXAMPLE.

If the bung diameter be 32, the head 24, and the length 40 inches.

6 S

Having

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\* The above gage points are found thus, viz.  $\sqrt{\frac{282}{.785398 \&c.}} = 18.95$ , and  $\sqrt{\frac{231}{.785398 \&c.}} = 17.15$ .



Having found the mean diameters as in the last problem, and set 40 on *C* to 18.95 or 17.15 on *D*,  
 against  $\left\{ \begin{array}{l} 29.57 \\ 29.43 \\ 28.28 \\ 28.10 \end{array} \right\}$  on *D* is  $\left\{ \begin{array}{l} 97.4 \\ 96.5 \\ 89.1 \\ 88.0 \end{array} \right\}$  or  $\left\{ \begin{array}{l} 119.5 \\ 118 \\ 108.8 \\ 107.3 \end{array} \right\}$  on *C*, as near as can be judged; which agree nearly with the contents determined in the preceding chapter.

## CHAPTER V.

*Of the gauging of all casks in general by means of four dimensions, viz. the length, the bung and head diameters, and the diameter taken in the middle between the bung and head.*

## GENERAL PROBLEM.

*To find the content of any cask in ale or wine gallons, by four dimensions.*

## RULE.\*

Add into one sum the square of the bung diameter, the square of the head diameter, and 4 times the

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\* This rule is taken from prop. 3 sect. 1 part 4; the number .0005 $\frac{2}{3}$  being =  $\frac{.785398 \text{ \&c.}}{6 \times 231}$  extremely near, and  $\frac{.785398 \text{ \&c.}}{6 \times 282} = .0004641844$ .

And with regard to the operation by the sliding rule, it may be observed that 42 is =  $\sqrt{\frac{6 \times 231}{.785398 \text{ \&c.}}}$ , and 46.4 =  $\sqrt{\frac{6 \times 282}{.785398 \text{ \&c.}}}$ .

The above rule, by the said prop. 3 sect. 1 part 4, was proved to be accurately true, not only for all the four different varieties of casks, but also for all casks and solids generated from any conic section whatever; and although the cask be not precisely in the form of any such curve, the rule will give the content extremely near the truth;



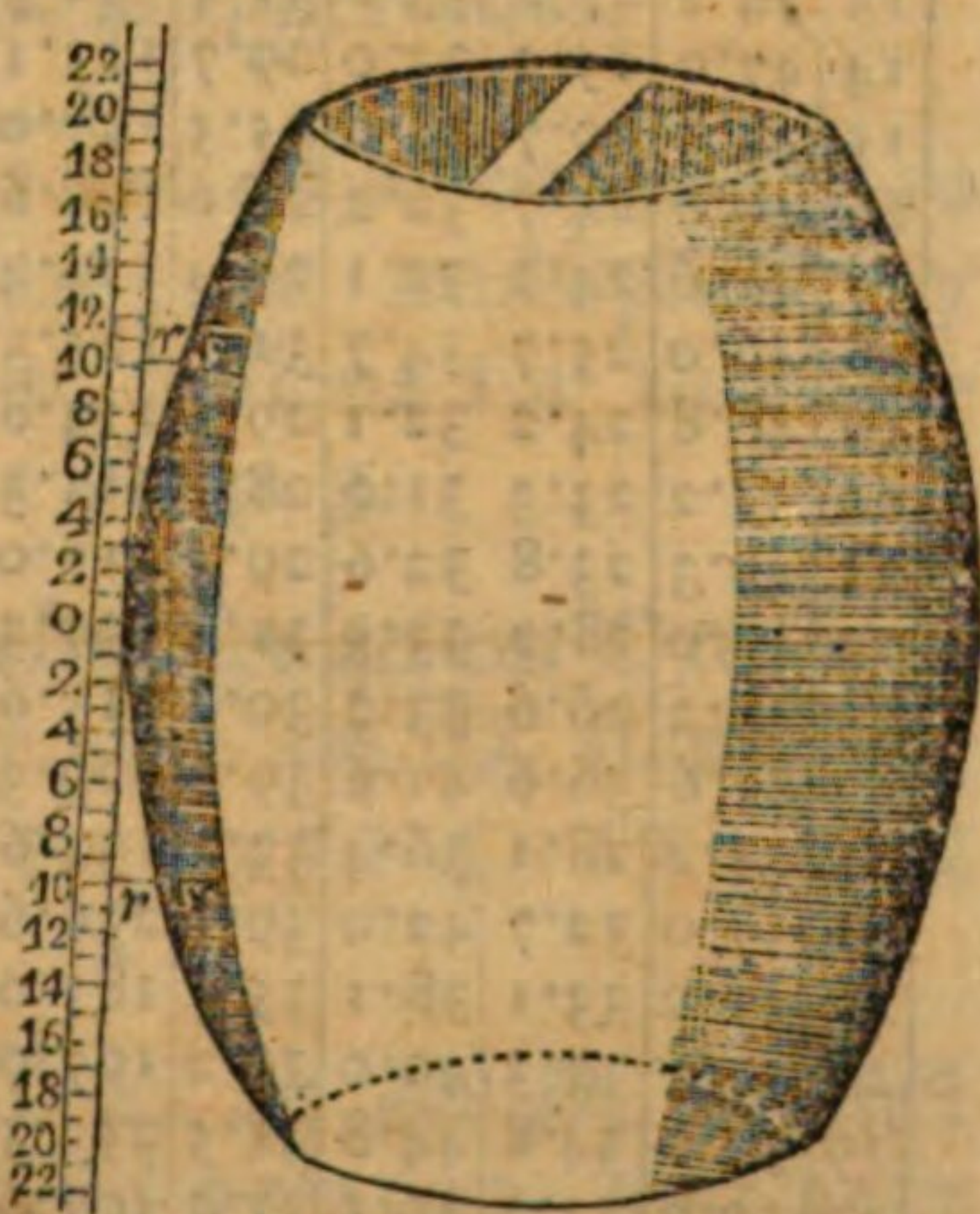
the square of the middle diameter between the bung and head, and multiply that sum by the length of the cask; then the product

mult.  $\left. \begin{array}{l} .00046418 \\ \text{by } .0005\frac{2}{3} \end{array} \right\} \begin{array}{l} \text{for ale} \\ \text{for wine} \end{array} \right\}$  will give the content.

*Note.*

truth; so that whatever be the form of the cask, we may in all cases be pretty sure of the content to within  $\frac{1}{10}$  of a gallon, or perhaps less, supposing the dimensions to be truly taken. So that more perfect than this, both with respect to truth and expedition, nothing can be expected or indeed wished for in gauging: which makes me hope that one day this method will come into general use with the practitioners in the excise; and till then, I am fully persuaded that much of their practice must be mere guessed work. The pretended difficulty of taking the middle diameter may perhaps deter many from using this method, but there cannot be any real difficulty in taking this diameter, except when a wooden hoop may happen to be on at the part where the diameter ought to be taken; but that will very rarely happen.

To find the fourth dimension, or diameter in the middle between the bung and head; upon one side of a square rule let be drawn a scale of quarter inches, numbered both ways from nothing; and let the middle or 0 division be applied to the bung or middle of the cask, as in the margin, parallel or nearly parallel to the axe, and in the direction of the staff; then whatever number of inches are in the whole length of the cask, it is evident that, from the nature of the scale, the same division or number of quarter inches will be opposite to the part where the middle diameter must be taken, which here suppose to be 10; and at 10, on each part of the rule, measure the distance  $rs$ ; then if the sum of  $rs$  and  $rs$  be taken from the bung diameter, there will remain the required middle diameter, excepting the allowance for the thickness of the staff, which must be subducted.



*Casks*



*Note.* To abbreviate the operation,  $\frac{5}{11}$  may be used instead of 4.6418, to which it is nearly equal. Also

| <i>Casks gauged by four dimensions.</i> |        |            |            |            |             |                       |
|-----------------------------------------|--------|------------|------------|------------|-------------|-----------------------|
| No.                                     | Length | Head diam. | Bung diam. | Mid. diam. | Ale gallons | Diff. less than sphr. |
| 1                                       | 28.3   | 23.2       | 27.7       | 26.3       | 53.6        | 1.0                   |
| 2                                       | 29.8   | 22.2       | 26.0       | 24.8       | 50.2        | 1.1                   |
| 3                                       | 30.8   | 23.2       | 27.5       | 26.1       | 57.7        | 1.1                   |
| 4                                       | 32.2   | 24.5       | 30.1       | 28.4       | 70.6        | 1.3                   |
| 5                                       | 30.0   | 24.7       | 29.2       | 27.6       | 62.6        | 1.8                   |
| 6                                       | 32.5   | 23.8       | 28.2       | 26.8       | 63.6        | 1.9                   |
| 7                                       | 34.3   | 26.3       | 33.5       | 31.1       | 90.4        | 2.9                   |
| 8                                       | 34.5   | 26.4       | 33.0       | 30.7       | 89.0        | 3.0                   |
| 9                                       | 41.0   | 26.3       | 32.2       | 30.2       | 102.2       | 3.0                   |
| 10                                      | 37.0   | 26.1       | 31.8       | 29.9       | 90.3        | 3.1                   |
| 11                                      | 44.5   | 34.4       | 40.8       | 38.8       | 183.8       | 3.2                   |
| 12                                      | 47.0   | 26.3       | 33.8       | 31.4       | 126.3       | 3.5                   |
| 13                                      | 34.2   | 27.2       | 33.8       | 31.4       | 92.3        | 3.8                   |
| 14                                      | 47.0   | 25.3       | 32.0       | 29.7       | 113.1       | 4.3                   |
| 15                                      | 45.5   | 30.7       | 38.0       | 35.5       | 157.0       | 4.7                   |
| 16                                      | 44.6   | 24.7       | 32.2       | 29.6       | 106.6       | 4.7                   |
| 17                                      | 48.6   | 24.2       | 32.1       | 29.4       | 114.4       | 4.9                   |
| 18                                      | 46.0   | 25.7       | 34.7       | 31.7       | 125.3       | 5.5                   |
| 19                                      | 48.8   | 24.2       | 32.1       | 29.4       | 114.8       | 5.5                   |
| 20                                      | 51.2   | 23.3       | 31.0       | 28.2       | 111.3       | 5.8                   |
| 21                                      | 49.3   | 23.8       | 32.6       | 29.5       | 117.0       | 6.1                   |
| 22                                      | 48.0   | 28.2       | 33.8       | 31.4       | 137.3       | 6.1                   |
| 23                                      | 45.2   | 26.6       | 33.2       | 30.4       | 115.6       | 6.5                   |
| 24                                      | 51.6   | 36.6       | 41.6       | 39.6       | 223.3       | 6.7                   |
| 25                                      | 44.2   | 28.1       | 36.4       | 33.3       | 134.6       | 6.8                   |
| 26                                      | 57.0   | 32.7       | 42.0       | 39.1       | 236.6       | 6.8                   |
| 27                                      | 51.0   | 33.1       | 38.1       | 35.7       | 181.0       | 8.0                   |
| 28                                      | 51.5   | 33.3       | 40.0       | 37.2       | 197.0       | 8.7                   |
| 29                                      | 54.0   | 34.8       | 44.8       | 41.5       | 253.5       | 8.8                   |
| 30                                      | 50.0   | 34.3       | 40.5       | 37.7       | 197.6       | 9.4                   |
| 31                                      | 49.0   | 29.5       | 36.0       | 33.0       | 148.1       | 9.4                   |
| 32                                      | 51.0   | 33.5       | 39.2       | 36.4       | 189.6       | 9.7                   |
| 33                                      | 51.0   | 33.4       | 39.8       | 37.0       | 194.0       | 9.8                   |
| 34                                      | 55.5   | 30.6       | 40.6       | 37.0       | 207.2       | 10.2                  |
| 35                                      | 45.6   | 28.0       | 34.6       | 32.4       | 134.8       | 12.0                  |
| 36                                      | 55.0   | 35.8       | 48.0       | 43.2       | 282.2       | 17.8                  |

The annexed collection of casks happened in real practice; and their dimensions were carefully taken; but their contents were computed by a sliding rule, and so may not all be precisely true.

From hence it appears that a spheroidal cask is a mere imaginary thing, the contents of real casks being less than is assigned to them by that form; as indeed they ought, from the nature of the curves; for a spheroidal cask would be least curved in the middle and the most at the ends, whereas a real cask is the least curved at the ends, if it be any thing curved there at all; and indeed I have reason to think it is not, as will appear in chap. 7.



Also instead of 4 times the square of the middle diameter, may be used the square of twice that diameter.

#### BY THE SLIDING RULE.

Set the length on *C* to  $\left\{ \begin{array}{l} 46.4 \text{ for ale} \\ 42.0 \text{ for wine} \end{array} \right\}$  on *D*; and find both the bung, head, and middle diameters on *D*, noting the three numbers against them on *C*; then the sum of the first and second with 4 times the third will be the content required.

#### EXAMPLE.

Let the length of a cask be 40 inches, the bung diameter 32, the head 24, and middle diameter 30.2 inches nearly =  $\sqrt{912}$ , which is taken upon the supposition that the cask is spheroidal.

Then  $32^2 + 24^2 + 4 \times 912 \times 40 \times .0004641844 = 97.44159$  &c. ale gallons; or multiplied by  $.0005\frac{2}{3}$ , gives  $118.954\frac{2}{3}$  wine gallons for the content, the same as at prob. 1 chap. 3.

*By the Sliding rule.* Having set 40 on *C* to 46.4 on *D*, against 32, 24, and 30.2 on *D*, stand 19, 10.5, and 17 on *C*; then  $19 + 10.5 + 4 \times 17 = 97.5$  ale gallons.

And, having set 40 on *C* to 42 on *D*, against 32, 24, and 30.2 on *D*, stand 23.2, 13, and 20.7 on *C*; then  $23.2 + 13 + 4 \times 20.7 = 119$  wine gallons, for the content as before nearly.



## CHAPTER VI.

*A new, easy, and expeditious method of computing the content of a cask from three dimensions only, of whatever variety it may be.*

## R U L E.\*

In the column of diameters, or middle column, in the following table, find the bung and head diameters, taking out the number on the left of the head

\* The method of gauging here proposed, is no other than the rule for one form of solids corrected so as to make it agree, in content, with the most common or general form of casks: and, for the more expedition, the squares of all diameters, within certain limits, are divided by the proper constant divisor, and the quotients ranged in the table opposite to their corresponding diameters; that in practice nothing more may be required than to multiply these quotients by any assigned length of a cask, for the content in gallons.

Now it hath been found that wine hogheads generally contain about a gallon and a quarter less than the content assigned to them by the rule for spheroidal casks; to correct the rule for that form of casks, then, we must increase the divisor 882.355 in the proportion of  $61\frac{1}{4}$  to 63; by which proportion we obtain 900, very nearly, to be used instead of 882.355, and then the rule will become  $\frac{H^2 + 2B^2}{900} \times L$  for the content in gallons.

Then as the least diameter of a wine hoghead is about 19 inches, and the greatest bung of a pipe about 35, to all diameters, in inches and tenths, from 18 to 36.3, I have computed the values of the quantities  $\frac{H^2}{900}$  and  $\frac{2B^2}{900}$ , and disposed them in the columns titled head and bung areas; and which it is evident need only be taken out of the table, for any example of bung and head diameters, and multiplying their sum by the length must give the content. So that we have here a general and easy rule by which any person, in half a minute, may compute the content of any cask whose bung and head diameters, and length are given, and that generally nearer to the truth than by any rule now commonly used.

I shall here insert the method in which I made this table; for although I have as above explained the property upon which the table



head diameter, and the number on the right of the bung diameter; then the sum of those two numbers multiplied by the length of the cask will give the content in wine gallons. And the wine gallons being multiplied by 77 and the product divided by 94, the quotient will be the ale gallons; or multiply by 9 and divide by 11 for ale gallons, because  $\frac{2\frac{3}{8}}{2} = \frac{7\frac{7}{8}}{9\frac{1}{4}} = \frac{9}{11}$  extremely near. *A*

table is founded, viz. that the numbers in it consist of every diameter squared and then divided by 900, I did not find the numbers in that manner, but by the following method in perhaps a tenth part of the time required by the former.

Since the 1st, 2d, 3d, 4th, &c. numbers are

$$\frac{18^2}{900} = .36 = A,$$

$$\frac{18.1^2}{900} = \frac{18^2}{900} + \frac{3.6}{900} + \frac{.01}{900} = \frac{18^2}{900} + \frac{3.61}{900} = A + \frac{3.61}{900} = A + .00401 = B,$$

$$\frac{18.2^2}{900} = \frac{18^2}{900} + \frac{2 \times 3.6}{900} + \frac{.04}{900} = \frac{18^2}{900} + \frac{3.61}{900} + \frac{3.63}{900} = B + \frac{3.63}{900} = B + .00403 = C,$$

$$\frac{18.3^2}{900} = \frac{18^2}{900} + \frac{3 \times 3.6}{900} + \frac{.09}{900} = \frac{18^2}{900} + \frac{.006}{19.8} + \frac{3.63}{900} + \frac{3.65}{900} = C + \frac{3.65}{900} = C + .00405 = D,$$

&c.

it is evident that the differences of the terms form an arithmetical series whose common difference is  $\frac{.02}{900} = .00002$ , and therefore by

writing down a column of differences, and to the first term adding the first difference, to obtain the second term; then to the second term adding the second difference, for the third term; and so on as appears in the margin; the terms will evidently be found in a very expeditious manner: also by doubling these head areas we obtain the corresponding bung areas.—In the above computations the last figures, having a point above them, are infinite repetends.

| Diff. | Areas. |
|-------|--------|
|       | 36000  |
| 401   | 36401  |
| 403   | 36804  |
| 405   | 37210  |
| 407   | 37617  |
| 410   | 38027  |
| 412   | 38440  |
| 414   | 38854  |

Farther, it is evident that in this method, the gauge point will be 30 ( $= \sqrt{900}$ ), which is to be used, in the rule for spheroidal casks, instead of 29.7, and the correct content will be obtained.

*A.*



## A Table of Areas.

| Head<br>area | Diam-<br>eter | Bung<br>area | Head<br>area | Diam-<br>eter | Bung<br>area | Head<br>area | Diam-<br>eter | Bung<br>area | Head<br>area | Diam-<br>eter | Bung<br>area |
|--------------|---------------|--------------|--------------|---------------|--------------|--------------|---------------|--------------|--------------|---------------|--------------|
| 3600         | 18.0          | 7200         | 5675         | 22.6          | 1.1350       | 8220         | 27.2          | 1.6441       | 1.1236       | 31.8          | 2.2472       |
| 3640         | 18.1          | 7280         | 5725         | 22.7          | 1.1451       | 8281         | 27.3          | 1.6562       | 1.1307       | 31.9          | 2.2614       |
| 3680         | 18.2          | 7361         | 5776         | 22.8          | 1.1552       | 8342         | 27.4          | 1.6684       | 1.1378       | 32.0          | 2.2756       |
| 3721         | 18.3          | 7442         | 5827         | 22.9          | 1.1654       | 8403         | 27.5          | 1.6806       | 1.1449       | 32.1          | 2.2898       |
| 3762         | 18.4          | 7524         | 5878         | 23.0          | 1.1756       | 8464         | 27.6          | 1.6928       | 1.1520       | 32.2          | 2.3041       |
| 3803         | 18.5          | 7606         | 5929         | 23.1          | 1.1858       | 8525         | 27.7          | 1.7051       | 1.1592       | 32.3          | 2.3184       |
| 3844         | 18.6          | 7688         | 5980         | 23.2          | 1.1961       | 8587         | 27.8          | 1.7174       | 1.1664       | 32.4          | 2.3328       |
| 3885         | 18.7          | 7771         | 6032         | 23.3          | 1.2064       | 8649         | 27.9          | 1.7298       | 1.1736       | 32.5          | 2.3472       |
| 3927         | 18.8          | 7854         | 6084         | 23.4          | 1.2168       | 8711         | 28.0          | 1.7422       | 1.1808       | 32.6          | 2.3617       |
| 3969         | 18.9          | 7938         | 6136         | 23.5          | 1.2272       | 8773         | 28.1          | 1.7547       | 1.1881       | 32.7          | 2.3762       |
| 4011         | 19.0          | 8022         | 6188         | 23.6          | 1.2377       | 8836         | 28.2          | 1.7672       | 1.1954       | 32.8          | 2.3908       |
| 4053         | 19.1          | 8107         | 6241         | 23.7          | 1.2482       | 8899         | 28.3          | 1.7798       | 1.2027       | 32.9          | 2.4054       |
| 4096         | 19.2          | 8192         | 6294         | 23.8          | 1.2588       | 8962         | 28.4          | 1.7924       | 1.2100       | 33.0          | 2.4200       |
| 4139         | 19.3          | 8278         | 6347         | 23.9          | 1.2694       | 9025         | 28.5          | 1.8050       | 1.2173       | 33.1          | 2.4347       |
| 4182         | 19.4          | 8364         | 6400         | 24.0          | 1.2800       | 9088         | 28.6          | 1.8177       | 1.2247       | 33.2          | 2.4494       |
| 4225         | 19.5          | 8450         | 6453         | 24.1          | 1.2907       | 9152         | 28.7          | 1.8304       | 1.2321       | 33.3          | 2.4642       |
| 4268         | 19.6          | 8537         | 6507         | 24.2          | 1.3014       | 9216         | 28.8          | 1.8432       | 1.2395       | 33.4          | 2.4790       |
| 4312         | 19.7          | 8624         | 6561         | 24.3          | 1.3122       | 9280         | 28.9          | 1.8560       | 1.2469       | 33.5          | 2.4939       |
| 4356         | 19.8          | 8712         | 6615         | 24.4          | 1.3230       | 9344         | 29.0          | 1.8689       | 1.2544       | 33.6          | 2.5088       |
| 4400         | 19.9          | 8800         | 6669         | 24.5          | 1.3339       | 9409         | 29.1          | 1.8818       | 1.2619       | 33.7          | 2.5238       |
| 4444         | 20.0          | 8889         | 6724         | 24.6          | 1.3448       | 9474         | 29.2          | 1.8948       | 1.2694       | 33.8          | 2.5388       |
| 4489         | 20.1          | 8978         | 6779         | 24.7          | 1.3558       | 9539         | 29.3          | 1.9078       | 1.2769       | 33.9          | 2.5538       |
| 4534         | 20.2          | 9068         | 6834         | 24.8          | 1.3668       | 9604         | 29.4          | 1.9208       | 1.2844       | 34.0          | 2.5689       |
| 4579         | 20.3          | 9158         | 6889         | 24.9          | 1.3778       | 9669         | 29.5          | 1.9339       | 1.2920       | 34.1          | 2.5840       |
| 4624         | 20.4          | 9248         | 6944         | 25.0          | 1.3888       | 9735         | 29.6          | 1.9470       | 1.2996       | 34.2          | 2.5992       |
| 4669         | 20.5          | 9339         | 7000         | 25.1          | 1.4000       | 9801         | 29.7          | 1.9602       | 1.3072       | 34.3          | 2.6144       |
| 4715         | 20.6          | 9430         | 7056         | 25.2          | 1.4112       | 9867         | 29.8          | 1.9734       | 1.3148       | 34.4          | 2.6297       |
| 4761         | 20.7          | 9522         | 7112         | 25.3          | 1.4224       | 9933         | 29.9          | 1.9867       | 1.3225       | 34.5          | 2.6450       |
| 4807         | 20.8          | 9614         | 7168         | 25.4          | 1.4337       | 1.0000       | 30.0          | 2.0000       | 1.3302       | 34.6          | 2.6604       |
| 4853         | 20.9          | 9707         | 7225         | 25.5          | 1.4450       | 1.0067       | 30.1          | 2.0134       | 1.3379       | 34.7          | 2.6758       |
| 4900         | 21.0          | 9800         | 7282         | 25.6          | 1.4564       | 1.0134       | 30.2          | 2.0268       | 1.3456       | 34.8          | 2.6912       |
| 4947         | 21.1          | 9894         | 7339         | 25.7          | 1.4678       | 1.0201       | 30.3          | 2.0402       | 1.3533       | 34.9          | 2.7067       |
| 4994         | 21.2          | 9988         | 7396         | 25.8          | 1.4792       | 1.0268       | 30.4          | 2.0537       | 1.3611       | 35.0          | 2.7222       |
| 5041         | 21.3          | 1.0082       | 7453         | 25.9          | 1.4907       | 1.0336       | 30.5          | 2.0772       | 1.3689       | 35.1          | 2.7378       |
| 5088         | 21.4          | 1.0177       | 7511         | 26.0          | 1.5022       | 1.0404       | 30.6          | 2.0808       | 1.3767       | 35.2          | 2.7534       |
| 5136         | 21.5          | 1.0272       | 7569         | 26.1          | 1.5138       | 1.0472       | 30.7          | 2.0944       | 1.3845       | 35.3          | 2.7691       |
| 5184         | 21.6          | 1.0368       | 7627         | 26.2          | 1.5254       | 1.0540       | 30.8          | 2.1081       | 1.3924       | 35.4          | 2.7848       |
| 5232         | 21.7          | 1.0464       | 7685         | 26.3          | 1.5371       | 1.0609       | 30.9          | 2.1218       | 1.4003       | 35.5          | 2.8006       |
| 5280         | 21.8          | 1.0561       | 7744         | 26.4          | 1.5488       | 1.0678       | 31.0          | 2.1356       | 1.4082       | 35.6          | 2.8164       |
| 5329         | 21.9          | 1.0658       | 7803         | 26.5          | 1.5606       | 1.0747       | 31.1          | 2.1494       | 1.4161       | 35.7          | 2.8322       |
| 5378         | 22.0          | 1.0756       | 7862         | 26.6          | 1.5724       | 1.0816       | 31.2          | 2.1632       | 1.4240       | 35.8          | 2.8481       |
| 5427         | 22.1          | 1.0854       | 7921         | 26.7          | 1.5842       | 1.0885       | 31.3          | 2.1771       | 1.4320       | 35.9          | 2.8640       |
| 5476         | 22.2          | 1.0952       | 7980         | 26.8          | 1.5961       | 1.0955       | 31.4          | 2.1910       | 1.4400       | 36.0          | 2.8800       |
| 5525         | 22.3          | 1.1051       | 8040         | 26.9          | 1.6080       | 1.1025       | 31.5          | 2.2050       | 1.4480       | 36.1          | 2.8960       |
| 5575         | 22.4          | 1.1150       | 8100         | 27.0          | 1.6200       | 1.1095       | 31.6          | 2.2190       | 1.4560       | 36.2          | 2.9121       |
| 5625         | 22.5          | 1.1250       | 8160         | 27.1          | 1.6320       | 1.1165       | 31.7          | 2.2331       | 1.4641       | 36.3          | 2.9282       |



## EXAMPLE.

Required the content of a cask whose length is 40 inches, and bung and head diameters 32 and 24 inches.

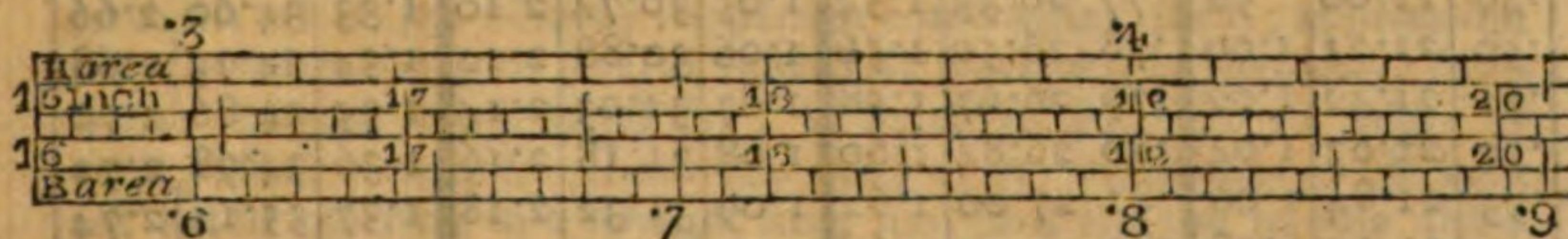
To the right of 32, the } 2.2756 the bung area;  
bung diameter, is }  
and to the left of 24, the } .6400 the head area,  
head diameter, is }

the sum 2.9156  
multiplied by 40 the length

produces 116.624 wine gallons  
for the content required.

To work by the sliding rule, we may proceed as in prob. 1 chap. 3, using 30 as the gauge point instead of 29.7; thus, having set 40 on C to 30 on D,

A still farther improvement is to be made of this method, by transferring the table to scales upon a rod or the inside of a Branan's rule instead of the two useless diagonal scales, a scale of inches and tenths corresponding to the scale of areas. A part of such a scale is represented below.



And for the making of such a scale I shall here subjoin a table of diameters computed from assumed equi-different areas, which is much fitter for this purpose than the foregoing table of areas which was computed from assumed equi-different diameters. This table of diameters was computed by *Mr. AB. CROCKER, Master of a boarding school in Ilminster, Somersetshire*, and who also communicated the substance of this chapter, together with the collection of casks at the end of the last.



$D$ , against 32 and 24 on  $D$  stand 45.6 and 25.5 on  $C$ ; then  $2 \times 45.6 + 25.5 = 91.2 + 25.5 = 116.7$  wine gallons the content nearly as before.

And for the content in ale gallons, we have  $116.6 \times \frac{9}{11} = 95.4$ .

CHAP-

*A table of diameters for the making of scales.*

| Head<br>area | Diam-<br>eter | Bung<br>area | Head<br>area | Diam-<br>eter | Bung<br>area | Head<br>area | Diam-<br>eter | Bung<br>area | Head<br>area | Diam-<br>eter | Bung<br>area |
|--------------|---------------|--------------|--------------|---------------|--------------|--------------|---------------|--------------|--------------|---------------|--------------|
| .30          | 16.43         | .60          | .58          | 22.85         | 1.16         | .86          | 27.82         | 1.72         | 1.14         | 32.03         | 2.28         |
| .31          | 16.70         | .62          | .59          | 23.04         | 1.18         | .87          | 27.98         | 1.74         | 1.15         | 32.17         | 2.30         |
| .32          | 16.97         | .64          | .60          | 23.24         | 1.20         | .88          | 28.14         | 1.76         | 1.16         | 32.31         | 2.32         |
| .33          | 17.24         | .66          | .61          | 23.43         | 1.22         | .89          | 28.30         | 1.78         | 1.17         | 32.45         | 2.34         |
| .34          | 17.50         | .68          | .62          | 23.62         | 1.24         | .90          | 28.45         | 1.80         | 1.18         | 32.59         | 2.36         |
| .35          | 17.76         | .70          | .63          | 23.81         | 1.26         | .91          | 28.61         | 1.82         | 1.19         | 32.73         | 2.38         |
| .36          | 18.00         | .72          | .64          | 24.00         | 1.28         | .92          | 28.78         | 1.84         | 1.20         | 32.86         | 2.40         |
| .37          | 18.26         | .74          | .65          | 24.18         | 1.30         | .93          | 28.93         | 1.86         | 1.21         | 33.00         | 2.42         |
| .38          | 18.50         | .76          | .66          | 24.37         | 1.32         | .94          | 29.09         | 1.88         | 1.22         | 33.14         | 2.44         |
| .39          | 18.73         | .78          | .67          | 24.56         | 1.34         | .95          | 29.24         | 1.90         | 1.23         | 33.27         | 2.46         |
| .40          | 18.97         | .80          | .68          | 24.74         | 1.36         | .96          | 29.40         | 1.92         | 1.24         | 33.41         | 2.48         |
| .41          | 19.21         | .82          | .69          | 24.92         | 1.38         | .97          | 29.55         | 1.94         | 1.25         | 33.54         | 2.50         |
| .42          | 19.44         | .84          | .70          | 25.10         | 1.40         | .98          | 29.70         | 1.96         | 1.26         | 33.67         | 2.52         |
| .43          | 19.67         | .86          | .71          | 25.28         | 1.42         | .99          | 29.85         | 1.98         | 1.27         | 33.81         | 2.54         |
| .44          | 19.90         | .88          | .72          | 25.45         | 1.44         | 1.00         | 30.00         | 2.00         | 1.28         | 33.94         | 2.56         |
| .45          | 20.12         | .90          | .73          | 25.63         | 1.46         | 1.01         | 30.15         | 2.02         | 1.29         | 34.07         | 2.58         |
| .46          | 20.35         | .92          | .74          | 25.81         | 1.48         | 1.02         | 30.30         | 2.04         | 1.30         | 34.20         | 2.60         |
| .47          | 20.57         | .94          | .75          | 25.98         | 1.50         | 1.03         | 30.45         | 2.06         | 1.31         | 34.33         | 2.62         |
| .48          | 20.79         | .96          | .76          | 26.15         | 1.52         | 1.04         | 30.60         | 2.08         | 1.32         | 34.47         | 2.64         |
| .49          | 21.00         | .98          | .77          | 26.33         | 1.54         | 1.05         | 30.74         | 2.10         | 1.33         | 34.60         | 2.66         |
| .50          | 21.21         | 1.00         | .78          | 26.50         | 1.56         | 1.06         | 30.88         | 2.12         | 1.34         | 34.73         | 2.68         |
| .51          | 21.42         | 1.02         | .79          | 26.67         | 1.58         | 1.07         | 31.03         | 2.14         | 1.35         | 34.86         | 2.70         |
| .52          | 21.63         | 1.04         | .80          | 26.84         | 1.60         | 1.08         | 31.18         | 2.16         | 1.36         | 34.98         | 2.72         |
| .53          | 21.84         | 1.06         | .81          | 27.00         | 1.62         | 1.09         | 31.32         | 2.18         | 1.37         | 35.11         | 2.74         |
| .54          | 22.05         | 1.08         | .82          | 27.17         | 1.64         | 1.10         | 31.47         | 2.20         | 1.38         | 35.25         | 2.76         |
| .55          | 22.25         | 1.10         | .83          | 27.33         | 1.66         | 1.11         | 31.61         | 2.22         | 1.39         | 35.37         | 2.78         |
| .56          | 22.45         | 1.12         | .84          | 27.50         | 1.68         | 1.12         | 31.75         | 2.24         | 1.40         | 35.50         | 2.80         |
| .57          | 22.65         | 1.14         | .85          | 27.66         | 1.70         | 1.13         | 31.90         | 2.26         | 1.41         | 35.62         | 2.82         |

Now as to the method of forming the preceding table; since any area is equal to the square of its diameter divided by 900, the diameter must be equal to the root of the product of 900 drawn into the



## CHAPTER VII.

*A new and very exact method of computing the content of a cask of any form from three dimensions only.*

## R U L E.\*

Add into one sum 488 times the square of the bung diameter, 313 times the square of the head diameter,

the area; so that to the first area  $\cdot 30$  will belong the diameter  $\sqrt{\cdot 30 \times 900}$  or  $\sqrt{30 \times 9} = \sqrt{A}$ , to the 2d area  $\cdot 31$  the diameter  $\sqrt{31 \times 9} = \sqrt{A+9} = \sqrt{B}$ , to the 3d area  $\cdot 32$  the diameter  $\sqrt{32 \times 9} = \sqrt{B+9} = \sqrt{C}$ , &c. Hence it is evident that the successive products, out of which the square root is to be extracted, will be obtained by the continual addition of 9; and then extracting the roots, by means of the table of logarithms, will give the several diameters required.

But if you have a table of square roots by you, since the root of 9 is 3, the most expeditious method of making the table, is to take the square roots of 30, 31, 32, &c. out of the table, and then multiply them by 3 for the several corresponding diameters.

\* This rule is taken from a method, of investigating the contents of casks, hinted by Mr. JAMES DAVIDSON, *Teacher of the Mathematics at Dundee in North Briton*, the invention of which take as below.

It appearing reasonable that, in determining a proper rule for computing the contents of casks, we ought to have regard to the general methods which are used by coopers in constructing them; application was therefore made to several ingenious workmen in that way, from whose account it appeared that, ordinarily, the sides or edges of each stave of a cask, for about one-third of its length next each end, are made tapering in a straight line, and that for the middle third part they were curved or made convex to form the bulge or middle of the cask.

From this plain account then we are naturally led to consider one-third of a cask at each end as the frustum of a cone; and the middle part is most properly considered as in the form of a parabolic curve, for these reasons; viz. 1st, that, on account of the shortness of this curved part, it may be considered but as a small part near the vertex of the curve; 2d, that all curves near the vertex differ insensibly







## EXAMPLE.

Required the content of a cask whose length is 40, bung diameter 32, and head diameter 24 inches.

Here  $488 \times 32^2 + 324 \times 32 \times 24 + 313 \times 24^2 \times 40 \times .0000272 = 1010.569$ , which being divided by 9 and by 11, we obtain 112.3 wine gallons, or 91.9 ale gallons for the content required.

## CHAPTER VIII.

*Of the Ullage of Casks.*

The ullage of a cask is now generally understood to be either the empty part or the part filled, of a cask which is not quite full.

6 X

In

tity  $\frac{.785398 \&c.}{1125}$  being divided by 231 gives  $.000003\frac{2}{3} = \frac{.0000272}{9}$  the multiplier for wine gallons; and since 231 is to 282 as 9 to 11 extremely near,  $\frac{.0000272}{11}$  will be the multiplier for ale gallons, as in the rule.

And thus we have a rule which, although it be not the best accommodated to expedition, hath not only the property of agreeing with the content as computed from the method by four dimensions when those dimensions are actually taken, but also agrees surprizingly well with the real contents of casks; as hath been proved by several casks which have actually been filled with a true gallon measure, after their contents have been computed by this method.

To shew its agreement with the four-dimension method, take any one of the examples out of the collection at the end of chapter 5, as for instance No. 14, whose content by that method is 113.1 gallons; and by this method it will be found to come out nearly 112. Again, if we take any other, as suppose the last whose content was 282.2, it will come out by this method 283.2. So then in these two instances, taken at random, although the contents be very large, they agree to within a gallon; the one method exceeding in the former case and the other in the latter.



In this business casks are considered as in two positions, viz. with their axes either parallel or perpendicular to the horizon, that is, either lying along or standing upright upon one end. The ullage of a cask, either standing or lying, may be determined for any particular variety by the preceding parts of this work; but the rules thence obtained are too complex for ordinary practice, especially for a lying cask; I shall not, therefore, introduce those rules into this place, but shall content myself with putting down such approximations as have been found to accord best with expedition and accuracy, together with the universal method by three diameters, which is not only sufficiently easy but also the most accurate for this purpose.

#### P R O B L E M I.

*To find the ullage of a standing and lying cask by the lines SS and SL on the sliding rule.*

#### R U L E.

By some of the preceding chapters, find the whole content of the cask; then for a cask  
 $\left\{ \begin{array}{l} \text{standing} \\ \text{lying} \end{array} \right\} \text{ set } \left\{ \begin{array}{l} \text{length} \\ \text{the bung diam.} \end{array} \right\} \text{ on } N \text{ to } 100 \text{ on } \left\{ \begin{array}{l} SS \\ SL \end{array} \right\},$   
 and on the same  $\left\{ \begin{array}{l} SS \\ SL \end{array} \right\}$  against the wet or dry inches on  $N$  is a number to be reserved; then set 100 on  $B$  to the reserved number on  $A$ , and against the whole content on  $B$  will be found the ullage on  $A$ ; and this ullage will be either the empty part or the part filled, according as the dry or wet inches were used in finding the reserved number.

E x-



## E X A M P L E I.

Required the ullage of a standing cask whose length is 40, bung diameter 32, head diameter 24, wet inches 10, and, consequently, dry inches 30 inches.

By the last chapter, the content was nearly 92 ale gallons: Hence, having set 40 on *N* to 100 on *SS*, and against 10 on *N* found 23 on *S*; I then set 100 on *B* to 23 on *A*, and against 92 on *B* find 21.2 ale gallons on *A*, for the quantity remaining in the cask.

If 30 the dry inches be used, the reserved segment will be found to be 76.7, and then the corresponding ullage is 70.2 for the gallons drawn off.

And the sum of these two parts is 91.4 gallons, which is about half a gallon less than the whole content; which error would be inconsiderable if it were to be divided between the two parts; but instead of that generally the one part is too little, and the other too much, by which it happens that the error in each part commonly exceeds that of their sum.

## E X A M P L E II.

Let the dimensions and content be the same in the case of the lying cask, also the wet inches 8, and consequently the dry inches 24.

Having set 32 on *N* to 100 on *SL*, against 8 and 24 on *N* are the reserved segments 17.8 and 82.5 on *SL*; then having set 100 on *B* to 92 on *A*, against 17.8 and 82.5 on *B* are 16.4 and 76 on *A*, which



which are the parts filled and empty respectively, and whose sum is 92.4, near half a gallon too much.

### PROBLEM II.

*To find the ullage of a standing cask by the pen.*

### RULE I.

Take here the rule in chapter 5, using the \* diameter at the surface of the liquor instead of the bung diameter, the diameter in the middle between the surface of the liquor and its nearest head instead of the middle diameter, and the distance between the surface and nearest head instead of the length of the cask; and you will obtain the part empty or filled according as the cask is more or less than half full.

### EXAMPLE.

Taking the above diameters 24, 27, and 29 inches, and the wet inches 10; we shall have  $24^2 + 54^2 + 29^2 \times 10 \times \frac{51}{11000} = 4333 \times \frac{51}{11000} = 20.9$  gallons for ullage required.

### RULE II.

As the square of the length of the cask is to the square of the difference between the said length and  
wet

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\* The diameters at the surface of the liquor and in the middle between it and the nearest head, will easily be obtained by suspending a plummet by a rule laid over the middle of the head, so as just to touch the bulge of the cask, and then measuring the distance of the string from the side of the cask at the surface of the liquor and in the middle between it and the nearest head; for then the doubles of these measures taken from the bung diameter will leave the said required diameters.



wet or dry inches, viz. the less of them, so is the difference between the bung and head diameters to a number; which being taken from the bung diameter, will leave the diameter of a cylinder of the same length with, and nearly equal to the part filled or empty, viz. the less of them.\*

## E X A M P L E.

Taking the same example as in problem I, we have as  $40^2 : 40 - 10|^2 :: 32 - 24 : \frac{8 \times 30^2}{40^2} = \frac{8 \times 9}{16} = 4\frac{1}{2}$ ; and  $32 - 4\frac{1}{2} = 27\frac{1}{2}$  the mean diameter.

Then  $27\frac{1}{2}^2 \times 10 \times .0027851 = 21$  ale gallons = the ullage required.

*Note.* The number .0027851 is the constant multiplier  $\frac{.785398 \text{ \&c.}}{282}$ .

## P R O B L E M III.

*To find the ullage of a lying cask by the pen.*

## R U L E.

Divide the wet inches by the bung diameter, find the quotient in the first column of the table of circular

6 Y

cular

\* This rule is founded upon the reasonable supposition that for the small part of a cask, the diameter in the middle may be considered as a mean diameter; which is strictly true when the cask is paraboloidal, and exceedingly near the truth when parabolic-spindular; but the said middle diameter, as given in the rule, is computed from the latter form, because the rule is thereby rendered not only easier but generally more exact.

Thus, by the property of the parabola,  $L^2 : L - I|^2 :: B - H : \frac{B - H}{L^2} \times L - I|^2$ , and  $B - \frac{B - H}{L^2} \times L - I|^2 = M$ ; where  $B$  is the bung,  $H$  the head, and  $M$  the mean diameter, also  $L$  is the length of the cask, and  $I$  the wet or dry inches.



cular segments (at the end of the work), and take out the segment standing opposite to it, then multiply this segment by the whole content of the cask, and then the product being either divided by  $\cdot 785398$  &c. or multiplied by  $1\cdot 2732395$  will give the ullage nearly.\*

#### E X A M P L E.

Taking the same cask as before, whose length is 40, bung diameter 32, head diameter 24; and supposing the wet inches to be 8.

By chapter 7, the whole content is nearly 92 ale gallons. Then  $\frac{8}{32} = \frac{1}{4} = \cdot 25$ ; opposite to which, in the table of segments, is the segment  $\cdot 15354621$ ; hence  $92 \times \cdot 15354621 \times 1\cdot 2732395 = 18$  ale gallons = the ullage required.

#### S C H O L I U M.

Having delivered the necessary rules for measuring casks, &c. I do not suppose that any thing more of the subject of gauging is necessary to be given in this book; for as to cisterns, couches, &c. tuns, coolers, &c. coppers, stills, &c. which are first supposed to be in the form of some of the solids in the former parts of this work, and then measured accordingly, no person can be at a loss concerning them who knows any thing of such solids in general; and to treat of them here would induce me to  
a long

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\* This method is evidently nothing else than taking the whole content, in such proportion to the ullage as the whole bung circle bears to the segment of it cut off by the surface of the liquor; and is nearer to the truth than any other practical rule which I can find.



a long and tedious repetition, only for the sake of pointing out the proper multipliers, or divisors, which is, I think, a reason very inadequate to so cumbersome an increase of my book.

I shall only just observe, that when tuns, &c. of oval bases are to be gauged, as those bases really measure to more than true ellipses of the same length and breadth, they ought to be measured by the equi-distant ordinate method delivered in section 2 of part 4.

And that when casks are met with which have different head diameters, they may be deemed incomplete casks, and their contents considered and measured as the ullage of a cask.

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### SECTION III.

#### *Of the works of Artificers.*

THE artificers whose works are here to be treated of, are 1 Bricklayers, 2 Carpenters and Joiners, 3 Glaziers, 4 Masons, 5 Painters, 6 Pavers, 7 Plasterers, 8 Plumbers, and 9 Slaters and Tilers, &c.

Different branches of artificers estimate the value of their work by measures of different sizes, such as by the foot, the yard, the rod, the square, &c. but they generally take the dimensions in yards, or feet, inches and tenths or eighths, or feet, tenths, and hundredths; which last is certainly the best method of taking the dimensions, because that the computation will be performed by the common multiplication or by the sliding rule described below. For taking measures, the most common instrument is  
what



what is called the carpenters rule, of which it will be necessary here to give a description.

#### CHAPTER I.

##### *Of the common or carpenters rule.*

This instrument consists of two pieces each of a foot in length, and connected together by a folding joint, and one of them having a slider running in it.

The one face is for taking dimensions in inches and half-quarters or eighths, the whole length being divided into inches and eighths. Upon this face also are plain scales divided by diagonals into twelves, for planning dimensions taken in feet and inches.

For taking dimensions in feet, tenths, and hundredths, the edge of the rule is adapted, it being divided after that manner, viz. each foot into ten parts, and each of these tenths into ten parts again; that so the rule may suit the different inclinations of measurers, viz. those who use twelves and those who use tens; but as I prefer the latter method, for the reasons above-mentioned, I would fain persuade all measurers to do the same; and therefore in the following work I shall take the dimensions and make the calculations in that manner as well as in feet, inches, and parts by cross multiplication. For the duodecimal multiplication, or multiplication of feet, inches, and parts, runs all their notions of the parts of the products into confusion, habituating them to account those parts square inches, &c. which are really quite different things.

Upon



Upon one part of the other face are four lines marked *A*, *B*, *C*, and *D*; the two middle ones, *B* and *C*, being on a slider. These four lines are logarithmic ones, the first three being all of the same radius, but the last of a radius double to that of the former; so that this line is but of one radius in length while each of the three former is of two; or the divisions on it answer to the square roots of the divisions on the others; that is, when any number on the slider is set opposite to its root on *D*, then also are all the other numbers opposite to their square roots: it is necessary to shift the slider out of its natural position to make these squares and roots correspond, because they begin with different numbers, *D* beginning with 4 and the rest with the number 1. Upon the line *D* are marked *WG* and *AG*, the wine and ale gauge points, the former at 17.15 and the latter at 18.95, to make this instrument answer the end of a gauging rule. This line is also called and marked the *Girt Line*, because it serves for the measuring of timber.

Upon the other part of this face is a table of the value of a load or 50 cubic feet of timber at all prices from 6 pence to 2 shillings a foot.

The particular uses of this instrument will be seen in the following examples, where such of them are wrought by the rule as were judged necessary to explain its use.

stone



## CHAPTER II.

*Of the different measures used by different artificers.*

144 square inches = a square foot,

9 square feet = a square yard,

63 square feet = 7 square yards = a rood,

100 square feet = a square,

$272\frac{1}{4}$  square feet =  $30\frac{1}{4}$  square yards = a rod, perch, or square pole.

So that if the content of any work be first computed in feet, which is generally the method, it will be found in yards, roods, squares, or rods, by dividing the feet by the corresponding divisors 9, 63, 100 or  $272\frac{1}{4}$ .—Instead of the divisor 63, divide first by 9 and then by 7, because 9 times 7 make 63. You easily divide by 100, by only cutting off two figures on the right hand. To divide by  $272\frac{1}{4}$ , first multiply by 4 and then divide by 9, 11, and 11; because  $272\frac{1}{4} = \frac{9 \times 11 \times 11}{4}$ . It may be noted here that this rod is the statute perch used in land measuring, being the square of the pole of  $5\frac{1}{2}$  yards or  $16\frac{1}{2}$  feet.

The above denominations are those by which most kinds of work are valued; some particular articles indeed are valued by the foot running or lineal measure: By the square foot are valued glazing and masonry; by the square yard, joinery, painting, paving, and plastering; by the square, slating, tiling, and carpentry or flooring, partitioning, and roofing; and by the rod, brick work. The rood is very little used except in some country places for very common  
stone



stone walls: but it may be noted that this rood of 63 square feet is not the rood used in land measuring; for this last rood is a quarter of an acre = 40 square poles = 10890 square feet.

*A table for changing hundredth parts of feet into inches and parts, and the contrary; by means of which dimensions taken in one of these may be readily changed to the other.*

| Centesms, or hundredth parts of a foot | Inches. | 12th pts. | Inches. | 8th pts, or half quarters. | Centesms. |
|----------------------------------------|---------|-----------|---------|----------------------------|-----------|
| 1                                      | 0       | 1         | 0       | 1                          | 1         |
| 2                                      | 0       | 3         | 0       | 2                          | 2         |
| 3                                      | 0       | 4         | 0       | 3                          | 3         |
| 4                                      | 0       | 6         | 0       | 4                          | 4         |
| 5                                      | 0       | 7         | 0       | 5                          | 5         |
| 6                                      | 0       | 9         | 0       | 6                          | 6         |
| 7                                      | 0       | 10        | 0       | 7                          | 7         |
| 8                                      | 1       | 0         | 1       | 0                          | 8         |
| 9                                      | 1       | 1         | 2       | 0                          | 17        |
| 10                                     | 1       | 2         | 3       | 0                          | 25        |
| 20                                     | 2       | 5         | 4       | 0                          | 33        |
| 30                                     | 3       | 7         | 5       | 0                          | 42        |
| 40                                     | 4       | 10        | 6       | 0                          | 50        |
| 50                                     | 6       | 0         | 7       | 0                          | 58        |
| 60                                     | 7       | 2         | 8       | 0                          | 67        |
| 70                                     | 8       | 5         | 9       | 0                          | 75        |
| 80                                     | 9       | 7         | 10      | 0                          | 83        |
| 90                                     | 10      | 10        | 11      | 0                          | 92        |
| 100                                    | 12      | 0         | 12      | 0                          | 100       |

Ex-



## EXAMPLE I.

If it be required to change 44 centesms into inches and 12ths.—Against 4 and 40 in the first column stand respectively 6 parts and 4 in. 10 pts in the second; then the sum of these two is 5 inches 4 parts, the number required.

## EXAMPLE II.

If it be required to change 5 inches and  $\frac{3}{4}$  or 6 eighths to centesms.—Against 6 eighths and 5 inches the third column stand respectively 6 and 42 in the last column; the sum of which is 48 centesms.

The reason why I have set 8ths in the third column, and not 12ths, is that the measurers who calculate by 12ths take the dimensions in 8ths and change them into 12ths, because that the inches upon the rule are divided into 8ths, and not 12ths.

*Note.* You may convert centesms into inches and parts, and the contrary, without the table, by the following rule, viz.

Multiply centesms by 12 for inches, cutting off two figures; and multiply these two figures by 12 again for parts, cutting off two figures here also. And divide 8ths or 12ths, with two cyphers affixed, by 8 or by 12 for inches; and divide inches by 12 for centesms.

## EXAMPLE I.

Thus, taking the first example above.

|  |        |             |                       |
|--|--------|-------------|-----------------------|
|  |        | 44 centesms |                       |
|  |        | <u>12</u>   |                       |
|  | Inches | 5.28        | That is 44 centesms = |
|  |        | <u>12</u>   | 5 inches 3 parts.     |
|  | Parts  | 3.36        |                       |

Ex-



## EXAMPLE II.

And in the second example above,

$$\begin{array}{r|l} 8 & 6.00 \\ 12 & 5.75 \\ \hline \end{array}$$

48 centesms.

I shall now proceed to the different kinds of works, in treating of which separately I shall note the methods of measuring the several articles, with the allowances, deductions, &c. and then at the end of the whole illustrate them all in the several parts of a real building, whose plan and elevation shall be laid down, with its scale annexed for comparing the dimensions, &c.

## CHAPTER III.

*Of Bricklayers Work.*

Bricklayers work, properly speaking, is only such work as is built with bricks, but in many places the bricklayers also perform some or all of these other works, viz. masonry, tiling, plastering, and paving; but, for the sake of perspicuity and distinction, I shall consider each of the articles separately, throughout the several works.

Brick work hath been commonly estimated by the rod of  $272\frac{1}{4}$  square feet, but I think that by the yard of 9 square feet, as is now generally done, is the more plain and intelligible method to both proprietor and builder, and answers every purpose with more ease and exactness.

As brick work is estimated by this superficial or square measure, and not by solid measure, it is



necessary that some particular thickness of wall be fixed upon as a standard to estimate this area by; for it is evident that the area must be accounted more or less in proportion to the thickness of the wall; and this standard thickness is the thickness of 3 half bricks, that is the length of a brick and half the length of another, or three times the breadth of a brick, and is called accordingly three half bricks thick, or a brick and a half thick, or 14 or 15 inches thick. And before any area in feet be brought to yards or rods, if the wall be different from the standard thickness, it must be reduced to it by multiplying the said area by the particular number of half bricks in the thickness of the wall, and then dividing by 3, so is the quotient the measure when reduced to the standard thickness, and is called the reduced content or measure. And hence it is evident that all the parts of the same wall that are of different thicknesses must be measured and reduced separately: but in measuring buildings in which are many dimensions of each of several thicknesses, let the measures which are all of the same thickness be first added together, then reduce their sum, and lastly add all the reduced sums together.

It is the common or plain work only which is valued by the yard of 9 square feet; for ornamental &c. work is valued by the foot square, such as arches, over windows, doors, &c. facias, architraves, rubbed returns, frizes, cornices, &c. And by ornamental work in general is meant such as is ground, gauged, or polished. But carved mouldings, &c. are often agreed for by the foot running or lineal measure.

1. When



1. When walls join in an angle, measure the length of the one on the outside, and the other on the inside; or else from the sum of their outside lengths deduct the thickness of the wall.—Thus, to a  $2\frac{1}{2}$  brick wall, whose length is 34 feet 6 inches, joins another, of the same thickness, whose outside length also is 24 feet 9 inches; required the reduced measure of the whole, the common height being 24 feet 6 inches. Here

| F     | I |                   |
|-------|---|-------------------|
| 34    | 6 | } outside lengths |
| 24    | 9 |                   |
| <hr/> |   |                   |
| 59    | 3 | sum               |
| 2     | 0 | thickness deduct  |
| <hr/> |   |                   |
| 57    | 3 | true length       |

*Duodecimally*

| F  | I |        |
|----|---|--------|
| 57 | 3 | length |
| 24 | 6 | height |

---

228

114

28 7 6

6 0 0

---

1402 7 6

5 half bricks thick

---

3) 7013 1 6

9) 2337 8 6 reduced cont. 9) 2337 7083

---

Yards 259 6 8 6*Decimally*

Feet

57.25 length

24.5 height

---

28625

22900

11450

---

1402.625

---

5

3) 7013.125

---

Yards 259.745 &c.*By*



*By the Sliding Rule.*

$B \quad A \quad B \quad A$   
 $1 : 24\frac{1}{2} :: 57\frac{1}{4} : 1403$ . The meaning of which is, that 1 on  $B$  is set under  $24\frac{1}{2}$  on  $A$ , then above  $57\frac{1}{4}$  on  $B$  is found 1403 on  $A$ .

2. In measuring a gable end, because it is a triangle, multiply the breadth at the bottom by the perpendicular height, and take half the product; or multiply one of these dimensions by half of the other.—Thus if a gable of  $17\frac{1}{2}$  feet high were raised upon the end wall in the last example, whose length is 24 feet 9 inches, required the reduced content, supposing its thickness to be 2 bricks.

| <i>Duodecimally</i> |   |                     | <i>Decimally</i> |
|---------------------|---|---------------------|------------------|
| F                   | I |                     | Feet             |
| 24                  | 9 | length              | 24.75            |
| 8                   | 9 | half height         | 8.75             |
| <hr/>               |   |                     | <hr/>            |
| 198                 | 0 | P                   | 12375            |
| 12                  | 4 | 6                   | 17325            |
| 6                   | 2 | 3                   | 19800            |
| <hr/>               |   |                     | <hr/>            |
| 216                 | 6 | 9                   | 216.5625         |
|                     |   | 4 half bricks thick | 4                |
| <hr/>               |   |                     | <hr/>            |
| 3) 866              | 3 | 0                   | 3) 866.25        |
| <hr/>               |   |                     | <hr/>            |
| 9) 288              | 9 | 0 reduced cont.     | 9) 288.75        |
| <hr/>               |   |                     | <hr/>            |
| Yards 32            | 0 | 9                   | Yards 32.083     |
| <hr/>               |   |                     | <hr/>            |

*By the Sliding Rule.*

$B \quad A \quad B \quad A$   
 $1 : 24\frac{3}{4} :: 8\frac{3}{4} : 216\frac{1}{2}$ .

3. Chim-



3. Chimneys are measured as if they were solid, on account of the trouble attending them, deducting only the vacuity between the jambs from the hearth to the mantle. So that when the chimney doth not stand in an angle, the breadth of the breast or front multiplied by the height will be the content, the thickness being the depth of the jambs. So if the breadth of the breast be 7 feet 3 inches, depth of the jambs three bricks, and height equal to the sum of the two heights in the last two articles, required the reduced content, deducting for two fire places each 5 feet square.

| <i>Duodecimally</i> |   |                           | <i>Decimally</i> |               |
|---------------------|---|---------------------------|------------------|---------------|
| F                   | I |                           | F                |               |
| 24                  | 6 |                           | 24.5             |               |
| 17                  | 6 |                           | 17.5             |               |
| <hr/>               |   |                           | <hr/>            |               |
| 42                  |   | sum of the heights        | 42               |               |
| 7                   | 3 |                           | 7.25             |               |
| <hr/>               |   |                           | <hr/>            |               |
| 294                 |   |                           | 294              |               |
| 10                  | 6 |                           | 10.5             |               |
| <hr/>               |   |                           | <hr/>            |               |
| 304                 | 6 |                           | 304.5            |               |
| 50                  | 0 | two opens deduct          | 50               |               |
| <hr/>               |   |                           | <hr/>            |               |
| 254                 | 6 |                           | 254.5            |               |
|                     | 2 | double standard thickness | 2                |               |
| <hr/>               |   |                           | <hr/>            |               |
| 9) 509              |   | reduced content           | 9) 509           |               |
| <hr/>               |   |                           | <hr/>            |               |
| Yards 56            | 5 |                           | Yards 56         | $\frac{5}{9}$ |

If the chimney stand in a square corner or angle, and be equally distant both ways from the angle,

7 B

half



half the product of the height multiplied by the breadth in front will be the content, the thickness being half the breast or front.—So if the height be 9 feet 3 inches, and the front 8 feet 6 inches. Then

| <i>Duodecimally</i> |     |                            | <i>Decimally</i> |
|---------------------|-----|----------------------------|------------------|
| F                   | I   |                            |                  |
| 9                   | 3   | height                     | 9'25             |
| 4                   | 3   | half the front             | 4'25             |
| <hr/>               |     |                            | <hr/>            |
| 37                  | 0   |                            | 37'00            |
| 2                   | 3   | 9                          | 2'3125           |
| <hr/>               |     |                            | <hr/>            |
| 39                  | 3   | 9                          | 39'3125          |
| 12                  | 6   | 0 fire place 5 feet square | 12'5             |
| <hr/>               |     |                            | <hr/>            |
| 26                  | 9   | 9                          | 26'8125          |
|                     |     | 11 half bricks thick       | 11               |
| <hr/>               |     |                            | <hr/>            |
| 3)                  | 294 | 11 3                       | 3) 294'9375      |
| <hr/>               |     |                            | <hr/>            |
| 9)                  | 98  | 3 9 reduced content        | 9) 98'3125       |
| <hr/>               |     |                            | <hr/>            |
| Yds.                | 10  | 8 3 9                      | Yds. 10'9236     |
| <hr/>               |     |                            | <hr/>            |

If the angle be not square, or if the breast at each side be not equally distant from the angle, the content will be found as before, but the thickness here will be the perpendicular distance of the breast from the angle.

If that part of a chimney which is without or above the roof be built with brick, it is common to omit it in measuring, the bricklayers generally measuring the height only to the top of the roof or top of the gable, on account of the trouble of getting to them to measure them: but then so much  
a-piece



a-piece is allowed for each funnel, more or less according to the height to which they are raised.

4. As to the ornamental &c. parts; in taking the dimensions, the length is measured where it is greatest; and for the height or breadth, make a line ply close over all the projections and returns that can be seen; these two multiplied together give the superficial content. For instance, in arches, whether straight or curved, the length is taken along the upper side of the arch; and for the breadth is taken both the surface and soffit, that is the distance from the top of the arch straight down to the under part of it, and returned across the bottom to the back part of the bricks. In the same manner, fascias, cornices, &c. are girt close over both upper and under returns as well as the surfaces: in short, all that appear are girt. All which will be illustrated in the general examples near the end of these works.

#### *Of the Deductions.*

It must be remarked that all windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed; or rather, that the sum of all these deductions, when reduced, are to be taken from the sum of the reduced contents of the walls. But it is to be understood that this deduction is made only with regard to materials; for the value of the workmanship of this quantity of deductions is added to the workman's bill, at the stated rate of workmanship only. There are also other allowances for workmanship made to the workman, such as for returns or angles made  
by



by two joining walls, and double measure for feathered gable ends. That is, the quantity deducted for the thickness of the walls where two meet in an angle, as in the first article, is to be added for workmanship; for all outside measure is allowed for workmanship, the true measure taking place only with regard to materials: and when a gable is feathered, besides the charge for the true measure for both work and materials, the content is charged over again for work only, on account of the trouble of them. Examples of these things will appear in the general abstract at the end.

*Of arched Roofs.*

Arched roofs are either *vaults*, *domes*, *saloons*, or *groins*.

*Vaulted* roofs are formed by an arch or arches springing from the opposite or side walls and meeting in a line at the top. And may be represented by a hollow cylinder cut in two through its axe, and one half placed with the section parallel to the horizon.

*Domes* consist of an arch or arches springing from a circular or polygonal base, and meeting in a point at the top. Like to a semi-sphere, or semi-spheroid.

*Saloons* are formed by arches connecting the side walls to a flat roof or ceiling in the middle.

*Groins* are formed by the intersection of vaults with one another. These mostly happen in cellars.

Domes and saloons are things that very rarely occur in the practice of measuring; but vaults or groins cover the cellars of most houses, and it is chiefly of them that I shall here treat. What regards their surfaces will be given in plasterers or painters work.

The



The general rule for measuring all arches is thus: From the content of the whole, considered as solid from the springing of the arch to the outside of it, deduct the vacuity contained between the said springing and the underside of it, and the remainder will be the real content of the solid part.

And because the upper side of the arches of all cellars, whether vaults or groins, are built up solid, above the hances, to the level or same height with the crown, it is evident that the area of the base will be the content of the whole above mentioned, taking for its thickness the whole height from the springing to the top; and for the content of the vacuity to be deducted, take the area of its base, accounting its thickness to be two-thirds of the greatest inside height, that is  $\frac{2}{3}$  of the former whole height lessened by the thickness of the arch at the crown. This  $\frac{2}{3}$  of the height is sufficiently exact for the purpose, and is used by the most intelligent measurers. But it may be noted that the area used in the vacuity is not exactly the same with that which is used in the solid; for the diameter of the former, by being taken on the inside, is twice the thickness of the arch less than that of the latter.

But although I have mentioned the deduction of the vacuity as common to both vault and groin, it is reasonable to make it only in the former; for in the groin there is made a very considerable waste of materials, by cutting them to fit uniformly and regularly at the angles or diagonal intersections; and this also causes a very considerable labour and waste of time to the workman; and since to omit the deduction of the vacuity in the groin seems to



be the most easy and reasonable method of compensating for both, I shall suppose the groins to be solid from the springing to the whole height.

### EXAMPLE.

Suppose the length of a cellar, within or below the arch, to be 17 feet, and breadth 15 feet; required the reduced content of the arching, supposing the perpendicular height of the under side of the arch to be 2 feet 2 inches, and the thickness of the arch 10 inches.

#### I. For the Vault.

Here twice the thickness of the arch must be added to the breadth.

| <i>The solid</i> |       |                           | <i>The vacuity</i>                |         |
|------------------|-------|---------------------------|-----------------------------------|---------|
| 17               | 0     |                           | 17                                |         |
| 16               | 8     |                           | 15                                |         |
| <hr/>            |       |                           | <hr/>                             |         |
| 272              |       |                           | 85                                |         |
| 11               | 4     |                           | 17                                |         |
| <hr/>            |       |                           | <hr/>                             |         |
| 283              | 4     | — area of the base —      | 255                               |         |
|                  | 7     | — half bricks thick —     | $3\frac{1}{3} = \frac{2}{3}$ of 5 |         |
| <hr/>            |       |                           | <hr/>                             |         |
| 1983             | 4     |                           | 765                               |         |
| 850              | 0     | vacuity deduct            | 85                                |         |
| <hr/>            |       |                           | <hr/>                             |         |
| 3) 1133          | 4     |                           | 850                               | vacuity |
| <hr/>            |       |                           | <hr/>                             |         |
| 9) 377           | 9 4   |                           |                                   |         |
| <hr/>            |       |                           | <hr/>                             |         |
| Yards 41         | 8 9 4 | the neat reduced content. |                                   |         |
| <hr/>            |       |                           |                                   |         |

#### II. For



II. *For the Groin.*

Here twice the thickness of the arch must be added to both length and breadth.

$$\begin{array}{r} 18 \quad 8 \\ 16 \quad 8 \\ \hline \end{array}$$

$$\begin{array}{r} 298 \quad 8 \\ 12 \quad 5 \quad 4 \\ \hline \end{array}$$

$$311 \quad 1 \quad 4$$

7 half bricks thick

$$3) 2177 \quad 9 \quad 4$$

$$9) 725 \quad 11 \quad 1$$

Yards 80 5 11 1 reduced content.

*By the Sliding Rule.*

$$\begin{array}{ccccc} B & A & B & A & \\ 3\frac{6}{7} & : & 16\frac{2}{3} & :: & 18\frac{2}{3} : 80\frac{2}{3} \text{ feet.} \end{array}$$

$$\text{The first term is} = \frac{9 \times 3}{7} = \frac{27}{7}.$$

## CHAPTER IV.

*Of Masons Work.*

To masonry belongs all stone work. Masonry is measured by the foot, either superficial or solid. By the solid foot are measured solid walls, blocks of marble or stone, columns, &c. and by the foot superficial are measured pavements, flabs, chimney pieces, with coping and ornamental stone work set in brick buildings. The solid measure is for materials,



terials, and the superficial for workmanship. In the solid measure, the true length, breadth, and thickness are taken, and multiplied continually together; and in the superficial measure, are taken the length and breadth of every part or projection that appears without the general upright face of the building.

I shall put down only an example or two here, as this branch is fully illustrated in the general example.

### EXAMPLE I.

Required the solid content of a wall whose length is 53 feet 6 inches, its height 12 feet 3 inches, and thickness two feet.

*Duodecimally.*

|       |   |   |
|-------|---|---|
| 53    | 6 |   |
| 12    | 3 |   |
| <hr/> |   |   |
| 13    | 4 | 6 |
| 642   | 0 | 0 |
| <hr/> |   |   |
| 655   | 4 | 6 |
| 2     | 0 | 0 |
| <hr/> |   |   |
| 1310  | 9 | 0 |

— the solid content —

*Decimally.*

|                  |
|------------------|
| 53.5             |
| 12 $\frac{1}{4}$ |
| <hr/>            |
| 642              |
| 13.375           |
| <hr/>            |
| 655.375          |
| 2                |
| <hr/>            |
| 1310.75          |

*By the Sliding Rule.*

|          |          |          |          |
|----------|----------|----------|----------|
| <i>B</i> | <i>A</i> | <i>B</i> | <i>A</i> |
| 1        | 53½      | 12¼      | 655,     |
| 1        | 655      | 2        | 1310.    |

**Ex-**



## EXAMPLE II.

| In a chimney piece, suppose the    |   | F            | I | Products |    |
|------------------------------------|---|--------------|---|----------|----|
| Length of the mantle and slab each |   | 4            | 6 | 14       | 3  |
| Breadth of both together           | — | 3            | 2 |          |    |
|                                    |   | <hr/>        |   |          |    |
| Length of each jamb                | — | 4            | 4 | 7        | 7  |
| Breadth of both together           | — | 1            | 9 |          |    |
|                                    |   | <hr/>        |   |          |    |
|                                    |   | The whole is |   | 21       | 10 |

## CHAPTER V.

*Of Carpenters and Joiners Work.*

By carpenters and joiners is performed the wood work in a house, such as joisting and flooring, partitions, roofs, centering, staircases, wainscoting, doors, windows, cornices, cove-eaves, &c. the heavy and coarse articles, formerly, being the work of the carpenter, and the others performed by the joiner: But now the two are considered but as one branch, and all the articles are performed by the same workmen.

The large and plain articles are estimated by square or superficial measure, either by the foot, or the yard, or the square; enriched mouldings, and some other things, are often estimated by running or lineal measure; and some things are rated by the piece.

1. *Floors.* When the boarding, or flooring, properly so called, is measured by itself, the dimensions are taken the whole inside length and breadth of the room.—But when the joisting is measured separately



rately, only one of its dimensions will be the same with the flooring; and the other, which is taken the length way of the joists to their full extent, will exceed the length of the room by the thickness of the wall and  $\frac{1}{3}$  of the same, because each end is let into the wall about  $\frac{2}{3}$  of its thickness.—And when the joisting and flooring are measured together, as one article, the inside breadth is taken, as before, for the one dimension; and for the other dimension is taken a mean between the length of the joists and length of the floor, that is, the length of the room with  $\frac{2}{3}$  of the thickness of the wall added, because the flooring is generally of about the same value with the joisting.

No deductions are made for hearths on account of the additional trouble and waste of materials.

2. *Partitions.* These are measured from wall to wall for the one dimension, and from floor to floor as far as they extend for the other.

No deductions in partitioning for door-ways, on account of the trouble of framing them.

3. *Centering for cellars.* For common vaulted centering, the upper surface of the center, or the under side of the arch, is the measure; and the dimensions of this measure are found by making a line ply close over the surface of the arch for the breadth, and taking the length of the cellar for the length. But in groin-centering double measure is to be taken, on account of their extraordinary trouble, &c.

4. *Roofing.* Here the length of the house on the inside together with  $\frac{2}{3}$  of the thickness of one gable, is considered as the length; for though the timbers be



be let into both gables  $\frac{2}{3}$  of their thickness, it is only the principal timbers that are so let in, and  $\frac{2}{3}$  of the thickness is but once taken for the same reason as when joisting and flooring are measured together; and for the breadth is taken double the length of a line stretched from the ridge down to the rafter, along the eaves board, and down the thickness of the same till the line meet with the top of the wall.

In case of any interfections forming hips or valleys, and causing triangular parts in the roof, by means of the ridge being longer than the eaves, or the eaves longer than the ridge; having taken the true area, it seems to be a reasonable allowance, for the waste of materials and time attending the interfections, to take  $\frac{1}{3}$  of the triangular parts more.

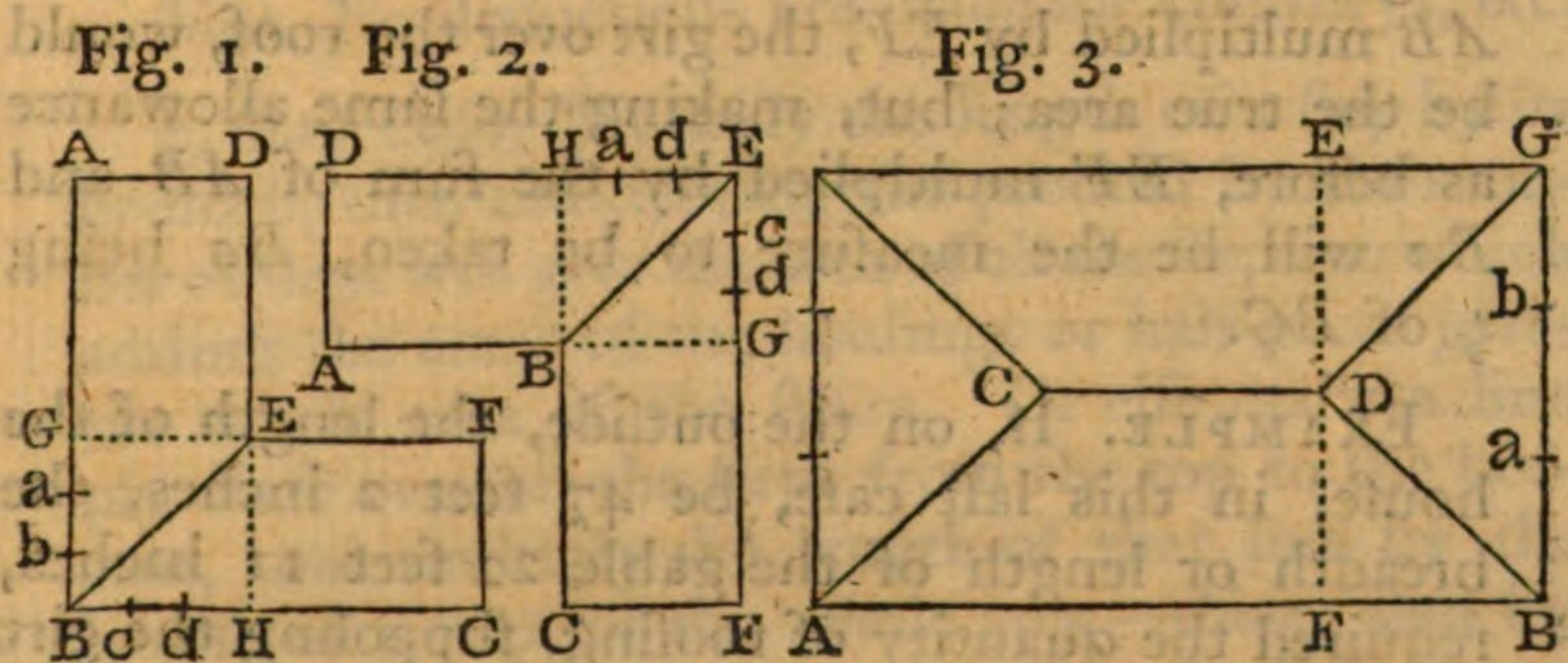


Figure 1 represents a hip, in which the eaves  $AB$ ,  $BC$ , are longer than the ridges  $DE$ ,  $EF$ ; and figure 2 represents a valley, in which the eaves  $AB$ ,  $BC$ , are shorter than the ridges  $DE$ ,  $EF$ : In each  $BE$  is the interfection of the two roofs; and  $BGE$ ,  $BHE$  are two triangular parts of which the  $\frac{1}{3}$  part more must be taken besides their true area, with those



those of the rectangular parts  $CFEH$ ,  $ADEG$  in figure 1, or  $CFGB$ ,  $ADHB$  in figure 2. Consequently, if, in figure 1,  $GB$  and  $BH$  be each divided into three equal parts at the points  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $HE$ ,  $EG$  in the same manner divided in figure 2; then, in figure 1,  $Ab$  multiplied by  $AD$  will give  $ABED$ , and  $Cc$  multiplied by  $CF$  will give  $CBEF$ ; or, as  $AD$  and  $CF$  are supposed to be equal to each other,  $AD$  multiplied by the sum of  $Ab$  and  $cC$  will give a reasonable measure for the whole. And, in figure 2,  $AD$  multiplied by the sum of  $Db$  and  $cF$  will give  $ADEFGB$ .

Also figure 3 represents a roof of four sides, two of which are triangles springing from the gable ends, and whose tops meet with the ends of the ridge  $CD$  formed by the other two sides. And here  $AB$  multiplied by  $EF$ , the girt over the roof, would be the true area; but, making the same allowance as before,  $EF$  multiplied by the sum of  $AB$  and  $Ba$  will be the measure to be taken,  $Ba$  being  $\frac{1}{3}$  of  $BG$ .

EXAMPLE. If, on the outside, the length of the house, in this last case, be 47 feet 2 inches, the breadth or length of the gable 22 feet 11 inches, required the quantity of roofing, supposing the girt  $EF$  over from side to side to be 33 feet 3 inches, and the thickness of the walls 2 feet.—Here 4 feet taken from both length and breadth leaves 43 feet 2 inches and 18 feet 11 inches for the inside length and breadth, to each of which adding 1 foot 4 inches or  $\frac{2}{3}$  of the thickness of the wall, produces 44 feet 6 inches for the length, and 20 feet 3 inches for the breadth,  $\frac{1}{3}$  of which is 6 feet 9 inches.

*Duo-*



| <i>Duodecimally</i> |   |   |                                        | <i>Decimally</i> |   |           |
|---------------------|---|---|----------------------------------------|------------------|---|-----------|
| 44                  | 6 | — | <i>AB</i>                              | —                | — | 44.5      |
| 6                   | 9 | — | <i>Ba = <math>\frac{1}{3}BG</math></i> | —                | — | 6.75      |
| <hr/>               |   |   |                                        | <hr/>            |   |           |
| 51                  | 3 | — | sum                                    | —                | — | 51.25     |
| 33                  | 3 | — | girt                                   | —                | — | 33.25     |
| <hr/>               |   |   |                                        | <hr/>            |   |           |
| 153                 |   |   |                                        |                  |   | 25625     |
| 153                 |   |   |                                        |                  |   | 10250     |
| 8                   | 3 |   |                                        |                  |   | 15375     |
| 12                  | 9 | 9 |                                        |                  |   | 15375     |
| <hr/>               |   |   |                                        | <hr/>            |   |           |
| 1704                | 0 | 9 | — content —                            |                  |   | 1704.0625 |
| <hr/>               |   |   |                                        | <hr/>            |   |           |

Hence the content is 1704 square feet, or, by cutting off two figures, 17 squares and 4 feet.

*Note.* No deductions are made for chimneys, &c.

5. *Stair Cafes.* Multiply the length of a step by its breadth, then multiplying the product by the number of steps will give the area of all the steps, to which adding the areas of the landings or half-paces gives the whole area of the stairs. Or else make a line ply close over all the steps from the top to the bottom, and multiply the length of that line by the length of a step for the whole area.—*Note,* that by the length of a step is meant the length of the front together with the return or returns at the end or ends; and by the breadth is meant the girt of its two fur-faces, or the tread and riser, or horizontal and perpendicular faces together.

In the balustrade, take for the length the whole length of the upper part of the hand-rail, from the top to the bottom of the stairs, girt over its end



till it meet the top of the newel post; and for the breadth take the girt up the longer baluster, over the hand-rail, and down the same baluster to the step again; or twice the length of the baluster upon the landing with the girt of the hand-rail added.

6. *Wainscoting*. In this branch take for the length the compass of the room; and for the breadth take the height from the floor to the top of the wainscoting, making the line ply close into all the mouldings in both cases.—Out of this must be made deductions for chimneys, doors, and windows, which are measured separately. But for workmanship only no deduction ought to be made on account of the extraordinary trouble.

7. *Doors*. Into both the dimensions of length and breadth take once the thickness of the door. If the door be pannelled on both sides, take double the measure of it for the workmanship; but if it be pannelled only on one side, the other side being simply planed, take the area and the half of it for workmanship.

For the length of the surrounding architrave, girt it about the external or outermost part; and for the breadth, girt it over as far as is seen when the door is open.

8. *Window architraves* and *window-shutters* are measured in the same manner; as are also *bases*, *dados*, *sub-bases*, &c. *Windows* themselves are measured by multiplying their greatest lengths by the greatest breadths.

Examples are unnecessary, as there are plenty in the general one at the end.



## CHAPTER VI.

*Of Slaters and Tilers Work.*

This chapter will be very short, for there is nothing more to be done but just to multiply the length of the tiling, &c. by the girt over from the ridge to the eaves. And in taking this girt the line is made to ply over the lowest row of slates and returned up the underside of the same till it meet with the wall or eaves board, because the lowest row is laid double. But in tiling, the line is stretched down only to the lowest part, without returning it up the under side.

No deductions are made for chimneys, &c. also the content is computed either in yards of 9 square feet, or in squares of 100 feet; and the same allowances for hips and valleys are to be made as in roofing in the last chapter, to which therefore I refer for particular directions.

EXAMPLE. The length of a slated roof being 45 feet 9 inches, and girt 34 feet 3 inches, required the content.

| <i>Duodecimally.</i> |      |        |             | <i>Decimally.</i> |      |      |
|----------------------|------|--------|-------------|-------------------|------|------|
| 45                   | 9    | —      | length      | —                 | 45   | 75   |
| 34                   | 3    | —      | girt        | —                 | 34   | 25   |
| <hr/>                |      |        |             | <hr/>             |      |      |
| 180                  |      |        |             | 228               | 75   |      |
| 135                  |      |        |             | 915               | 0    |      |
| 17                   |      |        |             | 183               | 00   |      |
| 8                    | 6    |        |             | 137               | 25   |      |
| 11                   | 5    | 3      |             |                   |      |      |
| <hr/>                |      |        |             | <hr/>             |      |      |
| 9)                   | 1566 | 11 3   | — content — | 9)                | 1566 | 9375 |
| <hr/>                |      |        |             | <hr/>             |      |      |
|                      | 174  | 0 11 3 |             |                   | 174  | 104  |

So the content is 15 squares and 67 feet nearly,  
or 174 yards.

*By*



*By the Sliding Rule.*

$$\begin{array}{cccc} B & A & B & A \\ 1 & : 34\frac{1}{4} :: 45\frac{3}{4} : 1567. \end{array}$$

## CHAPTER VII.

### *Of Plasterers Work.*

Plasterers plain work is measured by the square foot, or yard of 9 square feet, or square of 100 feet; and enriched mouldings, &c. by running or lineal measure.

The work is chiefly of two kinds, viz. 1. plastering upon laths, called ceiling; and 2. plastering upon walls, called rendering. All the different kinds, such as ceiling, rendering, cornices, &c. or which are of different prices, by having more or fewer coats of plaster, are to be measured separately; and then all the contents of the same kind, or same number of coats, are to be collected together at the last, as is done in the general example.

Deductions must be made out of rooms, &c. for chimneys, doors, windows, and other parts that are not plastered; but there will seldom need any deductions for windows, on account of the plastered returns both at the top and sides, which will generally be fully as large as the window itself. But it is to be understood that these deductions are made only when the workman finds the materials.

In arches it is evident that the length multiplied by the girt round them will give the content.

EXAMPLE. The length of a room being 14 feet 5 inches, its breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, whose



whose girt is  $8\frac{1}{2}$  inches, and projects 5 inches from the wall on the upper part next the ceiling; required the quantity of plastering, the cornice having one enriched moulding in it, there being no deductions but for one door whose size is 7 feet by 4 feet.

|        |   |         |                          |                             |                |                  |                        |
|--------|---|---------|--------------------------|-----------------------------|----------------|------------------|------------------------|
| 14     | 5 | length  |                          | 55                          | 2              | compass          |                        |
| 13     | 2 | breadth |                          |                             | $8\frac{1}{2}$ | girt of cornice  |                        |
| <hr/>  |   |         |                          | I                           | <hr/>          |                  |                        |
| 27     | 7 | fum     |                          | $6 = \frac{1}{2}$           | 27             | 7                |                        |
| 27     | 7 | add     |                          | $2 = \frac{1}{3}$           | 9              | 2                | 4                      |
| <hr/>  |   |         |                          | $\frac{1}{2} = \frac{1}{4}$ | 2              | 3                | 7                      |
| 55     | 2 | compass |                          | <hr/>                       |                |                  |                        |
| 9      | 3 | height  |                          | 39                          | 0              | 11               | cont. of the cornice   |
| <hr/>  |   |         |                          | <hr/>                       |                |                  |                        |
| 496    | 6 |         |                          | 13                          | 7              | length ceiling   |                        |
| 13     | 9 | 6       |                          | 12                          | 4              | breadth of ditto |                        |
| <hr/>  |   |         |                          | <hr/>                       |                |                  |                        |
| 510    | 3 | 6       |                          | 163                         | 0              |                  |                        |
| 28     | 0 | 0       | door deduct              | 4                           | 6              | 4                |                        |
| <hr/>  |   |         |                          | <hr/>                       |                |                  |                        |
| 9) 482 | 3 | 6       |                          | 9) 167                      | 6              | 4                |                        |
| <hr/>  |   |         |                          | <hr/>                       |                |                  |                        |
| 53     | 5 | 3       | 6 cont. of the rendering | 18                          | 5              | 6                | 4 cont. of the ceiling |
| <hr/>  |   |         |                          | <hr/>                       |                |                  |                        |

So that there are

| Yd  | F  | I | P  |               |
|-----|----|---|----|---------------|
| 53  | 5  | 3 | 6  | of rendering  |
| 18  | 5  | 6 | 4  | of ceiling    |
| and | 39 | 0 | 11 | plain cornice |

together with 55 feet 2 inches of enriched moulding.



## CHAPTER VIII.

*Of Painters Work.*

Painters work is done by the square yard ; and every part where the colour lies must be measured ; so that, in taking the dimensions, the line must be forced close into all mouldings, &c.

Here balustrades and most other things are measured as in joiners work, to which I refer for particular directions ; excepting indeed in some cases in which they differ a little, for although sometimes for doors and window-shutters the joiner hath only the area and half area, yet the painter hath always double the area of one side, because both sides are coloured, and every part that is so painted must be measured.

Windows are done at so much a-piece.

For carved mouldings, &c. it is customary to allow double the measure which results from multiplying the length by the girt.

## EXAMPLE.

Suppose a room were to be painted, and that its length is 24 feet 6 inches, its breadth 16 feet 3 inches, and its height 12 feet 9 inches ; also the size of the door 7 feet by 3 feet 6 inches, and the size of the window-shutters to each of two windows 7 feet 9 inches by 3 feet 6 inches, but the breaks of the windows themselves are 8 feet 6 inches high and 1 foot 3 inches deep ; what will the expence be, giving it three coats at 2d. a yard each ; the size of the chimney to be deducted being 5 feet by 5 feet 6 inches, and all the dimensions being taken according to the methods above mentioned.



|                         |   |            |    |   |                      |
|-------------------------|---|------------|----|---|----------------------|
| 24                      | 6 | length     | 7  | 0 | height of the door   |
| 16                      | 3 | breadth    | 7  | 9 | height of wind. sh.  |
|                         |   |            | 7  | 9 | the other            |
| 40                      | 9 | their sum  |    |   |                      |
| 40                      | 9 | add        | 22 | 6 | their sum            |
|                         |   |            | 3  | 6 | common breadth       |
| 81                      | 6 | compass    |    |   |                      |
| 12                      | 9 | height     | 67 | 6 |                      |
|                         |   |            | 11 | 3 |                      |
| 978                     | 0 |            |    |   |                      |
| 40                      | 9 | P          | 78 | 9 | content of the doors |
| 20                      | 4 | 6          |    |   | and window-shut-     |
|                         |   |            |    |   | ters once more.      |
| 1039                    | 1 | 6 the con- |    |   |                      |
| tent of the room inclu- |   |            | 8  | 6 | height of wind. br.  |
| ding the door and win-  |   |            | 3  | 6 | breadth of the same  |
| dows once.              |   |            |    |   |                      |
|                         |   |            | 12 | 0 | their sum            |
| 5                       | 6 | } chimney  | 12 | 0 | the same             |
| 5                       | 0 |            |    |   |                      |
|                         |   |            | 24 | 0 | the compass          |
| 27                      | 6 | content    | 1  | 3 | depth                |

Hence, by collecting the sums, we shall obtain as below:

|    |   |                    |
|----|---|--------------------|
| 30 | 0 | content of 1 wind. |
| 30 | 0 | the same           |
| 60 | 0 | content of both.   |

1039 1 6 whole room with door, &c. once  
 78 9 0 door and wind. shut. once more  
 60 0 0 window jambs, &c.

1177 10 6 their sum  
 27 6 0 chimney deduct

9) 1150 4 6 whole content

6d. =  $\frac{1}{2}$ s.) 127  $\frac{7}{9}$  content in yards

63 10  $\frac{1}{2}$  or £ 3 3 10  $\frac{1}{2}$  = the expence re-  
 quired.



## CHAPTER IX.

*Of Glaziers Work.*

Glaziers take their dimensions in feet, inches, and eighths or tenths, or else in feet and hundredth parts of a foot, and estimate their work by the square foot.

Windows are sometimes measured by taking the dimensions of one pane, and multiplying its content by the number of panes. But more generally, for the whole length of the glazing in a window, is taken the length from the top of the highest pane to the bottom of the lowest one; and the breadth is also taken at once over all the panes from outside to outside, the cross-bars between panes being included.

Circular, or triangular windows, or windows of any other figure, are measured as square, taking for the dimensions their extreme lengths and breadths, on account of the waste of glass attending the cutting of it to the necessary forms.

## EXAMPLE.

At 13 pence a foot it is required to find the expence of glazing the front of a house having 3 tier of windows and 3 windows in each tier; the height of those in the lowest tier being 7 feet 9 inches, of those in the middle 6 feet 6 inches, and of those in the upper tier 5 feet  $3\frac{1}{4}$  inches; and having also a circular window above the door, the greatest height of which is 1 foot  $10\frac{1}{2}$  inches; and the common breadth of all the windows 3 feet 9 inches.



|                       |                 |                                     |
|-----------------------|-----------------|-------------------------------------|
| 7                     | 9               |                                     |
| 6                     | 6               |                                     |
| 5                     | $3\frac{1}{4}$  |                                     |
| <hr/>                 |                 |                                     |
| 19                    | $6\frac{1}{4}$  | sum of heights 1 wind. in each tier |
|                       | 3               | multiply                            |
| <hr/>                 |                 |                                     |
| 58                    | $6\frac{3}{4}$  | sum of heights of the 9 windows     |
| 1                     | $10\frac{1}{2}$ | height of that above the door       |
| <hr/>                 |                 |                                     |
| 60                    | $5\frac{1}{4}$  | sum of all the heights              |
|                       | 3               | 9 common breadth                    |
| <hr/>                 |                 |                                     |
| 181                   | $3\frac{3}{4}$  |                                     |
| 30                    | $2\frac{1}{2}$  |                                     |
| 15                    | $1\frac{1}{4}$  |                                     |
| <hr/>                 |                 |                                     |
| 226                   | $7\frac{1}{2}$  | content of the whole                |
| <hr/>                 |                 |                                     |
| 1 s. = $\frac{1}{20}$ | 11              | $6\ 7\frac{1}{2}$                   |
| 1 d. = $\frac{1}{12}$ | 0               | $18\ 10\frac{1}{2}$                 |
| <hr/>                 |                 |                                     |
| £                     | 12              | 5 6 the expence required.           |
| <hr/>                 |                 |                                     |

## CHAPTER X.

*Of Pavers Work.*

Pavers work is done by the square yard. And the length is multiplied by the breadth for the content.

## EXAMPLE.

What will be the expence of paving a rectangular court yard, its length being 63 feet and breadth 45 feet; and in which is laid a foot path

7 G

the



the whole length of it, and  $5\frac{1}{4}$  feet broad, with broad stones, at 3 shillings a foot; the rest being paved with pebbles at 2 shillings and six-pence a foot.

$$\begin{array}{r} 63 \text{ the length} \\ 45 \text{ the breadth} \\ \hline 315 \\ 252 \\ \hline 9) 2835 \end{array}$$

$2s. 6d. = \frac{1}{8}) 315$  yds. = the content of the court

£ 39 7 6 expence of the whole at  $2s. 6d.$   
a foot.

But the walk is 6 pence a foot more than the rest; hence

$$\begin{array}{r} 63 \text{ length} \\ 5\frac{1}{4} \text{ breadth} \\ \hline 315 \\ 15\frac{3}{4} \\ \hline 9) 330\frac{3}{4} \\ \hline 36\frac{3}{4} \\ \hline 6d. = \frac{1}{40} \left| \begin{array}{r} 0 \quad 18 \quad 4\frac{1}{2} \text{ additional part} \\ 39 \quad 7 \quad 6 \text{ add} \end{array} \right. \\ \hline \hline \text{£ } 40 \quad 5 \quad 10\frac{1}{2} \text{ whole expence required.} \end{array}$$



## CHAPTER XI.

*Of Plumbers Work.*

Plumbers work is generally done by the pound or hundred weight; but I shall here insert a method of discovering the weight by the measure of it.

Sheet-lead used in roofing, guttering, &c. is generally between 7 and 12 pound weight to the square foot; but the following table shews by inspection the particular weight of a square foot for each of several thickneffes.

| Thick-<br>nefs | lb. to a sq.<br>foot | Thick-<br>nefs | lb. to a sq.<br>foot | Thick-<br>nefs | lb. to a sq.<br>foot | Thick-<br>nefs    | lb. to a sq.<br>foot |
|----------------|----------------------|----------------|----------------------|----------------|----------------------|-------------------|----------------------|
| ·10            | 5·899                | $\frac{1}{8}$  | 7·373                | ·15            | 8·848                | ·18               | 10·618               |
| ·11            | 6·489                | ·13            | 7·668                | ·16            | 9·438                | ·19               | 11·207               |
| $\frac{1}{9}$  | 6·554                | ·14            | 8·258                | $\frac{1}{6}$  | 9·831                | $2 = \frac{1}{5}$ | 11·797               |
| ·12            | 7·078                | $\frac{1}{7}$  | 8·427                | ·17            | 10·028               | ·21               | 12·387               |

In this table the thicknefs is fet down in tenths and hundredths, &c. of an inch; and the annexed corresponding numbers are the weights in avoirdupois pounds and thousandth parts of a pound. So the weight of a square foot of  $\frac{1}{10}$  or  $\frac{10}{1000}$  of an inch thick is 5 pounds and 899 thousandth parts of a pound; and the weight of a square foot to  $\frac{1}{9}$  of an inch thicknefs is 6 pounds and  $\frac{554}{1000}$  of a pound.

## EXAMPLE.

What will be the expence of covering a roof with lead  $\frac{1}{8}$  of an inch in thicknefs, the length being 43 feet, and breadth or girt over 32 feet; together with 57 feet of guttering of 2 feet wide, with lead  
of



of  $\frac{1}{8}$  of an inch thick; after the rate of 18 shillings a hundred weight.

|          |   |                 |   |        |
|----------|---|-----------------|---|--------|
| 43       | — | length          | — | 57     |
| 32       | — | breadth         | — | 2      |
| <hr/>    |   |                 |   | <hr/>  |
| 1376     | — | content in feet | — | 114    |
| 9831     | — | lb. to a foot   | — | 9373   |
| <hr/>    |   |                 |   | <hr/>  |
| 1376     |   |                 |   | 342    |
| 4128     |   |                 |   | 798    |
| 11008    |   |                 |   | 342    |
| 12384    |   |                 |   | 798    |
| <hr/>    |   |                 |   | <hr/>  |
| 13527456 | — | weight in lb.   | — | 840522 |
| 840522   |   |                 |   |        |
| <hr/>    |   |                 |   |        |

14367.978 whole weight in pounds, which being divided by 112 gives 128.2855 cwt. for the weight of the whole, which at 18 shillings the cwt. comes to £ 115 9 1 $\frac{1}{2}$  for the expence required.

## CHAPTER XII.

### GENERAL ILLUSTRATION.

Having gone through the measurable parts of a building, &c. and noted the methods of measuring them, and of computing the contents; I proceed now to the general illustration of the whole, by assigning the dimensions of a house, and from thence computing the contents of the several different kinds of works supposed to be used in it; in the performing of which are shewn the methods of ruling the book and entering the dimensions with the contents, then the method of abstracting the contents, and



and lastly of forming the bills of the expences of the work and materials. At the end of this chapter is given a plate containing the elevation and plans of the several storys of the house, accurately delineated from a particular scale, inserted in the plate; by which the reader may, with a pair of compasses, measure and compare the dimensions of most of the articles as he proceeds through them. In this example is used a house of only two storys and two rooms on each floor, as those are very sufficient for explaining the several methods and works here treated of.

The columns of numbers in the following forms are sufficiently explained by the titles at the tops of them; excepting the figures 2, 3, &c. in the first column, and prefixed to the dimensions, the meaning of which figures is, that the contents arising from the dimensions to which they are prefixed, are to be multiplied by 2, 3, &c. and then set down in the column of contents.

*The DIGGING of the CELLARS.*

*Feet In.*

54 0 length

28 6 breadth

8 0 depth

which three dimensions, being multiplied continually together, produce 12096 cubic feet, which being divided by 27, the cubic feet in a yard, there result 448 cubic yards of digging.

7 H

*The*



*The BRICK WORK.*

| Dimen-<br>sions |            | Half<br>bricks<br>thick | Contents    |            | Titles                     |
|-----------------|------------|-------------------------|-------------|------------|----------------------------|
| <i>Feet</i>     | <i>In.</i> |                         | <i>Feet</i> | <i>In.</i> |                            |
| 155             | 0          | 6                       | 1550        | 0          | The cellar walls.          |
| 10              | 0          |                         |             |            |                            |
| 23              | 6          | 8                       | 470         | 0          | Middle walls of ditto.     |
| 10              | 0          |                         |             |            |                            |
| 6               | 8          | 8                       | 48          | 10         | Deduct doors in ditto.     |
| 3               | 8          |                         |             |            |                            |
| 23              | 6          | 1                       | 799         | 0          | Paving the cellars, brick  |
| 17              | 0          |                         |             |            | on edge.                   |
| 23              | 6          | 1                       | 211         | 6          | Ditto between the cellars  |
| 9               | 0          |                         |             |            |                            |
| 24              | 0          | 8                       | 840         | 0          | Groin arches over the two  |
| 17              | 6          |                         |             |            | cellars.                   |
| 24              | 0          | 8                       | 228         | 0          | Common arch under the      |
| 9               | 6          |                         |             |            | passage.                   |
| 23              | 6          | 4                       | 211         | 6          | Deduct vacuity of ditto.   |
| 9               | 0          |                         |             |            |                            |
| 155             | 0          | 5                       | 3797        | 6          | Out walls of the principal |
| 24              | 6          |                         |             |            | and attic story.           |
| 24              | 0          | 7                       | 1176        | 0          | Middle walls of ditto.     |
| 24              | 6          |                         |             |            |                            |
| 7               | 0          | 6                       | 112         | 0          | Chimney shafts within the  |
| 8               | 0          |                         |             |            | roof.                      |

Di-



|   | Dimenfi-<br>ons |     | Half<br>bricks<br>thick | Products |     | Titles                 |
|---|-----------------|-----|-------------------------|----------|-----|------------------------|
|   | Feet            | In. |                         | Feet     | In. |                        |
| 2 | 8               | 8   | 5                       | 80       | 10  | Windows in front.      |
|   | 4               | 8   |                         |          |     |                        |
| 3 | 4               | 8   | 5                       | 65       | 4   | Ditto.                 |
|   | 4               | 8   |                         |          |     |                        |
|   | 11              | 0   | 5                       | 66       | 0   | Front door.            |
|   | 6               | 0   |                         |          |     |                        |
| 4 | 8               | 0   | 5                       | 128      | 0   | End windows.           |
|   | 4               | 0   |                         |          |     |                        |
| 2 | 4               | 0   | 5                       | 32       | 0   | Ditto.                 |
|   | 4               | 0   |                         |          |     |                        |
| 2 | 4               | 0   | 1                       | 32       | 0   | Ditto blind or false.  |
|   | 4               | 0   |                         |          |     |                        |
|   | 4               | 0   | 5                       | 32       | 0   | Stair-case window.     |
|   | 8               | 0   |                         |          |     |                        |
| 6 | 6               | 8   | 7                       | 146      | 8   | Doors in middle walls. |
|   | 3               | 8   |                         |          |     |                        |
| 2 | 6               | 8   | 1                       | 48       | 10  | Ditto false.           |
|   | 3               | 8   |                         |          |     |                        |
| 2 | 4               | 6   | 5                       | 40       | 6   | Recess of chimneys.    |
|   | 4               | 6   |                         |          |     |                        |
| 2 | 3               | 10  | 5                       | 29       | 4   | Ditto.                 |
|   | 3               | 10  |                         |          |     |                        |
| 2 | 3               | 6   | 5                       | 24       | 6   | Ditto.                 |
|   | 3               | 6   |                         |          |     |                        |

To



To abstract the foregoing particulars, that is, to collect together the contents of the same kind, reduce their sums to the standard thickness of a brick and a half, and then, having made the deductions, to collect the reduced sums, to obtain the whole in one sum, I proceed thus: I make only two columns for the whole contents, and two for the deductions of the same thickness, viz. one column for the one brick contents, and the other for the brick and half contents; and dispose the superior denominations in one or both of these columns by placing them down more than once; thus, the contents of 2 bricks thick I set down twice in the one brick column; those of  $2\frac{1}{2}$  bricks, I set down once in the one brick column and once in the brick-and-half column; those of 3 bricks, twice in the brick-and-half column; those of  $3\frac{1}{2}$  bricks, once in the brick-and-half, and twice in the one brick column; and those of four bricks, twice in the brick-and-half and once in the one brick column; and so on if there were higher denominations; and, lastly, for the contents of half a brick thick, I take the halves of them and set in the one brick column. Then, having added up every column, I reduce the sums of the one brick thick to the brick and half thickness; and add this reduced content to the brick and half sums, both in the solids and deductions; then, lastly, take the one sum from the other for the whole content of the brick work, as below.



*The Abstract of Brick Work.*

| 1 $\frac{1}{2}$ br. thick |     | 1 brick thick |     | 1 $\frac{1}{2}$ deduct |     | 1 br. deduct |     |
|---------------------------|-----|---------------|-----|------------------------|-----|--------------|-----|
| Feet                      | In. | Feet          | In. | Feet                   | In. | Feet         | In. |
| 1550                      | 0   | 470           | 0   | 48                     | 10  | 48           | 10  |
| 1550                      | 0   | 399           | 6   | 48                     | 10  | 211          | 6   |
| 470                       | 0   | 105           | 9   | 80                     | 10  | 211          | 6   |
| 470                       | 0   | 840           | 0   | 65                     | 4   | 80           | 10  |
| 840                       | 0   | 228           | 0   | 66                     | 0   | 65           | 4   |
| 840                       | 0   | 3797          | 6   | 128                    | 0   | 66           | 0   |
| 228                       | 0   | 1176          | 0   | 32                     | 0   | 128          | 0   |
| 228                       | 0   | 1176          | 0   | 32                     | 0   | 32           | 0   |
| 3797                      | 6   |               |     | 146                    | 8   | 16           | 0   |
| 1176                      | 0   | 8192          | 9   | 40                     | 6   | 32           | 0   |
| 112                       | 0   |               | 2   | 29                     | 4   | 146          | 8   |
| 112                       | 0   | 3)16385       | 6   | 24                     | 6   | 146          | 8   |
|                           |     |               |     |                        |     | 24           | 5   |
| 11373                     | 6   |               |     | 742                    | 10  | 40           | 6   |
| 5461                      | 10  | 5461          | 10  | 869                    | 5   | 29           | 4   |
|                           |     |               |     |                        |     | 24           | 6   |
| 16835                     | 4   |               |     | 1612                   | 3   |              |     |
| 1612                      | 3   |               |     |                        |     | 1304         | 1   |
|                           |     |               |     |                        |     |              | 2   |
| 9)15223                   | 1   |               |     |                        |     | 3)2608       | 2   |
| 1691 $\frac{4}{9}$ yds.   |     |               |     |                        |     |              |     |
|                           |     |               |     |                        |     | 869          | 5   |



| <i>Feet In.</i> |                                               | <i>Feet In.</i>     |
|-----------------|-----------------------------------------------|---------------------|
| 2               | 0 thick. of walls.                            | 211 6 vacu. arch.   |
| 4               | multiply                                      | 4 half br. thick    |
| 8               | 0                                             | 3) 846 0            |
| 24              | 6 height of 2 storys                          | 282 0 reduced con-  |
| 196             | 0                                             | tent of the vacuity |
| 5               | half bricks thick.                            | of the com. arch.   |
| 3) 980          | 0                                             |                     |
| 326             | 8 reduced cont. of the 4 quoins of the house. |                     |
| 1612            | 3 the deductions.                             |                     |
| 1933            | 11 sum.                                       |                     |
| 282             | 0 vacuity of arch under the passage, deduct.  |                     |
| 9) 1656         | 11                                            |                     |
| 184             | yards to be added for workmanship only.       |                     |

*So that in the bill we shall have*

|                                     |   |   |   |   |
|-------------------------------------|---|---|---|---|
|                                     |   | £ | s | d |
| 1691 yards 4 feet of brick-and-half | } | — | — | — |
| wall, work and materials at         |   |   |   |   |
| — per yard                          |   |   |   |   |

|                               |   |   |   |   |
|-------------------------------|---|---|---|---|
| 184 yards for workmanship, in | } | — | — | — |
| doors and windows, &c. at     |   |   |   |   |
| — per yard                    |   |   |   |   |

£ — — —

*The*



*The MASON'S WORK.*

| Dimensions. |            | Contents    |            | Titles                           |
|-------------|------------|-------------|------------|----------------------------------|
| <i>Feet</i> | <i>In.</i> | <i>Feet</i> | <i>In.</i> |                                  |
| 157         | 0          | 340         | 2          | Stone base.                      |
| 2           | 2          |             |            |                                  |
| 157         | 0          | 104         | 8          | Facia.                           |
| 0           | 8          |             |            |                                  |
| 4           | 8          | 21          | 0          | Window soles.                    |
| 0           | 9          |             |            |                                  |
| 27          | 0          | 63          | 0          | Jambs and head of front door,    |
| 2           | 4          |             |            | plain.                           |
| 9           | 0          | 54          | 0          | Columns to front door.           |
| 3           | 0          |             |            |                                  |
| 3           | 8          | 22          | 0          | Base and capitals to ditto (com- |
| 1           | 6          |             |            | monly taken double measure.)     |
| 11          | 6          | 10          | 6          | Architrave.                      |
| 0           | 11         |             |            |                                  |
| 9           | 6          | 7           | 11         | Soffit of ditto.                 |
| 0           | 10         |             |            |                                  |
| 11          | 6          | 7           | 8          | Frize.                           |
| 0           | 8          |             |            |                                  |
| 13          | 10         | 20          | 9          | Level cornice.                   |
| 1           | 6          |             |            |                                  |
| 12          | 0          | 30          | 0          | Cornice of pediment.             |
| 2           | 6          |             |            |                                  |
| 1           | 6          | 1           | 0          | Returns of cima-recta.           |
| 0           | 4          |             |            |                                  |

Di-



|   | Dimensions |                 | Contents |     | Titles                                  |
|---|------------|-----------------|----------|-----|-----------------------------------------|
|   | Feet       | In.             | Feet     | In. |                                         |
| 2 | 22         | 8               | 98       | 2   | Architraves of low windows.             |
|   | 2          | 2               |          |     |                                         |
| 2 | 5          | 8               | 6        | 1   | Frize of ditto.                         |
|   | 0          | 6 $\frac{1}{2}$ |          |     |                                         |
| 2 | 7          | 9               | 21       | 11  | Cornice of ditto.                       |
|   | 1          | 5               |          |     |                                         |
| 3 | 21         | 4               | 138      | 8   | Architrave of attic windows.            |
|   | 2          | 2               |          |     |                                         |
|   | 120        | 2               | 252      | 4   | Cornice to front and ends of the house. |
|   | 2          | 1               |          |     |                                         |
|   | 109        | 6               | 200      | 9   | Blocking course to front and ends.      |
|   | 1          | 10              |          |     |                                         |
| 4 | 5          | 6               | 34       | 10  | Steps to front.                         |
|   | 1          | 7               |          |     |                                         |
| 2 | 8          | 6               | 34       | 0   | Ends of ditto.                          |
|   | 2          | 0               |          |     |                                         |
| 2 | 3          | 0               | 8        | 0   | Ditto.                                  |
|   | 1          | 4               |          |     |                                         |
|   | 9          | 0               | 27       | 0   | Top of the landing.                     |
|   | 3          | 0               |          |     |                                         |
| 2 | 14         | 0               | 98       | 0   | Chimney tops of stone.                  |
|   | 3          | 6               |          |     |                                         |
| 2 | 15         | 4               | 51       | 1   | Base and facias of ditto.               |
|   | 1          | 8               |          |     |                                         |

Di-



| Dimensions |              | Contents |     | Titles                                              |
|------------|--------------|----------|-----|-----------------------------------------------------|
| Feet       | In.          | Feet     | In. |                                                     |
| 2          | 5 0<br>2 2   | 21       | 8   | Polished slabs to the chimneys<br>in the low rooms. |
| 2          | 3 9<br>1 10  | 13       | 9   | Back slabs to ditto.                                |
| 2          | 12 1<br>0 11 | 22       | 1   | Jambs and mantles to ditto.                         |
| 4          | 4 4<br>1 6   | 26       | 0   | Coves to ditto.                                     |
| 2          | 4 8<br>2 0   | 18       | 8   | Slabs to chimneys in the attic<br>story.            |
| 2          | 3 6<br>1 10  | 12       | 10  | Back slabs.                                         |
| 2          | 11 2<br>0 11 | 20       | 5   | Jambs and mantles.                                  |
| 4          | 4 0<br>1 6   | 24       | 0   | Coves.                                              |
| 2          | 4 2<br>2 0   | 16       | 8   | Slabs.                                              |
| 2          | 3 0<br>1 10  | 11       | 0   | Back slabs.                                         |
| 2          | 10 4<br>0 10 | 17       | 2   | Jambs and mantles.                                  |
| 4          | 3 6<br>1 6   | 21       | 0   | Coves.                                              |
| 12         | 3 6<br>1 4   | 56       | 0   | Steps down to the cellar.                           |



*The Abstract of the Masonry.*

In making this abstract, the articles of the same price must be collected together; so in the first column are collected together the base, facias, window soles, jambs and head of door, blocking course and chimney tops, as being of the same value; in the 2d are collected architraves, frizes, cornices, mouldings about the door, columns, bases and capitals; and so for other things of another value.

| Base, facias, &c. |          | Archi-traves, &c. |          | Chimney flabs, &c. |          | Front steps |          | Chimney jambs, &c. |          | Cornice of the house |          | Cellar steps |          |
|-------------------|----------|-------------------|----------|--------------------|----------|-------------|----------|--------------------|----------|----------------------|----------|--------------|----------|
| <i>Feet</i>       | <i>I</i> | <i>Feet</i>       | <i>I</i> | <i>Feet</i>        | <i>I</i> | <i>Feet</i> | <i>I</i> | <i>Feet</i>        | <i>I</i> | <i>Feet</i>          | <i>I</i> | <i>Feet</i>  | <i>I</i> |
| 340               | 2        | 54                | 0        | 21                 | 8        | 34          | 10       | 22                 | 1        | 252                  | 4        | 56           | 0        |
| 104               | 8        | 22                | 0        | 13                 | 9        | 34          | 0        | 20                 | 5        |                      |          |              |          |
| 21                | 0        | 10                | 6        | 26                 | 0        | 8           | 0        | 17                 | 2        |                      |          |              |          |
| 63                | 0        | 7                 | 11       | 18                 | 8        | 27          | 0        |                    |          |                      |          |              |          |
| 200               | 9        | 7                 | 8        | 12                 | 10       |             |          | 59                 | 8        |                      |          |              |          |
| 98                | 0        | 20                | 9        | 24                 | 0        | 103         | 10       |                    |          |                      |          |              |          |
| 51                | 1        | 30                | 0        | 16                 | 8        |             |          |                    |          |                      |          |              |          |
| 878               | 8        | 1                 | 0        | 11                 | 0        |             |          |                    |          |                      |          |              |          |
|                   |          | 98                | 2        | 21                 | 0        |             |          |                    |          |                      |          |              |          |
|                   |          | 6                 | 1        |                    |          |             |          |                    |          |                      |          |              |          |
|                   |          | 21                | 11       | 165                | 7        |             |          |                    |          |                      |          |              |          |
|                   |          | 138               | 8        |                    |          |             |          |                    |          |                      |          |              |          |
|                   |          | 418               | 8        |                    |          |             |          |                    |          |                      |          |              |          |

*The Mason's Bill.*

| <i>Feet</i> | <i>In.</i> |                                      | £ | s | d |
|-------------|------------|--------------------------------------|---|---|---|
| 878         | 8          | Base, facias, &c. at — a foot        | — | — | — |
| 418         | 8          | Columns, mouldings, &c. at — a foot  | — | — | — |
| 165         | 7          | Chimney flabs and coves, at — a foot | — | — | — |
| 103         | 10         | Steps to front door, at — a foot     | — | — | — |
| 59          | 8          | Chim. jambs and mantles, at — a foot | — | — | — |
| 252         | 4          | Cornice of the house, at — a foot    | — | — | — |
| 56          | 0          | Cellar steps, at — a foot            | — | — | — |

£ — — —



*The CARPENTERS and JOINERS WORK.*

| Dimen-<br>sions |              | Contents    | Titles     |                                                                                                    |
|-----------------|--------------|-------------|------------|----------------------------------------------------------------------------------------------------|
| <i>Feet</i>     | <i>In.</i>   | <i>Feet</i> | <i>In.</i> |                                                                                                    |
| 2               | 24 0<br>17 6 | 840 0       |            | Common joist. in the low rooms.                                                                    |
| 2               | 24 6<br>17 6 | 857 6       |            | Framing the upper floors, with<br>girders, binding & bridging joist.                               |
| 2               | 11 9<br>12 0 | 282 0       |            | Common joisting of the stair-case,<br>allowing the joist one foot hold<br>of the wall at each end. |
|                 |              | 180 0       |            | Plates over doors and windows,<br>running measure.                                                 |
|                 | 59 3<br>36 0 | 2133 0      |            | Roofing.                                                                                           |
|                 | 109 0<br>2 0 | 218 0       |            | Gutter boarding at front & ends.                                                                   |
| 4               | 24 0<br>17 6 | 1680 0      |            | Ceiling joists to both storys.                                                                     |
| 2               | 17 6<br>11 0 | 385 0       |            | Stoothed partitions.                                                                               |
| 2               | 24 0<br>17 6 | 840 0       |            | Deal flooring.                                                                                     |
| 2               | 23 6<br>17 6 | 822 6       |            | Ditto upper story.                                                                                 |
| 2               | 12 0<br>9 9  | 234 0       |            | Ditto in stair-case.                                                                               |

Di-



|    | Dimenfi-<br>ons |     | Contents |     | Titles                            |
|----|-----------------|-----|----------|-----|-----------------------------------|
|    | Feet            | In. | Feet     | In. |                                   |
| 11 | 4               | 8   | 51       | 4   | Ditto in window ways.             |
|    | 1               | 0   |          |     |                                   |
| 7  | 3               | 8   | 64       | 2   | Ditto in door ways.               |
|    | 2               | 6   |          |     |                                   |
| 11 | 4               | 0   | 66       | 0   | Deal steps to stair-case, 1st and |
|    | 1               | 6   |          |     | 2d flights.                       |
| 11 | 1               | 0   | 6        | 5   | Ends of ditto.                    |
|    | 0               | 7   |          |     |                                   |
| 2  | 4               | 0   | 36       | 0   | Half paces to ditto.              |
|    | 4               | 6   |          |     |                                   |
| 9  | 4               | 0   | 108      | 0   | Steps of the 3d flight.           |
|    | 3               | 0   |          |     |                                   |
| 9  | 1               | 0   | 5        | 3   | Ends of ditto.                    |
|    | 0               | 7   |          |     |                                   |
|    | 9               | 9   | 5        | 8   | Face of the landing.              |
|    | 0               | 7   |          |     |                                   |
| 7  | 8               | 4   | 272      | 2   | Safhes.                           |
|    | 4               | 8   |          |     |                                   |
| 7  | 4               | 8   | 152      | 5   | Ditto.                            |
|    | 4               | 4   |          |     |                                   |
| 6  | 16              | 8   | 250      | 0   | Inside door-casings, panned.      |
|    | 2               | 6   |          |     |                                   |
| 3  | 16              | 8   | 33       | 4   | Ditto, plain.                     |
|    | 0               | 8   |          |     |                                   |

Di-



|    | Dimenfi-<br>ons |     | Contents |     | Titles                                                       |
|----|-----------------|-----|----------|-----|--------------------------------------------------------------|
|    | Feet            | In. | Feet     | In. |                                                              |
|    | 25              | 0   | 18       | 9   | Frame to front door.                                         |
|    | 0               | 9   |          |     |                                                              |
|    | 25              | 0   | 29       | 2   | Inside casing to ditto, panneled.                            |
|    | 1               | 2   |          |     |                                                              |
| 9  | 6               | 8   | 200      | 0   | Doors, 6 panneled.                                           |
|    | 3               | 4   |          |     |                                                              |
|    | 10              | 0   | 50       | 0   | Front door.                                                  |
|    | 5               | 0   |          |     |                                                              |
| 7  | 8               | 0   | 224      | 0   | Window shutters on the first<br>floor and stair-case window. |
|    | 4               | 0   |          |     |                                                              |
| 5  | 4               | 0   | 80       | 0   | Ditto to attic windows.                                      |
|    | 4               | 0   |          |     |                                                              |
| 12 | 4               | 6   | 63       | 0   | Soffits to windows, panneled.                                |
|    | 1               | 2   |          |     |                                                              |
| 7  | 22              | 8   | 145      | 5   | Architraves to low windows<br>and stair-case window.         |
|    | 0               | 11  |          |     |                                                              |
|    | 28              | 4   | 28       | 4   | Ditto to front door.                                         |
|    | 1               | 0   |          |     |                                                              |
| 5  | 20              | 8   | 94       | 8   | Ditto to upper windows.                                      |
|    | 0               | 11  |          |     |                                                              |
| 15 | 18              | 9   | 210      | 11  | Ditto to all the doors.                                      |
|    | 0               | 9   |          |     |                                                              |
| 2  | 72              | 0   | 288      | 0   | Plain dados to low rooms.                                    |
|    | 2               | 0   |          |     |                                                              |

7 L

Di-



| Dimensions |             | Contents |     | Titles                                   |
|------------|-------------|----------|-----|------------------------------------------|
| Feet       | In.         | Feet     | In. |                                          |
| 2          | 72 0<br>0 9 | 108 0    |     | Base and fur-base to ditto.              |
| 2          | 72 0<br>0 6 | 72 0     |     | Plain plinth to ditto.                   |
|            | 22 6<br>8 2 | 183 9    |     | Balustrade in the stair-case.            |
|            | 68 0<br>0 3 | 17 0     |     | Base moulding to stair-case.             |
|            | 68 0<br>0 6 | 34 0     |     | Plain plinth to ditto.                   |
| 5          | 6 4<br>3 0  | 95 0     |     | Plain backs and elbows to attic windows. |

*The Abstract of the Carpenters and Joiners Work.*

| Joist-<br>ing | Ceiling<br>joists,<br>&c. | Floor-<br>ing | Stairs | Sashes | Window<br>shutters,<br>&c. | Dados,<br>&c. | 6 pan-<br>neled<br>doors | Archi-<br>traves,<br>&c. |
|---------------|---------------------------|---------------|--------|--------|----------------------------|---------------|--------------------------|--------------------------|
| Feet I        | Feet I                    | Feet I        | Feet I | Feet I | Feet I                     | Feet I        | Feet I                   | Feet I                   |
| 840 0         | 1680 0                    | 840 0         | 66 0   | 272 2  | 250 0                      | 33 4          | 200 0                    | 145 5                    |
| 282 0         | 385 0                     | 822 6         | 6 5    | 152 5  | 29 2                       | 18 9          | 50 0                     | 28 4                     |
|               |                           | 234 0         | 36 0   |        | 224 0                      | 288 0         |                          | 94 8                     |
| 1122 0        | 2065 0                    | 51 4          | 108 0  | 424 7  | 80 0                       | 72 0          | 250 0                    | 210 11                   |
|               |                           | 64 2          | 5 3    |        | 63 0                       | 34 0          |                          | 108 0                    |
|               |                           |               | 5 8    |        |                            | 95 0          |                          | 17 0                     |
|               | 224 0                     | 2012 0        |        |        | 646 2                      |               |                          |                          |
|               | 80 0                      |               | 227 4  |        |                            | 541 1         |                          | 604 4                    |

2) 304 0

152 0 Half of the wind. shut. being work & half.

250 0 Doors, being double work.

402 0 Sum to be charged for workmanship, be-  
sides being charged once for work and  
materials.

*The*



*The Carpenters and Joiners Bill.*

| <i>Sq. Feet</i> | <i>I</i> |                                                                                                           | <i>£ s d</i>               |
|-----------------|----------|-----------------------------------------------------------------------------------------------------------|----------------------------|
| 11              | 22       | o Common joisting, at — a square                                                                          | - - -                      |
| 8               | 57       | 6 Framing the upper floors, at — a sq.                                                                    | - - -                      |
| 180             | o        | Plates over doors and windows, }<br>running measure, at — a foot }                                        | - - -                      |
| 21              | 33       | o Roofing, at — a square —                                                                                | - - -                      |
| 218             | o        | Gutter boarding, at — a foot                                                                              | - - -                      |
| 20              | 65       | o Ceiling joists and stoothed par- }<br>titions, at — a square }                                          | - - -                      |
| 20              | 12       | o Flooring, at — a square —                                                                               | - - -                      |
| 227             | 4        | Steps in the stair-case, at — a foot                                                                      | - - -                      |
| 424             | 7        | Sashes, at — a foot — —                                                                                   | - - -                      |
| 646             | 2        | Window-shutters, soffits, and pan- }<br>neled door-cases, at — a foot }                                   | - - -                      |
| 541             | 1        | Plain door-cases, frame to front }<br>door, plain plinths, and plain }<br>backs and elbows, at — a foot } | - - -                      |
| 250             | o        | Doors, 6 panned, at — a foot                                                                              | - - -                      |
| 604             | 4        | Architraves, &c. at — a foot                                                                              | - - -                      |
| 402             | o        | Workmanship only in doors and }<br>window-shutters }                                                      | - - -                      |
|                 |          |                                                                                                           | <hr/> <i>£</i> - - - <hr/> |

*Note.* All the articles except the last are for both work and materials.

*The SLATERS WORK.*

The Slater's work will be the same with the article of roofing in the carpenter's work, which is 2 133 square feet, or

*Sqrs. Feet*

21 33 at — a square — — *£* - - -  
*Th*



*The* PLASTERERS WORK.

| Dimensions  |              | Contents    |            | Titles                                                                                                         |  |
|-------------|--------------|-------------|------------|----------------------------------------------------------------------------------------------------------------|--|
| <i>Feet</i> | <i>In.</i>   | <i>Feet</i> | <i>In.</i> |                                                                                                                |  |
| 2           | 23 0<br>17 0 | 782 0       |            | Ceil. to the low rooms, 3 coats.                                                                               |  |
| 2           | 83 0<br>1 0  | 166 0       |            | Plain cornice to ditto.                                                                                        |  |
| 8           | 83 0         | 664 0       |            | Enriched mouldings in ditto, running measure.                                                                  |  |
|             | 9 0<br>12 0  | 108 0       |            | Ceiling in the lower part of the stair-case, 3 coats.                                                          |  |
|             | 43 6<br>0 9  | 32 7        |            | Plain cornice in ditto.                                                                                        |  |
|             | 83 0<br>9 0  | 747 0       |            | Walls in one of the low rooms, hard-finish.                                                                    |  |
| 2           | 4 4<br>4 4   | 37 6        |            | Deduct doors from ditto.                                                                                       |  |
| 3           | 8 8<br>5 4   | 138 8       |            | Deduct windows.                                                                                                |  |
|             | 5 0<br>2 0   | 10 0        |            | Deduct part of chimney-piece.                                                                                  |  |
|             |              | 747 0       |            | The walls plastering of the other low rooms, 2 coats, the dimensions and deductions the same as in the former. |  |
|             |              | 186 2       |            | The deductions together.                                                                                       |  |
|             | 22 2<br>11 6 | 254 11      |            | Walls of the stair-case, hard-finish.                                                                          |  |

Di-



| Dimen-<br>sions | Contents        | Titles                                       |
|-----------------|-----------------|----------------------------------------------|
| <i>Feet In.</i> | <i>Feet In.</i> |                                              |
| 7 6<br>3 0      | 22 6            | Walls of the stair-case, hard-<br>finishing. |
| 7 6<br>9 6      | 71 3            | Ditto.                                       |
| 6 0<br>5 9      | 34 6            | Ditto.                                       |
| 17 9<br>7 0     | 124 3           | Ditto.                                       |
| 67 0<br>8 6     | 569 6           | Walls of upper part, ditto.                  |
| 10 0<br>6 8     | 66 8            | Deduct front door from ditto.                |
| 6 0<br>4 0      | 168 0           | Ditto other doors.                           |
| 8 8<br>5 4      | 46 2            | Ditto window.                                |
| 7 4<br>5 4      | 39 1            | Ditto attic window.                          |
| 23 0<br>8 9     | 201 3           | Stair-case ceiling, 3 coats.                 |
| 67 8<br>1 0     | 67 8            | Plain cornice to ditto.                      |



|   | Dimenfi-<br>ons |         | Contents | Titles                           |
|---|-----------------|---------|----------|----------------------------------|
|   | Feet            | In.     | Feet In. |                                  |
| 2 | 17<br>14        | 0<br>6  | 493 0    | Ceilings of upper rooms, 3 coats |
| 2 | 17<br>7         | 0<br>3  | 246 6    | Ditto.                           |
| 2 | 65<br>0         | 0<br>10 | 108 4    | Plain cornices.                  |
| 2 | 50<br>0         | 6<br>10 | 84 2     | Ditto.                           |
| 4 | 17<br>8         | 6<br>6  | 595 0    | Partitions, 2 coats on laths.    |
| 2 | 47<br>8         | 6<br>6  | 807 0    | Ditto walls, 2 coats.            |
| 2 | 33<br>8         | 6<br>6  | 569 6    | Ditto.                           |
| 4 | 7<br>5          | 4<br>4  | 156 5    | Deduct windows from ditto.       |
| 4 | 6<br>4          | 0<br>4  | 104 0    | Deduct doors.                    |
| 2 | 4<br>4          | 6<br>0  | 36 0     | Deduct chimney-pieces.           |
|   | 4<br>3          | 0<br>4  | 13 4     | Ditto.                           |

The



*The Abstract of the Plastering.*

| Ceiling       |               | Rendering     |               | Deductions    |               | Plain         | Enrich'd      |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 3 Coats       | 2 Coats       | Hard-finish   | 2 Coats       | Hard-finish   | 2 Coats Walls | Cornices      | Mouldings     |
| <i>Feet I</i> | <i>Feet I</i> | <i>Feet I</i> | <i>Feet I</i> | <i>Feet I</i> | <i>Feet I</i> | <i>Feet I</i> | <i>Feet I</i> |
| 782 0         | 9) 595 0      | 747 0         | 747 0         | 37 6          | 186 2         | 166 0         | 664 0         |
| 108 0         |               | 254 0         | 807 0         | 138 8         | 156 5         | 32 7          |               |
| 201 3         | <i>Y F</i>    | 22 6          | 569 0         | 10 0          | 104 0         | 67 8          |               |
| 493 0         | 66 1          | 71 3          |               | 66 8          | 36 0          | 108 4         |               |
| 246 6         |               | 34 6          | 2123 0        | 168 0         | 13 4          | 84 2          |               |
|               |               | 124 3         | 495 11        | 46 2          |               |               |               |
| 9) 1830 9     |               | 569 6         | 9) 1627 1     | 39 1          | 495 11        | 458 9         |               |
| <i>Y F I</i>  |               | 1823 0        | <i>Y F I</i>  | 506 1         |               |               |               |
| 203 3 9       |               | 506 1         | 180 7 1       |               |               |               |               |
|               |               | 9) 1316 11    |               |               |               |               |               |
|               |               | <i>Y F I</i>  |               |               |               |               |               |
|               |               | 146 2 11      |               |               |               |               |               |

*The Plasterers Bill.*

| <i>Yds</i> | <i>F</i> | <i>I</i> |                                     | <i>£ s d</i>   |
|------------|----------|----------|-------------------------------------|----------------|
| 203        | 3        | 9        | Ceiling, 3 coats, at — a yard       | — — —          |
| 66         | 1        | 0        | Ditto, 2 coats, at — a yard         | — — —          |
| 146        | 2        | 11       | Plastering, hard-finish, at — a yd. | — — —          |
| 180        | 7        | 1        | Ditto, 2 coats, at — a yard         | — — —          |
| 458        | 9        |          | Plain cornice, at — a foot          | — — —          |
| 664        | 0        |          | Enriched mouldings, at — a foot     | — — —          |
|            |          |          |                                     | <i>£</i> — — — |

I think it of no consequence to pursue these examples farther, viz. to the painting, glazing, &c. what hath been done being, I think, sufficient to illustrate the whole of this branch; and any more of



of it would only enlarge the book without any real use. I shall now deliver a short but compleat tract of timber measuring, with which I shall conclude these applications, as proposed.

The annexed plate contains the elevation and plans of the house to which belong all the foregoing examples.

## SECTION IV.

### *Timber Measuring.*

#### PROBLEM I.

*To find the area or superficial feet in a board or plank.*

#### RULE.

**M**ULTIPLY the length by the breadth for the content.

*Note.* If the board be not tapering, but equally thick at both ends, the breadth may be taken in any part of it; but if the board taper from the one end to the other, the buyer hath it in his option to take the breadth either in the middle or in any part between the middle and the broader end; or the breadths at the two ends may be added together, and half their sum taken; any of these is called the mean breadth.

#### *By the Sliding Rule.*

$$\begin{array}{ccccccc} B & & A & & B & & A \\ 12 : \text{Breadth in inches} :: \text{Length in feet} : \text{Cont. in feet.} \end{array}$$

That is, set 12 on *B* to the breadth in inches on *A*, then against the Length in feet on *B* is the content on *A*, in feet and fractional parts.

Ex-



## EXAMPLE I.

To how much, at  $1\frac{1}{2}d.$  a foot, amounts a board whose length is 12 feet 6 inches, and breadth throughout 11 inches.

$$\begin{array}{r}
 F \quad I \\
 12 \quad 6 \\
 \quad \quad 0 \quad 11 \\
 d \quad \quad \quad P \\
 1\frac{1}{2} = \frac{1}{8}) \quad 11 \quad 5 \quad 6 \\
 \hline
 I \quad P \quad 1s. \quad 4\frac{1}{2}d. \\
 5 \quad 6 = 0 \quad 0\frac{1}{2} \\
 \hline
 1s. \quad 5d. \text{ Answer.}
 \end{array}$$

*By the Sliding Rule.*

$$\begin{array}{cccc}
 B & A & B & A \\
 12 & : & 11 & :: 12\frac{1}{2} : 11\frac{1}{2}.
 \end{array}$$

## EXAMPLE II.

To find the value of 5 oaken planks, at  $3d.$  a foot, each  $17\frac{1}{2}$  feet long, and whose breadths are two of them each  $13\frac{1}{2}$  inches in the middle, two others each 18 inches at the broader end and  $11\frac{1}{4}$  at the narrower, and the fifth measures  $14\frac{1}{4}$  inches at the narrowest part between the middle and the broader end.

$$\begin{array}{r}
 13\frac{1}{2} \\
 13\frac{1}{2} \\
 11\frac{1}{4} \\
 18 \\
 14\frac{1}{4} \\
 \hline
 70\frac{1}{2} = 5 \quad 10\frac{1}{2} \text{ sum of breadths}
 \end{array}$$

7 N F I



|          |                 |                 |
|----------|-----------------|-----------------|
| <i>F</i> | <i>I</i>        |                 |
| 17       | 6               | common length   |
| 5        | $10\frac{1}{2}$ | sum of breadths |

|                                |    |   |          |
|--------------------------------|----|---|----------|
| $6 = \frac{1}{2} =$            | 87 | 6 |          |
| $3 = \frac{1}{2} =$            | 8  | 9 | <i>P</i> |
| $1\frac{1}{2} = \frac{1}{2} =$ | 4  | 4 | 6        |
|                                | 2  | 2 | 3        |

*d* 102 9 9 Content.

$3 = \frac{1}{4} =$  25s. 6d.

*I*

$6 = \frac{1}{2} =$  0  $1\frac{1}{2}$

$3 = \frac{1}{2} =$  0  $0\frac{3}{4}$

£ 1 5 8 $\frac{1}{4}$  Answer.

*By the Sliding Rule.*

|          |                 |                 |          |
|----------|-----------------|-----------------|----------|
| <i>B</i> | <i>A</i>        | <i>B</i>        | <i>A</i> |
| 12       | $70\frac{1}{2}$ | $17\frac{1}{2}$ | 102.     |

#### PROBLEM II.

*To find the solidity of squared or four-sided timber, or barks.*

#### RULE.

Multiply the mean breadth by the mean thickness, then the product multiplied by the length will give the solidity.

*Note 1.* If the balk be equally broad and thick throughout, the breadth and thickness any where taken are the mean breadth and thickness. And, in this case, if the breadth and thickness be also equal to each other, either of them is the quarter girt, which multiplied by itself, and the product multiplied by the length will give the solidity.

*Note 2.* But if the piece taper from the one end to the other, which is generally the case, the breadth and thickness taken in the middle will be the mean breadth and thickness; or half the sum of the



the extream breadths, or breadths taken at the two ends, will be the mean breadth, and half the sum of the extream thickneses will be the mean thickness. And if the two ends be squares, a side of each being added together, half the sum will be the side of a mean square, or quarter girt, to be used as in the former note. And if the piece do not taper regularly, but be diversly thick in some parts and small in others; either take the dimensions where they may be judged at about a mean; or else take several different dimensions, add them all together, and divide their sum by the number of them for the mean dimension.

*By the Sliding Rule.*

$\begin{array}{ccccccc} C & & D & & D & & C \\ \text{Length : } 12 & \text{or } 10 & :: & \text{quarter girt : } & \text{solidity.} \end{array}$

That is, set the length (in feet) on *C* to 12 or 10 on *D*, then against the quarter girt (in inches or in tenths of a foot) on *D* is the solidity on *C*.

*Note.* By the quarter girt above mentioned is not meant the 4th part of the circumference, but is the side of the mean square; and which, if the ends of the piece be squares, will be half the sum of the extream sides when the piece tapers, or the breadth any where taken when it does not taper; but if the breadths and thickneses be unequal, find the mean breadth and thickness as in the 2d note above, and take a mean proportional between them for the mean quarter girt, or side of a mean square. This mean proportional between them is not half their sum, but is the square root of their product; and is found upon the rule thus, as mentioned in page 549: Having set 16 on *C* to 4 on *D*, or the middle 1 on *C* to 10 on *D*, against the product of the mean breadth and thickness on *C* is its square root, or their mean proportional, on *D*. When the mean breadth and thickness are nearly equal to each other, half their sum may be used instead of the square root of their product without much error; but if the breadth differ much from the thickness, half their sum must by no means be used then, because it would produce a solidity exceedingly different from the truth, as is shewn at the end of the following examples.

#### EXAMPLE I.

Required the solidity of a piece of timber whose length is  $24\frac{5}{10}$  feet, and its ends equal squares whose sides are each 1 foot and  $\frac{4}{100}$  or 4 centesims or hundredth parts.

By the table on page 551, the  $\frac{4}{100}$  of a foot, are equal to  $\frac{1}{12}$  or  $\frac{1}{2}$  of an inch.

*De-*



*Decimally**Feet*

1'04

1'04

---

416

104

---

1'0816

24'5

---

54080

43264

---

21632

---

26'49920

— Solidity —

*Duodecimally**F**I*

1

0  $\frac{1}{2}$ 

1

0  $\frac{1}{2}$ 

---

10  $\frac{1}{2}$ 

0

0  $\frac{1}{2}$ 

---

1

1

24

6

---

26

0

0

6

---

26

6

*By the Sliding Rule.**C**D**D**C*As  $24\frac{1}{2} : 12$  or  $10 :: 12\frac{1}{2}$  or  $10\frac{1}{4} : 26\frac{1}{2}$  feet, the content.

EXAMPLE II.

If the length be 20 feet 4 inches and  $\frac{5}{8}$ ; and the ends unequal squares, the side of the greater being 19 inches and  $\frac{1}{8}$ , and the side of the less 9 and  $\frac{7}{8}$ ; required the solidity.

By the table, page 551, we find 4 in. = 33 centesms

$$\frac{5}{8} \text{ — } = 5 \text{ — }$$

$$19\frac{1}{8}$$

$$9\frac{7}{8}$$

$$\text{—}$$

$$2) 29$$

$$14\frac{4}{8}$$

$$\text{or } 14\frac{1}{2} = 1 \quad 2\frac{4}{8} = 1\frac{2}{4} = 1\frac{1}{2} = \text{the side of the}$$

mean square.

De-



| <i>Decimally</i> |              | <i>Duodecimally</i> |
|------------------|--------------|---------------------|
| <i>Feet</i>      |              | <i>F I</i>          |
| 1.21             |              | 1 2 $\frac{1}{2}$   |
| 1.21             |              | 1 2 $\frac{1}{2}$   |
| <hr/>            |              | <hr/>               |
| 121              |              | 1 2 $\frac{1}{2}$   |
| 242              |              | 0 2 $\frac{1}{2}$   |
| 121              |              | 0 0 $\frac{1}{2}$   |
| <hr/>            |              | <hr/>               |
| 1.4641           |              | 1 5 $\frac{1}{2}$   |
| 20.38            |              | 20 4 $\frac{5}{8}$  |
| <hr/>            |              | <hr/>               |
| 117128           |              | 29 2                |
| 43923            |              | 5 $\frac{3}{4}$     |
| 29282            |              | 1                   |
| <hr/>            |              | <hr/>               |
| 29.838358        | - Solidity - | 29 8 $\frac{3}{4}$  |

*By the Sliding Rule.*

As  $20.38 : 12 :: 14\frac{1}{2} : 29\frac{3}{4}$  the solidity.

EXAMPLE III.

The length being 27 feet and  $\frac{3.6}{100}$ ; at the greater end, the breadth 1 foot  $\frac{7.8}{100}$ , thickness  $1\frac{2.3}{100}$  foot; and at the less end, the breadth  $1\frac{4}{100}$ , and thickness  $\frac{2.1}{100}$ ; required the solidity.

|                      |                        |
|----------------------|------------------------|
| 1.78 greater breadth | 1.23 greater thickness |
| 1.04 less breadth    | 0.91 less thickness    |
| <hr/>                | <hr/>                  |
| 2) 2.82              | 2) 2.14                |
| <hr/>                | <hr/>                  |
| 1.41 mean breadth    | 1.07 mean thickness    |
| <hr/>                | <hr/>                  |

70

De-



| <i>Decimally</i> |             | <i>Duodecimally</i> |
|------------------|-------------|---------------------|
| <i>Feet</i>      |             | <i>F I P</i>        |
| 1'41             |             | 1 4 11              |
| 1'07             |             | 1 0 10              |
| <hr/>            |             | <hr/>               |
| 987              |             | 1 4 11              |
| 141              |             | 0 1 2               |
| <hr/>            |             | <hr/>               |
| 1'5087           |             | 1 6 1               |
| 27'36            |             | 27 4 4              |
| <hr/>            |             | <hr/>               |
| 90522            |             | 40 8 3              |
| 45261            |             | 0 6 0               |
| 105609           |             | 0 0 6               |
| 30174            |             | <hr/>               |
| <hr/>            |             | <hr/>               |
| 41'278032        | — content — | 41 2 9              |

*By the Sliding Rule.*

As  $\overset{B}{1} : \overset{A}{16\frac{1}{2}} :: \overset{B}{12\frac{1}{2}} : \overset{A}{217}$  the mean square.

As  $\overset{C}{16} : \overset{D}{4} :: \overset{C}{1'5} : \overset{D}{14'7}$  the side of the mean square.

As  $\overset{C}{27'36} : \overset{D}{12} :: \overset{C}{14'7} : \overset{D}{41\frac{1}{4}}$  feet, the content.

SCHOLIUM.

In measuring squared timber, unskilful measurers usually take  $\frac{1}{4}$  of the circumference or girt for the side of a mean square; which quarter girt therefore multiplied by itself, and the product multiplied by the length, they account the solidity or content; but it is really always above the truth. Indeed when the breadth and thickness are nearly equal, this method will give the solidity pretty near the truth; but if the breadth and thickness differ considerably from each other, the error will be so great as that it ought not by any means to be neglected.

Thus, suppose we take, for an example, a balk of 24 feet long, and a foot square throughout; and consequently its solidity 24 cubic feet. Now if this balk be slit exactly in-two from end to end, making each



each piece 6 inches broad and 12 inches thick; it is evident that the true solidity of each will be 12 feet; but by the quarter-girt method they would amount to much more; for the quarter girt, being equal to half the sum of the breadth and thickness, in this case will be 9 inches, the square of which is 81, which being divided by 144, and the quotient multiplied by 24 the length, we obtain  $13\frac{1}{2}$  feet for the solidity of each part; and consequently the two solidities together make 27 feet instead of 24.—Again, suppose the balk to be so cut as that the breadth of the one piece may be only 4 inches, and consequently that of the other 8 inches. Here the true content of

the less piece will be  $\left(\frac{24 \times 4}{12} =\right)$  8 feet, and that of the greater 16

feet. But, proceeding by the other method, the quarter girt of the less piece will be 8, whose square 64 multiplied by 24, and the product divided by 144 gives  $10\frac{2}{3}$  feet instead of 8. And, by the same method, the content of the greater piece will be  $16\frac{2}{3}$  feet instead of 16. And the sum of both is  $27\frac{1}{3}$  feet instead of 24.—Farther, if

the less piece be cut only 2 inches broad, and consequently the greater

10 inches; the true content of the less piece would be  $\left(\frac{24 \times 2}{12} =\right)$

4 feet, and that of the greater 20. But, by the other method, the quarter girt of the less piece would be 7 inches, whose square 49 being

divided by 6  $\left(= \frac{144}{24}\right)$  gives  $8\frac{1}{8}$  feet instead of 4 for the content. And by the same method the content of the greater piece would be  $20\frac{1}{8}$  instead of 20 feet. So that their sum would be  $28\frac{1}{8}$  instead of 24 feet.

Hence it is evident, that the greater the proportion is between the breadth and depth, the greater will the error be by using the false method; that the sum of the two parts, by the same method, is greater the greater the difference of the same two parts is, and consequently the sum is least when the two parts are equal to each other, or when the balk is cut equally in-two; and lastly, that when the sides of a balk differ not above an inch or two from each other, the quarter-girt method may then be used without inducing an error that will be of any material consequence.

### P R O B L E M III.

*To find the solidity of round timber or unsquared trees.*

#### R U L E I.

Multiply the square of  $\frac{1}{5}$  of the girt or circumference by 2 times the length, and the product will be the content (extremely near the truth.)

B



*By the Sliding Rule.*

$C \qquad D \qquad D \qquad C$

Twice Length : 12 or 10 ::  $\frac{1}{5}$  girt : content.

That is, set twice the length (in feet) on  $C$  to 12 or 10 on  $D$ , then against  $\frac{1}{5}$  of the girt (in inches or in tenths of a foot) on  $D$  is the content on  $C$ .

### R U L E II.

Multiply the square of the quarter girt (or  $\frac{1}{4}$  of the circumference) by the length, and the product will be the content (but not near the truth).

*By the Sliding Rule.*

$C \qquad D \qquad D \qquad C$

Length :  $\left\{ \begin{smallmatrix} 12 \\ 10 \end{smallmatrix} \right\} :: \frac{1}{4}$  girt : content.

*Note 1.* The second rule is that which is commonly used, although it differs from the truth about  $\frac{1}{4}$  of its content; that is, when this rule produces 4 for the content, it should be above 5, and so in proportion. But the first rule, which is a new one, is above 50 times nearer to the truth than the other, it differing from the truth by only about 1 foot in 190\*; and as it is full as easy in practice as the latter, both by the pen and by the sliding rule, and hath in every other respect the advantage of it, it ought to be brought into general use by the measurers of timber, who should certainly prefer truth to such gross errors as are always introduced by the other method.

*Note 2.* If the tree be not equally thick throughout, but taper from the one end to the other, which is generally the case, the girt may

---

\* Putting  $c$  for the circumference or girt, and  $l$  for the length; the true content is known to be  $\frac{ccl}{4 \times 3.1416} = \frac{ccl}{12.5664}$ . Now the first rule is  $\frac{2ccl}{25} = \frac{ccl}{12.5}$ , which is but about the 190th part distant from the truth. But the latter rule  $\frac{ccl}{16}$  is about a fourth part from the truth.



may be taken in the middle for the mean girt; but the buyer generally hath it in his option to girt it either in the middle or any where between the middle and the thicker end, because of the inequalities that may happen there. Or half the sum of the girts of the two ends may be taken for the mean girt; or else girt it in several parts, and divide the sum of the girts by the number of them, for the mean girt.

*Note 3.* That part of a tree which is less than 2 feet in circumference, or 6 inches quarter girt, is cut off; it not being accounted timber. And as so much of the tree as measures 2 feet in girt is accounted timber, so also as much of the branches as measures 2 feet in girt is accounted and measured as timber.

*Note 4.* When trees, having their bark on, are measured for sale, an allowance is made for the bark, by deducting from the girt so much as may be judged sufficient to reduce it to such girt as the tree would have without its bark. In oak this allowance is about  $\frac{1}{10}$  or  $\frac{1}{12}$  of the girt; but in other wood, whose bark is not so thick, the deduction ought to be less.

*Note 5.* Fifty cubic feet of timber make a load; and therefore to reduce feet to loads, divide them by 50.

## EXAMPLE I.

If the length of a tree be 24 feet, and the girt throughout 8 feet, required the content.

| <i>True Method</i>          |
|-----------------------------|
| $1.6 = \frac{1}{5}$ of girt |
| 1.6                         |
| —                           |
| 96                          |
| 16                          |
| —                           |
| 2.56 square                 |
| 48 double length            |
| —                           |
| 2048                        |
| 1024                        |
| —                           |
| 122.88 content              |

| <i>False Method</i>            |
|--------------------------------|
| 2 quarter girt                 |
| 2                              |
| —                              |
| 4 square                       |
| 24 length                      |
| —                              |
| 96 content, 26 feet too little |



*By the Sliding Rule.*

1st. As  $\begin{matrix} C & D & D & C \\ \{ 48 : 10 :: 16 : 123 \} \\ \{ 48 : 12 :: 19\frac{1}{5} : 123 \} \end{matrix}$  true content.

2d. As  $\begin{matrix} \{ 24 : 10 :: 20 : 96 \} \\ \{ 24 : 12 :: 24 : 96 \} \end{matrix}$  false content.

### EXAMPLE II.

If a tree girt 14 feet at the thicker end and 2 feet at the smaller, required the content, the length being 24 feet.

Here  $\frac{14+2}{2} = \frac{16}{2} = 8 =$  the mean girt, the same as in the example above, and therefore the content will likewise be the same.

### EXAMPLE III.

Required the content of a tree whose girt in the middle is 3 feet and  $\frac{15}{100}$  or 3.15 feet, and its length 14 $\frac{1}{2}$  feet.

*True Method*

•63 =  $\frac{1}{5}$  girt

•63

189

378

•3969 square

29 = twice length

35721

7938

11.5101

So that the true content is above 11 $\frac{1}{2}$  feet, and the false not quite 9 feet.

*False method*

•7875 = quarter girt

5787 inverted

5513

630

55

4

•6202 square

14 $\frac{1}{2}$  length

24808

6202

3101

8.9929

*By*



*By the Sliding Rule.*

$$\text{1ft. As } \left\{ \begin{array}{l} C \quad D \quad D \quad C \\ 29 : 10 :: 6.3 : 11\frac{1}{2} \\ 29 : 12 :: 7.56 : 11\frac{1}{2} \end{array} \right\} \text{ true content.}$$

$$\text{2d. As } \left\{ \begin{array}{l} 14\frac{1}{2} : 10 :: 7.875 : 9 \\ 14\frac{1}{2} : 12 :: 9.45 : 9 \end{array} \right\} \text{ false content.}$$

## EXAMPLE IV.

Required the content of a tree whose length is  $17\frac{1}{4}$  feet, and girts in five places, viz. in the first place 9.43 feet, in the second 7.92, in the third 6.15, in the fourth 4.74, and in the fifth 3.16 feet.

$$\begin{array}{r} 9.43 \\ 7.92 \\ 6.15 \\ 4.74 \\ 3.16 \\ \hline 5) 31.40 \text{ sum of the girts} \\ \hline 6.28 \text{ mean girt} \end{array}$$

*True Method*

$1.256 = \frac{1}{5} \text{ girt.}$

$6521 \text{ inverted}$

$1256$

$251$

$63$

$7$

$1.577 \text{ square}$

$34\frac{1}{2} \text{ twice length}$

$6308$

$4731$

$7885$

$54.4065 \text{ true content}$

*False Method*

$1.57 = \frac{1}{4} \text{ girt}$

$751 \text{ inverted}$

$157$

$79$

$11$

$2.47 \text{ square}$

$17\frac{1}{4} \text{ length}$

$1729$

$247$

$6175$

$42.6075 \text{ false content}$

*By*



*By the Sliding Rule.*

$$\begin{array}{l}
 \text{1ft. } \left\{ \begin{array}{l} C \quad D : D : C \\ 34\frac{1}{2} : 10 :: 12\cdot56 : 54\cdot4 \\ 34\frac{1}{2} : 12 :: 15\cdot07 : 54\cdot4 \end{array} \right\} \text{ true content.} \\
 \text{2d. } \left\{ \begin{array}{l} 17\frac{1}{4} : 10 :: 15\cdot70 : 42\cdot6 \\ 17\frac{1}{4} : 12 :: 18\cdot84 : 42\cdot6 \end{array} \right\} \text{ false content.}
 \end{array}$$

### EXAMPLE V.

Sold an oak tree whose girt at the low end was  $9\frac{1}{4}$  feet, and its length, to the part where it becomes of 2 feet girt, is  $27\frac{1}{2}$  feet; it hath also two boughs, the girt at the thicker end of the one is 4·3, and at the thicker end of the other 3·94, the length of the timber of the former, that is to the part where it becomes of only 2 feet girt, is 9 feet; and the length of the latter  $7\frac{3}{4}$  feet: required the price at 2*l.* 3*s.* 9*d.* a load of 50 feet, allowing  $\frac{1}{10}$  of the mean girt for the bark of the trunk, and  $\frac{1}{12}$  of the same in the boughs.

|                                             |                 |                                      |
|---------------------------------------------|-----------------|--------------------------------------|
| $9\frac{1}{4}$                              | 4·3             | 3·94                                 |
| <u>2</u>                                    | <u>2</u>        | <u>2</u>                             |
| 2) $11\frac{1}{4}$                          | 2) 6·3          | 2) 5·94                              |
| <u>10) 5·625</u>                            | <u>12) 3·15</u> | <u>12) 2·97</u> mean girts           |
| ·562                                        | ·262            | ·247 deduct for bark                 |
| <u>5·063</u>                                | <u>2·888</u>    | <u>2·723</u>                         |
| $\frac{1}{5} = 1\cdot012$                   | 0·577           | 0·544 the $\frac{1}{5}$ of the girts |
| <u><math>\frac{1}{4} = 1\cdot266</math></u> | <u>0·722</u>    | <u>0·681</u> the quart. girts        |

*By*



*By the true Method*

1'012 =  $\frac{1}{2}$  girt of trunk  
2101 inverted

1012

10

2

1'024 square

55 double length

5120

5120

56'320 content of trunk.

577 =  $\frac{1}{4}$  girt of 1st bough

775 inverted

2885

404

40

3329 square

18 double length

26632

3329

5'9922 content of 1st bough.

544 =  $\frac{1}{4}$  girt of 2d bough

445 inverted

2720

217

22

2959 square

15 $\frac{1}{2}$  double length

14795

2959

1479

4'5864 content of 2d bough.

56'3200 trunk

5'9922 first bough

4'5864 second bough

66'8986 sum

or 1 load 16 feet and  $\frac{9}{16}$  nearly

£ 2 18 6 true value.

*By the false Method*

1'266 =  $\frac{1}{4}$  girt of trunk

6621 inverted

1266

253

76

7

1'602 square

27 $\frac{1}{2}$  length

11214

3204

801

44'055 content of trunk.

722 =  $\frac{1}{4}$  girt of 1st bough

227 inverted

5054

144

14

5212 square

9 length

4'6908 content of 1st bough.

681 =  $\frac{1}{4}$  girt of 2d bough

186 inverted

4086

545

7

4638 square

7 $\frac{3}{4}$  length

32466

2319

1159

3'5944 content of 2d bough.

44'0550 trunk

4'6908 first bough

3'5944 second bough

52'3402 sum

or 1 load 2 feet and  $\frac{1}{4}$  nearly

£ 2 5 9 $\frac{1}{2}$  false value.



*By the Sliding Rule.*

| <i>C</i> | <i>D</i> | <i>D</i> | <i>C</i> | <i>C</i> | <i>D</i> | <i>D</i> | <i>C</i> |
|----------|----------|----------|----------|----------|----------|----------|----------|
| 55       | : 10 ::  | 10       | : 56.3   | 27½      | : 10 ::  | 12.66    | : 44.0   |
| 18       | : 10 ::  | 5.77     | : 6.0    | 9        | : 10 ::  | 7.22     | : 4.7    |
| 15½      | : 10 ::  | 5.44     | : 4.6    | 7¾       | : 10 ::  | 6.81     | : 3.6    |
| sum 66.9 |          |          |          | sum 52.3 |          |          |          |

## SCHOLIUM.

From the preceding examples it appears that this new method, which is extremely near the truth, is full as easy in practice as the common false one, and that therefore it ought to be used, since the ease of the false method is the only argument generally alledged for using it. But there are many other reasons for changing the method, and one in particular is the preventing of the fellers from playing any tricks with their timber by cutting trees into different lengths, so as to make them measure to more than the whole did; for, by the false method, this may be done in many respects, as will appear in the three following problems, which contain the chief cases of this artifice, but which however I do not explain to teach them to use these means.

## PROBLEM IV.

*To find where a piece of round tapering timber must be cut, so that the two parts, measured separately, according to the common method of measuring, shall produce a greater solidity than when cut in any other part, and greater than the whole.*

## RULE.\*

Cut it through exactly in the middle, or at  $\frac{1}{2}$  of the

## \* DEMONSTRATION.

Put  $G$  = the greatest girt,  $g$  = the least, and  $x$  = the girt at the section; also  $L$  = the whole length, and  $z$  = the length to be cut off the less end.

Then, by similar figures,  $L : z :: G - g : x - g$ , hence  $x = \frac{Gz - gz}{L} + g$ . But  $\frac{Gz - gz}{L} + g$  is a maximum; whose fluxion being put equal to nothing, and the value of  $x$  substituted instead of it, there results  $z = \frac{1}{2}L$ . Q.E.D.



the length, and the two parts will measure to the most possible by the common method.

E X A M P L E.

Supposing a tree to girt 14 feet at the greater end, 2 feet at the less, and consequently 8 feet in the middle; and that the length is 32 feet.

Then, by the common method, the whole tree measures to only 128 feet; but when cut through the middle, the greater part measures to 121, and the less part to 25 feet; whose sum is 146 feet; which exceeds the whole by 18 feet, and is the most that it can be made to measure to by cutting it into two parts.

P R O B L E M   V.

*To find where a tree must be cut, so that the part next the greater end may measure to the most possible.*

R U L E.\*

From the greater girt take 3 times the less; then as the difference of the girts is to the remainder, so is  $\frac{1}{3}$  of the whole length to the length from the less end to be cut off.

Or, cut it where the girt is  $\frac{1}{3}$  of the greatest girt.

*Note.*

\* D E M O N S T R A T I O N.

Using the same notation as in the last demonstration; we shall have here also  $x = \frac{Gz - gz}{L} + g$ , and  $G + x$  .  $L - x =$  a maximum;

which fluxed &c. as before, gives  $z = \frac{G - 3g}{G - g} \times \frac{1}{3} L$ . And  $x =$

$\frac{G - g}{L} z + g =$  (by substituting the above value of  $z$ )  $\frac{1}{3} G$ . Q.E.D.



*Note.* If the greatest girth do not exceed 3 times the least, the tree cannot be cut as is required by this problem. For, when the least girth is exactly equal to  $\frac{1}{3}$  of the greater, the tree already measures to the greatest possible; that is, none can be cut off, (nor indeed added to it, continuing the same taper) that the remainder (or sum) may measure to so much as the whole: And when the least girth exceeds  $\frac{1}{3}$  of the greater, the result by the rule shews how much in length must be *added*, that the result may measure to the most possible.

## E X A M P L E.

Taking here the same example as before, we shall have as  $12 : 8 :: \frac{3}{8}^2 : 7\frac{1}{9} =$  the length to be cut off, and consequently the length of the remaining part is  $24\frac{8}{9}$ , also  $\frac{1}{3}^4 = 4\frac{2}{3} =$  the girth at the section. Hence the content of the remaining part is  $135\frac{4}{9}$  feet; whereas the whole tree, by the same method, measures to but 128 feet.

## P R O B L E M VI.

*To cut a tree so as that the part next the greater end may measure, by the common method, to exactly the same quantity as the whole measures to.*

## R U L E.\*

Call the sum of the girths of the two ends  $s$ , and their difference  $d$ . Then multiply  $d$  by the sum of  $d$  and  $4s$ , and from the root of the product take the difference between  $d$  and  $2s$ ; then as  $2d$  is to the re-

## \* DEMONSTRATION.

Using still the same notation, we shall have  $s^2 L = L - z \cdot \overline{G+x}^2$ ; hence, instead of  $x$  substituting its value  $\frac{dz}{L} + g$ , we obtain  $z =$

$$\frac{L}{2d} \times \sqrt{4s+d \cdot d - 2s+d}. \text{ And hence } x = \frac{\sqrt{4s+d \cdot d - 2s+d}}{2} = \text{Q.E.D.}$$



remainder, so is the whole length to the length to be cut off the small end.

And if  $s$  be taken from the said root, half the remainder will be the girt at the section.

## EXAMPLE.

Using still the same example, we have  $s = 16$ ,  $d = 12$ , and the length  $L = 32$ ; hence

$$\frac{L}{2d} \times \sqrt{4s + d \cdot d - 2s + d} = \frac{32}{24} \times \sqrt{76 \times 12 - 20} = \frac{16}{3} \sqrt{57 - 26\frac{2}{3}} = 13.599118 = \text{the length to be cut off, and consequently the length of the remaining}$$

part is  $18.400882$ ; also  $\frac{1}{2} \sqrt{4s + d \cdot d - \frac{1}{2}s} = 2\sqrt{57 - 8} = 7.099669$  is the girt at the section. Hence the

girt in the middle of the greater part is  $\frac{14 + 7.099669}{2} = 10.549834$ , whose  $\frac{1}{4}$ th part is  $2.637458$ , and consequently the content of the same part is  $2.637458^2 \times 18.400882 = 128$  the very same as the whole tree measures to, notwithstanding above  $\frac{1}{3}$  part is cut off the length.

|     |     |     |     |     |     |     |     |     |      |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
| 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10   |
| 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20   |
| 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30   |
| 31  | 32  | 33  | 34  | 35  | 36  | 37  | 38  | 39  | 40   |
| 41  | 42  | 43  | 44  | 45  | 46  | 47  | 48  | 49  | 50   |
| 51  | 52  | 53  | 54  | 55  | 56  | 57  | 58  | 59  | 60   |
| 61  | 62  | 63  | 64  | 65  | 66  | 67  | 68  | 69  | 70   |
| 71  | 72  | 73  | 74  | 75  | 76  | 77  | 78  | 79  | 80   |
| 81  | 82  | 83  | 84  | 85  | 86  | 87  | 88  | 89  | 90   |
| 91  | 92  | 93  | 94  | 95  | 96  | 97  | 98  | 99  | 100  |
| 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110  |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120  |
| 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130  |
| 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140  |
| 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | 149 | 150  |
| 151 | 152 | 153 | 154 | 155 | 156 | 157 | 158 | 159 | 160  |
| 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 | 170  |
| 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180  |
| 181 | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190  |
| 191 | 192 | 193 | 194 | 195 | 196 | 197 | 198 | 199 | 200  |
| 201 | 202 | 203 | 204 | 205 | 206 | 207 | 208 | 209 | 210  |
| 211 | 212 | 213 | 214 | 215 | 216 | 217 | 218 | 219 | 220  |
| 221 | 222 | 223 | 224 | 225 | 226 | 227 | 228 | 229 | 230  |
| 231 | 232 | 233 | 234 | 235 | 236 | 237 | 238 | 239 | 240  |
| 241 | 242 | 243 | 244 | 245 | 246 | 247 | 248 | 249 | 250  |
| 251 | 252 | 253 | 254 | 255 | 256 | 257 | 258 | 259 | 260  |
| 261 | 262 | 263 | 264 | 265 | 266 | 267 | 268 | 269 | 270  |
| 271 | 272 | 273 | 274 | 275 | 276 | 277 | 278 | 279 | 280  |
| 281 | 282 | 283 | 284 | 285 | 286 | 287 | 288 | 289 | 290  |
| 291 | 292 | 293 | 294 | 295 | 296 | 297 | 298 | 299 | 300  |
| 301 | 302 | 303 | 304 | 305 | 306 | 307 | 308 | 309 | 310  |
| 311 | 312 | 313 | 314 | 315 | 316 | 317 | 318 | 319 | 320  |
| 321 | 322 | 323 | 324 | 325 | 326 | 327 | 328 | 329 | 330  |
| 331 | 332 | 333 | 334 | 335 | 336 | 337 | 338 | 339 | 340  |
| 341 | 342 | 343 | 344 | 345 | 346 | 347 | 348 | 349 | 350  |
| 351 | 352 | 353 | 354 | 355 | 356 | 357 | 358 | 359 | 360  |
| 361 | 362 | 363 | 364 | 365 | 366 | 367 | 368 | 369 | 370  |
| 371 | 372 | 373 | 374 | 375 | 376 | 377 | 378 | 379 | 380  |
| 381 | 382 | 383 | 384 | 385 | 386 | 387 | 388 | 389 | 390  |
| 391 | 392 | 393 | 394 | 395 | 396 | 397 | 398 | 399 | 400  |
| 401 | 402 | 403 | 404 | 405 | 406 | 407 | 408 | 409 | 410  |
| 411 | 412 | 413 | 414 | 415 | 416 | 417 | 418 | 419 | 420  |
| 421 | 422 | 423 | 424 | 425 | 426 | 427 | 428 | 429 | 430  |
| 431 | 432 | 433 | 434 | 435 | 436 | 437 | 438 | 439 | 440  |
| 441 | 442 | 443 | 444 | 445 | 446 | 447 | 448 | 449 | 450  |
| 451 | 452 | 453 | 454 | 455 | 456 | 457 | 458 | 459 | 460  |
| 461 | 462 | 463 | 464 | 465 | 466 | 467 | 468 | 469 | 470  |
| 471 | 472 | 473 | 474 | 475 | 476 | 477 | 478 | 479 | 480  |
| 481 | 482 | 483 | 484 | 485 | 486 | 487 | 488 | 489 | 490  |
| 491 | 492 | 493 | 494 | 495 | 496 | 497 | 498 | 499 | 500  |
| 501 | 502 | 503 | 504 | 505 | 506 | 507 | 508 | 509 | 510  |
| 511 | 512 | 513 | 514 | 515 | 516 | 517 | 518 | 519 | 520  |
| 521 | 522 | 523 | 524 | 525 | 526 | 527 | 528 | 529 | 530  |
| 531 | 532 | 533 | 534 | 535 | 536 | 537 | 538 | 539 | 540  |
| 541 | 542 | 543 | 544 | 545 | 546 | 547 | 548 | 549 | 550  |
| 551 | 552 | 553 | 554 | 555 | 556 | 557 | 558 | 559 | 560  |
| 561 | 562 | 563 | 564 | 565 | 566 | 567 | 568 | 569 | 570  |
| 571 | 572 | 573 | 574 | 575 | 576 | 577 | 578 | 579 | 580  |
| 581 | 582 | 583 | 584 | 585 | 586 | 587 | 588 | 589 | 590  |
| 591 | 592 | 593 | 594 | 595 | 596 | 597 | 598 | 599 | 600  |
| 601 | 602 | 603 | 604 | 605 | 606 | 607 | 608 | 609 | 610  |
| 611 | 612 | 613 | 614 | 615 | 616 | 617 | 618 | 619 | 620  |
| 621 | 622 | 623 | 624 | 625 | 626 | 627 | 628 | 629 | 630  |
| 631 | 632 | 633 | 634 | 635 | 636 | 637 | 638 | 639 | 640  |
| 641 | 642 | 643 | 644 | 645 | 646 | 647 | 648 | 649 | 650  |
| 651 | 652 | 653 | 654 | 655 | 656 | 657 | 658 | 659 | 660  |
| 661 | 662 | 663 | 664 | 665 | 666 | 667 | 668 | 669 | 670  |
| 671 | 672 | 673 | 674 | 675 | 676 | 677 | 678 | 679 | 680  |
| 681 | 682 | 683 | 684 | 685 | 686 | 687 | 688 | 689 | 690  |
| 691 | 692 | 693 | 694 | 695 | 696 | 697 | 698 | 699 | 700  |
| 701 | 702 | 703 | 704 | 705 | 706 | 707 | 708 | 709 | 710  |
| 711 | 712 | 713 | 714 | 715 | 716 | 717 | 718 | 719 | 720  |
| 721 | 722 | 723 | 724 | 725 | 726 | 727 | 728 | 729 | 730  |
| 731 | 732 | 733 | 734 | 735 | 736 | 737 | 738 | 739 | 740  |
| 741 | 742 | 743 | 744 | 745 | 746 | 747 | 748 | 749 | 750  |
| 751 | 752 | 753 | 754 | 755 | 756 | 757 | 758 | 759 | 760  |
| 761 | 762 | 763 | 764 | 765 | 766 | 767 | 768 | 769 | 770  |
| 771 | 772 | 773 | 774 | 775 | 776 | 777 | 778 | 779 | 780  |
| 781 | 782 | 783 | 784 | 785 | 786 | 787 | 788 | 789 | 790  |
| 791 | 792 | 793 | 794 | 795 | 796 | 797 | 798 | 799 | 800  |
| 801 | 802 | 803 | 804 | 805 | 806 | 807 | 808 | 809 | 810  |
| 811 | 812 | 813 | 814 | 815 | 816 | 817 | 818 | 819 | 820  |
| 821 | 822 | 823 | 824 | 825 | 826 | 827 | 828 | 829 | 830  |
| 831 | 832 | 833 | 834 | 835 | 836 | 837 | 838 | 839 | 840  |
| 841 | 842 | 843 | 844 | 845 | 846 | 847 | 848 | 849 | 850  |
| 851 | 852 | 853 | 854 | 855 | 856 | 857 | 858 | 859 | 860  |
| 861 | 862 | 863 | 864 | 865 | 866 | 867 | 868 | 869 | 870  |
| 871 | 872 | 873 | 874 | 875 | 876 | 877 | 878 | 879 | 880  |
| 881 | 882 | 883 | 884 | 885 | 886 | 887 | 888 | 889 | 890  |
| 891 | 892 | 893 | 894 | 895 | 896 | 897 | 898 | 899 | 900  |
| 901 | 902 | 903 | 904 | 905 | 906 | 907 | 908 | 909 | 910  |
| 911 | 912 | 913 | 914 | 915 | 916 | 917 | 918 | 919 | 920  |
| 921 | 922 | 923 | 924 | 925 | 926 | 927 | 928 | 929 | 930  |
| 931 | 932 | 933 | 934 | 935 | 936 | 937 | 938 | 939 | 940  |
| 941 | 942 | 943 | 944 | 945 | 946 | 947 | 948 | 949 | 950  |
| 951 | 952 | 953 | 954 | 955 | 956 | 957 | 958 | 959 | 960  |
| 961 | 962 | 963 | 964 | 965 | 966 | 967 | 968 | 969 | 970  |
| 971 | 972 | 973 | 974 | 975 | 976 | 977 | 978 | 979 | 980  |
| 981 | 982 | 983 | 984 | 985 | 986 | 987 | 988 | 989 | 990  |
| 991 | 992 | 993 | 994 | 995 | 996 | 997 | 998 | 999 | 1000 |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| *0001         | *00000133 | *0029         | *00020805 | *0057         | *00057281 | *0085         | *00104221 |
| *0002         | *00000377 | *0030         | *00021889 | *0058         | *00058793 | *0086         | *00106063 |
| *0003         | *00000693 | *0031         | *00022992 | *0059         | *00060318 | *0087         | *00107915 |
| *0004         | *00001067 | *0032         | *00024113 | *0060         | *00061856 | *0088         | *00109777 |
| *0005         | *00001490 | *0033         | *00025251 | *0061         | *00063407 | *0089         | *00111651 |
| *0006         | *00001959 | *0034         | *00026407 | *0062         | *00064971 | *0090         | *00113534 |
| *0007         | *00002469 | *0035         | *00027579 | *0063         | *00066547 | *0091         | *00115427 |
| *0008         | *00003016 | *0036         | *00028769 | *0064         | *00068135 | *0092         | *00117332 |
| *0009         | *00003599 | *0037         | *00029975 | *0065         | *00069736 | *0093         | *00119247 |
| *0010         | *00004215 | *0038         | *00031197 | *0066         | *00071350 | *0094         | *00121172 |
| *0011         | *00004863 | *0039         | *00032436 | *0067         | *00072975 | *0095         | *00123107 |
| *0012         | *00005541 | *0040         | *00033690 | *0068         | *00074613 | *0096         | *00125052 |
| *0013         | *00006247 | *0041         | *00034961 | *0069         | *00076263 | *0097         | *00127007 |
| *0014         | *00006981 | *0042         | *00036247 | *0070         | *00077924 | *0098         | *00128972 |
| *0015         | *00007742 | *0043         | *00037547 | *0071         | *00079597 | *0099         | *00130947 |
| *0016         | *00008529 | *0044         | *00038864 | *0072         | *00081282 | *0100         | *00132932 |
| *0017         | *00009341 | *0045         | *00040195 | *0073         | *00082979 | *0101         | *00134927 |
| *0018         | *00010177 | *0046         | *00041540 | *0074         | *00084688 | *0102         | *00136932 |
| *0019         | *00011036 | *0047         | *00042901 | *0075         | *00086407 | *0103         | *00138947 |
| *0020         | *00011919 | *0048         | *00044277 | *0076         | *00088139 | *0104         | *00140971 |
| *0021         | *00012823 | *0049         | *00045666 | *0077         | *00089881 | *0105         | *00143005 |
| *0022         | *00013749 | *0050         | *00047070 | *0078         | *00091635 | *0106         | *00145048 |
| *0023         | *00014697 | *0051         | *00048487 | *0079         | *00093400 | *0107         | *00147101 |
| *0024         | *00015665 | *0052         | *00049919 | *0080         | *00095176 | *0108         | *00149163 |
| *0025         | *00016654 | *0053         | *00051364 | *0081         | *00096963 | *0109         | *00151235 |
| *0026         | *00017663 | *0054         | *00052823 | *0082         | *00098762 | *0110         | *00153317 |
| *0027         | *00018691 | *0055         | *00054296 | *0083         | *00100571 | *0111         | *00155407 |
| *0028         | *00019738 | *0056         | *00055781 | *0084         | *00102391 | *0112         | *00157507 |



# A TABLE of CIRCULAR SEGMENTS. 619

| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 0113          | 00159617  | 0158          | 00263545  | 0203          | 00383283  | 0248          | 00516843  |
| 0114          | 00161735  | 0159          | 00266043  | 0204          | 00386107  | 0249          | 00519956  |
| 0115          | 00163863  | 0160          | 00268549  | 0205          | 00388938  | 0250          | 00523076  |
| 0116          | 00166000  | 0161          | 00271062  | 0206          | 00391775  | 0251          | 00526201  |
| 0117          | 00168146  | 0162          | 00273583  | 0207          | 00394620  | 0252          | 00529333  |
| 0118          | 00170301  | 0163          | 00276112  | 0208          | 00397471  | 0253          | 00532470  |
| 0119          | 00172466  | 0164          | 00278648  | 0209          | 00400328  | 0254          | 00535614  |
| 0120          | 00174639  | 0165          | 00281192  | 0210          | 00403193  | 0255          | 00538764  |
| 0121          | 00176821  | 0166          | 00283744  | 0211          | 00406064  | 0256          | 00541920  |
| 0122          | 00179012  | 0167          | 00286303  | 0212          | 00408941  | 0257          | 00545081  |
| 0123          | 00181212  | 0168          | 00288870  | 0213          | 00411826  | 0258          | 00548249  |
| 0124          | 00183421  | 0169          | 00291444  | 0214          | 00414717  | 0259          | 00551423  |
| 0125          | 00185639  | 0170          | 00284025  | 0215          | 00417614  | 0260          | 00554603  |
| 0126          | 00187865  | 0171          | 00296614  | 0216          | 00420518  | 0261          | 00557788  |
| 0127          | 00190100  | 0172          | 00299211  | 0217          | 00423429  | 0262          | 00560980  |
| 0128          | 00192344  | 0173          | 00301815  | 0218          | 00426346  | 0263          | 00564178  |
| 0129          | 00194597  | 0174          | 00304427  | 0219          | 00429270  | 0264          | 00567381  |
| 0130          | 00196858  | 0175          | 00307045  | 0220          | 00432201  | 0265          | 00570590  |
| 0131          | 00199128  | 0176          | 00309672  | 0221          | 00435137  | 0266          | 00573806  |
| 0132          | 00201406  | 0177          | 00312305  | 0222          | 00438081  | 0267          | 00577027  |
| 0133          | 00203693  | 0178          | 00314946  | 0223          | 00441031  | 0268          | 00580254  |
| 0134          | 00205988  | 0179          | 00317594  | 0224          | 00443987  | 0269          | 00583487  |
| 0135          | 00208292  | 0180          | 00320249  | 0225          | 00446950  | 0270          | 00586726  |
| 0136          | 00210604  | 0181          | 00322912  | 0226          | 00449919  | 0271          | 00589970  |
| 0137          | 00212925  | 0182          | 00325582  | 0227          | 00452895  | 0272          | 00593221  |
| 0138          | 00215254  | 0183          | 00328259  | 0228          | 00455877  | 0273          | 00596477  |
| 0139          | 00217591  | 0184          | 00330943  | 0229          | 00458866  | 0274          | 00599739  |
| 0140          | 00219937  | 0185          | 00333635  | 0230          | 00461861  | 0275          | 00603007  |
| 0141          | 00222291  | 0186          | 00336333  | 0231          | 00464862  | 0276          | 00606280  |
| 0142          | 00224653  | 0187          | 00339039  | 0232          | 00467869  | 0277          | 00609560  |
| 0143          | 00227024  | 0188          | 00341752  | 0233          | 00470883  | 0278          | 00612845  |
| 0144          | 00229402  | 0189          | 00344472  | 0234          | 00473904  | 0279          | 00616135  |
| 0145          | 00231789  | 0190          | 00347199  | 0235          | 00476930  | 0280          | 00619432  |
| 0146          | 00234184  | 0191          | 00349933  | 0236          | 00479963  | 0281          | 00622734  |
| 0147          | 00236587  | 0192          | 00352674  | 0237          | 00483002  | 0282          | 00626042  |
| 0148          | 00238998  | 0193          | 00355422  | 0238          | 00486047  | 0283          | 00629356  |
| 0149          | 00241417  | 0194          | 00358177  | 0239          | 00489099  | 0284          | 00632676  |
| 0150          | 00243844  | 0195          | 00360939  | 0240          | 00492157  | 0285          | 00636001  |
| 0151          | 00246279  | 0196          | 00363708  | 0241          | 00495221  | 0286          | 00639331  |
| 0152          | 00248722  | 0197          | 00366484  | 0242          | 00498291  | 0287          | 00642668  |
| 0153          | 00251173  | 0198          | 00369266  | 0243          | 00501368  | 0288          | 00646010  |
| 0154          | 00253631  | 0199          | 00372056  | 0244          | 00504451  | 0289          | 00649358  |
| 0155          | 00256098  | 0200          | 00374853  | 0245          | 00507539  | 0290          | 00652711  |
| 0156          | 00258573  | 0201          | 00377656  | 0246          | 00510634  | 0291          | 00656070  |
| 0157          | 00261055  | 0202          | 00380466  | 0247          | 00513736  | 0292          | 00659434  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 0293          | 00662804  | 0338          | 00820088  | 0383          | 00987831  | 0428          | 01165328  |
| 0294          | 00666180  | 0339          | 00823705  | 0384          | 00991672  | 0429          | 01169378  |
| 0295          | 00669561  | 0340          | 00827327  | 0385          | 00995517  | 0430          | 01173433  |
| 0296          | 00672948  | 0341          | 00830954  | 0386          | 00999368  | 0431          | 01177492  |
| 0297          | 00676341  | 0342          | 00834586  | 0387          | 01003224  | 0432          | 01181556  |
| 0298          | 00679739  | 0343          | 00838224  | 0388          | 01007083  | 0433          | 01185625  |
| 0299          | 00683142  | 0344          | 00841867  | 0389          | 01010948  | 0434          | 01189697  |
| 0300          | 00686551  | 0345          | 00845514  | 0390          | 01014818  | 0435          | 01193775  |
| 0301          | 00689966  | 0346          | 00849166  | 0391          | 01018692  | 0436          | 01197857  |
| 0302          | 00693386  | 0347          | 00852824  | 0392          | 01022571  | 0437          | 01201943  |
| 0303          | 00696811  | 0348          | 00856487  | 0393          | 01026455  | 0438          | 01206034  |
| 0304          | 00700242  | 0349          | 00860155  | 0394          | 01030343  | 0439          | 01210129  |
| 0305          | 00703679  | 0350          | 00863828  | 0395          | 01034237  | 0440          | 01214229  |
| 0306          | 00707120  | 0351          | 00867506  | 0396          | 01038135  | 0441          | 01218333  |
| 0307          | 00710568  | 0352          | 00871190  | 0397          | 01042038  | 0442          | 01222441  |
| 0308          | 00714021  | 0353          | 00874878  | 0398          | 01045945  | 0443          | 01226554  |
| 0309          | 00717479  | 0354          | 00878571  | 0399          | 01049857  | 0444          | 01230672  |
| 0310          | 00720942  | 0355          | 00882269  | 0400          | 01053774  | 0445          | 01234794  |
| 0311          | 00724411  | 0356          | 00885973  | 0401          | 01057695  | 0446          | 01238920  |
| 0312          | 00727886  | 0357          | 00889681  | 0402          | 01061622  | 0447          | 01243050  |
| 0313          | 00731366  | 0458          | 00893394  | 0403          | 01065553  | 0448          | 01247186  |
| 0314          | 00734851  | 0359          | 00897113  | 0404          | 01069488  | 0449          | 01251325  |
| 0315          | 00738342  | 0360          | 00900836  | 0405          | 01073428  | 0450          | 01255469  |
| 0316          | 00741838  | 0361          | 00904564  | 0406          | 01077373  | 0451          | 01259617  |
| 0317          | 00745339  | 0362          | 00908297  | 0407          | 01081323  | 0452          | 01263770  |
| 0318          | 00748846  | 0363          | 00912036  | 0408          | 01085277  | 0453          | 01267927  |
| 0319          | 00752358  | 0364          | 00915779  | 0409          | 01089236  | 0454          | 01272088  |
| 0320          | 00755875  | 0365          | 00919527  | 0410          | 01093199  | 0455          | 01276254  |
| 0321          | 00759398  | 0366          | 00923280  | 0411          | 01097167  | 0456          | 01280424  |
| 0322          | 00762926  | 0367          | 00927038  | 0412          | 01101140  | 0457          | 01284599  |
| 0323          | 00766459  | 0368          | 00930801  | 0413          | 01105117  | 0458          | 01288778  |
| 0324          | 00769997  | 0369          | 00934569  | 0414          | 01109099  | 0459          | 01292961  |
| 0325          | 00773541  | 0370          | 00938342  | 0415          | 01113086  | 0460          | 01297148  |
| 0326          | 00777090  | 0371          | 00942119  | 0416          | 01117077  | 0461          | 01301340  |
| 0327          | 00780645  | 0372          | 00945902  | 0417          | 01121073  | 0462          | 01305536  |
| 0328          | 00784204  | 0373          | 00949689  | 0418          | 01125073  | 0463          | 01309737  |
| 0329          | 00787769  | 0374          | 00953482  | 0419          | 01129078  | 0464          | 01313942  |
| 0330          | 00791339  | 0375          | 00957279  | 0420          | 01133088  | 0465          | 01318151  |
| 0331          | 00794915  | 0376          | 00961081  | 0421          | 01137102  | 0466          | 01322364  |
| 0332          | 00798495  | 0377          | 00964888  | 0422          | 01141120  | 0467          | 01326582  |
| 0333          | 00802081  | 0378          | 00968700  | 0423          | 01145144  | 0468          | 01330804  |
| 0334          | 00805672  | 0379          | 00972517  | 0424          | 01149171  | 0469          | 01335031  |
| 0335          | 00809268  | 0380          | 00976338  | 0425          | 01153203  | 0470          | 01339261  |
| 0336          | 00812869  | 0381          | 00980164  | 0426          | 01157240  | 0471          | 01343496  |
| 0337          | 00816476  | 0382          | 00983996  | 0427          | 01161282  | 0472          | 01347735  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 0473          | 01351979  | 0518          | 01547272  | 0563          | 01750760  | 0608          | 01962048  |
| 0474          | 01356226  | 0519          | 01551707  | 0564          | 01755372  | 0609          | 01966829  |
| 0475          | 01360478  | 0520          | 01556145  | 0565          | 01759988  | 0610          | 01971614  |
| 0476          | 01364735  | 0521          | 01560588  | 0566          | 01764608  | 0611          | 01976403  |
| 0477          | 01368995  | 0522          | 01565034  | 0567          | 01769231  | 0612          | 01981195  |
| 0478          | 01373260  | 0523          | 01569485  | 0568          | 01773859  | 0613          | 01986990  |
| 0479          | 01377529  | 0524          | 01573940  | 0569          | 01778490  | 0614          | 01991790  |
| 0480          | 01381802  | 0525          | 01578398  | 0570          | 01783125  | 0615          | 01996593  |
| 0481          | 01386079  | 0526          | 01582861  | 0571          | 01787763  | 0616          | 02001400  |
| 0482          | 01390361  | 0527          | 01587328  | 0572          | 01792406  | 0617          | 02006210  |
| 0483          | 01394647  | 0528          | 01591798  | 0573          | 01797052  | 0618          | 02011024  |
| 0484          | 01398937  | 0529          | 01596273  | 0574          | 01801702  | 0619          | 02015842  |
| 0485          | 01403231  | 0530          | 01600752  | 0575          | 01806356  | 0620          | 02020663  |
| 0486          | 01407530  | 0531          | 01605234  | 0576          | 01811014  | 0621          | 02025488  |
| 0487          | 01411833  | 0532          | 01609720  | 0577          | 01815676  | 0622          | 02030317  |
| 0488          | 01416140  | 0533          | 01614211  | 0578          | 01820341  | 0623          | 02035149  |
| 0489          | 01420451  | 0534          | 01618706  | 0579          | 01825010  | 0624          | 02039985  |
| 0490          | 01424766  | 0535          | 01623205  | 0580          | 01829683  | 0625          | 02044824  |
| 0491          | 01429085  | 0536          | 01627707  | 0581          | 01834360  | 0626          | 02049667  |
| 0492          | 01433409  | 0537          | 01632214  | 0582          | 01839041  | 0627          | 02054514  |
| 0493          | 01437737  | 0538          | 01636724  | 0583          | 01843725  | 0628          | 02059364  |
| 0494          | 01442069  | 0539          | 01641239  | 0584          | 01848413  | 0629          | 02064218  |
| 0495          | 01446405  | 0540          | 01645757  | 0585          | 01853105  | 0630          | 02069075  |
| 0496          | 01450745  | 0541          | 01650279  | 0586          | 01857800  | 0631          | 02073936  |
| 0497          | 01455090  | 0542          | 01654806  | 0587          | 01862500  | 0632          | 02078801  |
| 0498          | 01459438  | 0543          | 01659336  | 0588          | 01867203  | 0633          | 02083669  |
| 0499          | 01463791  | 0544          | 01663870  | 0589          | 01871910  | 0634          | 02088541  |
| 0500          | 01468148  | 0545          | 01668408  | 0590          | 01876621  | 0635          | 02093416  |
| 0501          | 01472509  | 0546          | 01672950  | 0591          | 01881335  | 0636          | 02098295  |
| 0502          | 01476874  | 0547          | 01677496  | 0592          | 01886053  | 0637          | 02103178  |
| 0503          | 01481243  | 0548          | 01682046  | 0593          | 01890775  | 0638          | 02108064  |
| 0504          | 01485616  | 0549          | 01686600  | 0594          | 01895500  | 0639          | 02112953  |
| 0505          | 01489994  | 0550          | 01691157  | 0595          | 01900230  | 0640          | 02117847  |
| 0506          | 01494375  | 0551          | 01695719  | 0596          | 01904963  | 0641          | 02122744  |
| 0507          | 01498761  | 0552          | 01700284  | 0597          | 01909700  | 0642          | 02127644  |
| 0508          | 01503151  | 0553          | 01704854  | 0598          | 01914440  | 0643          | 02132548  |
| 0509          | 01507544  | 0554          | 01709427  | 0599          | 01919184  | 0644          | 02137455  |
| 0510          | 01511942  | 0555          | 01714004  | 0600          | 01923932  | 0645          | 02142366  |
| 0511          | 01516344  | 0556          | 01718585  | 0601          | 01928684  | 0646          | 02147281  |
| 0512          | 01520750  | 0557          | 01723170  | 0602          | 01933439  | 0647          | 02152199  |
| 0513          | 01525161  | 0558          | 01727759  | 0603          | 01938198  | 0648          | 02157121  |
| 0514          | 01529575  | 0559          | 01732351  | 0604          | 01942961  | 0649          | 02162046  |
| 0515          | 01533993  | 0560          | 01736948  | 0605          | 01947727  | 0650          | 02166975  |
| 0516          | 01538415  | 0561          | 01741548  | 0606          | 01952497  | 0651          | 02171906  |
| 0517          | 01542842  | 0562          | 01746152  | 0607          | 01957271  | 0652          | 02176842  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 0653          | 02181781  | 0698          | 02406646  | 0743          | 02639348  | 0788          | 02878627  |
| 0654          | 02186724  | 0699          | 02411743  | 0744          | 02644598  | 0789          | 02884017  |
| 0655          | 02191671  | 0700          | 02416845  | 0745          | 02649845  | 0790          | 02889410  |
| 0656          | 02196620  | 0701          | 02421949  | 0746          | 02655098  | 0791          | 02894806  |
| 0657          | 02201574  | 0702          | 02427057  | 0747          | 02660355  | 0792          | 02900206  |
| 0658          | 02206531  | 0703          | 02432169  | 0748          | 02665614  | 0793          | 02905608  |
| 0659          | 02211491  | 0704          | 02437283  | 0749          | 02670877  | 0794          | 02911014  |
| 0660          | 02216455  | 0705          | 02442401  | 0750          | 02676144  | 0795          | 02916423  |
| 0661          | 02220422  | 0706          | 02447523  | 0751          | 02681413  | 0796          | 02921835  |
| 0662          | 02225393  | 0707          | 02452648  | 0752          | 02686686  | 0797          | 02927250  |
| 0663          | 02230368  | 0708          | 02457776  | 0753          | 02691962  | 0798          | 02932668  |
| 0664          | 02235346  | 0709          | 02462908  | 0754          | 02697241  | 0799          | 02938089  |
| 0665          | 02240327  | 0710          | 02468042  | 0755          | 02702526  | 0800          | 02943513  |
| 0666          | 02245312  | 0711          | 02473180  | 0756          | 02707809  | 0801          | 02948941  |
| 0667          | 02250300  | 0712          | 02478322  | 0757          | 02713097  | 0802          | 02954371  |
| 0668          | 02255292  | 0713          | 02483467  | 0758          | 02718389  | 0803          | 02959805  |
| 0669          | 02260287  | 0714          | 02488615  | 0759          | 02723684  | 0804          | 02965241  |
| 0670          | 02265286  | 0715          | 02493766  | 0760          | 02728982  | 0805          | 02970681  |
| 0671          | 02270288  | 0716          | 02498921  | 0761          | 02734284  | 0806          | 02976124  |
| 0672          | 02275294  | 0717          | 02504079  | 0762          | 02739589  | 0807          | 02981570  |
| 0673          | 02280303  | 0718          | 02509241  | 0763          | 02744897  | 0808          | 02987019  |
| 0674          | 02285316  | 0719          | 02514405  | 0764          | 02750208  | 0809          | 02992471  |
| 0675          | 02290332  | 0720          | 02519574  | 0765          | 02755523  | 0810          | 02997926  |
| 0676          | 02295351  | 0721          | 02524745  | 0766          | 02760840  | 0811          | 03003385  |
| 0677          | 02300374  | 0722          | 02529920  | 0767          | 02766161  | 0812          | 03008846  |
| 0678          | 02305401  | 0723          | 02535098  | 0768          | 02771485  | 0813          | 03014310  |
| 0679          | 02310431  | 0724          | 02540279  | 0769          | 02776812  | 0814          | 03019778  |
| 0680          | 02315484  | 0725          | 02545464  | 0770          | 02782142  | 0815          | 03025248  |
| 0681          | 02320500  | 0726          | 02550652  | 0771          | 02787475  | 0816          | 03030722  |
| 0682          | 02325540  | 0727          | 02555843  | 0772          | 02792812  | 0817          | 03036198  |
| 0683          | 02330584  | 0728          | 02561037  | 0773          | 02798152  | 0818          | 03041678  |
| 0684          | 02335631  | 0729          | 02566235  | 0774          | 02803495  | 0819          | 03047161  |
| 0685          | 02340681  | 0730          | 02571437  | 0775          | 02808841  | 0820          | 03052646  |
| 0686          | 02345735  | 0731          | 02576641  | 0776          | 02814190  | 0821          | 03058135  |
| 0687          | 02350792  | 0732          | 02581848  | 0777          | 02819543  | 0822          | 03063627  |
| 0688          | 02355852  | 0733          | 02587059  | 0778          | 02824898  | 0823          | 03069122  |
| 0689          | 02360916  | 0734          | 02592273  | 0779          | 02830257  | 0824          | 03074620  |
| 0690          | 02365984  | 0735          | 02597491  | 0780          | 02835619  | 0825          | 03080121  |
| 0691          | 02371055  | 0736          | 02602712  | 0781          | 02840984  | 0826          | 03085625  |
| 0692          | 02376129  | 0737          | 02607936  | 0782          | 02846352  | 0827          | 03091132  |
| 0693          | 02381206  | 0738          | 02613163  | 0783          | 02851723  | 0828          | 03096642  |
| 0694          | 02386287  | 0739          | 02618393  | 0784          | 02857097  | 0829          | 03102155  |
| 0695          | 02391372  | 0740          | 02623627  | 0785          | 02862475  | 0830          | 03107671  |
| 0696          | 02396459  | 0741          | 02628864  | 0786          | 02867856  | 0831          | 03113190  |
| 0697          | 02401550  | 0742          | 02634105  | 0787          | 02873240  | 0832          | 03118713  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 0833          | 03124238  | 0878          | 03375958  | 0923          | 03633578  | 0968          | 03896903  |
| 0834          | 03129766  | 0879          | 03381619  | 0924          | 03639368  | 0969          | 03902818  |
| 0835          | 03135297  | 0880          | 03387284  | 0925          | 03645161  | 0970          | 03908736  |
| 0836          | 03140832  | 0881          | 03392951  | 0926          | 03650957  | 0971          | 03914657  |
| 0837          | 03146369  | 0882          | 03398621  | 0927          | 03656756  | 0972          | 03920580  |
| 0838          | 03151909  | 0883          | 03404294  | 0928          | 03662558  | 0973          | 03926506  |
| 0839          | 03157452  | 0884          | 03409970  | 0929          | 03668362  | 0974          | 03932435  |
| 0840          | 03162999  | 0885          | 03415649  | 0930          | 03674169  | 0975          | 03938366  |
| 0841          | 03168548  | 0886          | 03421331  | 0931          | 03679980  | 0976          | 03944300  |
| 0842          | 03174100  | 0887          | 03427016  | 0932          | 03685792  | 0977          | 03950237  |
| 0843          | 03179655  | 0888          | 03432704  | 0933          | 03691608  | 0978          | 03956176  |
| 0844          | 03185214  | 0889          | 03438394  | 0934          | 03697426  | 0979          | 03962119  |
| 0845          | 03190775  | 0890          | 03444088  | 0935          | 03703248  | 0980          | 03968064  |
| 0846          | 03196339  | 0891          | 03449784  | 0936          | 03709072  | 0981          | 03974011  |
| 0847          | 03201906  | 0892          | 03455483  | 0937          | 03714899  | 0982          | 03979962  |
| 0848          | 03207476  | 0893          | 03461185  | 0938          | 03720728  | 0983          | 03985915  |
| 0849          | 03213050  | 0894          | 03466890  | 0939          | 03726561  | 0984          | 03991870  |
| 0850          | 03218626  | 0895          | 03472598  | 0940          | 03732396  | 0985          | 03997829  |
| 0851          | 03224205  | 0896          | 03478309  | 0941          | 03738234  | 0986          | 04003790  |
| 0852          | 03229787  | 0897          | 03484022  | 0942          | 03744074  | 0987          | 04009754  |
| 0853          | 03235372  | 0898          | 03489739  | 0943          | 03749918  | 0988          | 04015720  |
| 0854          | 03240960  | 0899          | 03495458  | 0944          | 03755764  | 0989          | 04021689  |
| 0855          | 03246551  | 0900          | 03501180  | 0945          | 03761613  | 0990          | 04027661  |
| 0856          | 03252145  | 0901          | 03506905  | 0946          | 03767465  | 0991          | 04033636  |
| 0857          | 03257742  | 0902          | 03512633  | 0947          | 03773320  | 0992          | 04039613  |
| 0858          | 03263342  | 0903          | 03518364  | 0948          | 03779177  | 0993          | 04045593  |
| 0859          | 03268945  | 0904          | 03524098  | 0949          | 03785037  | 0994          | 04051576  |
| 0860          | 03274550  | 0905          | 03529834  | 0950          | 03790900  | 0995          | 04057561  |
| 0861          | 03280159  | 0906          | 03535574  | 0951          | 03796766  | 0996          | 04063549  |
| 0862          | 03285771  | 0907          | 03541316  | 0952          | 03802634  | 0997          | 04069540  |
| 0863          | 03291386  | 0908          | 03547061  | 0953          | 03808506  | 0998          | 04075533  |
| 0864          | 03297003  | 0909          | 03552809  | 0954          | 03814380  | 0999          | 04081529  |
| 0865          | 03302624  | 0910          | 03558560  | 0955          | 03820256  | 1000          | 04087528  |
| 0866          | 03308247  | 0911          | 03564313  | 0956          | 03826136  | 1001          | 04093529  |
| 0867          | 03313874  | 0912          | 03570070  | 0957          | 03832018  | 1002          | 04099533  |
| 0868          | 03319503  | 0913          | 03575829  | 0958          | 03837903  | 1003          | 04105540  |
| 0869          | 03325135  | 0914          | 03581591  | 0959          | 03843791  | 1004          | 04111549  |
| 0870          | 03330771  | 0915          | 03587356  | 0960          | 03849681  | 1005          | 04117561  |
| 0871          | 03336409  | 0916          | 03593124  | 0961          | 03855574  | 1006          | 04123576  |
| 0872          | 03342050  | 0917          | 03598895  | 0962          | 03861470  | 1007          | 04129593  |
| 0873          | 03347694  | 0918          | 03604668  | 0963          | 03867369  | 1008          | 04135613  |
| 0874          | 03353341  | 0919          | 03610444  | 0964          | 03873270  | 1009          | 04141635  |
| 0875          | 03358991  | 0920          | 03616223  | 0965          | 03879174  | 1010          | 04147661  |
| 0876          | 03364643  | 0921          | 03622005  | 0966          | 03885081  | 1011          | 04153689  |
| 0877          | 03370299  | 0922          | 03627790  | 0967          | 03890991  | 1012          | 04159719  |



| Veri.<br>fine | Seg. area | Veri.<br>fine | Seg. area | Veri.<br>fine | Seg. area | Veri.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 1013          | 04165752  | 1058          | 04439954  | 1103          | 04719347  | 1148          | 05003779  |
| 1014          | 04171788  | 1059          | 04446107  | 1104          | 04725613  | 1149          | 05010155  |
| 1015          | 04177827  | 1060          | 04452262  | 1105          | 04731882  | 1150          | 05016535  |
| 1016          | 04183868  | 1061          | 04458420  | 1106          | 04738154  | 1151          | 05022916  |
| 1017          | 04189912  | 1062          | 04464581  | 1107          | 04744428  | 1152          | 05029300  |
| 1018          | 04195958  | 1063          | 04470744  | 1108          | 04750704  | 1153          | 05035687  |
| 1019          | 04202007  | 1064          | 04476910  | 1109          | 04756983  | 1154          | 05042076  |
| 1020          | 04208059  | 1065          | 04483078  | 1110          | 04763265  | 1155          | 05048467  |
| 1021          | 04214113  | 1066          | 04489249  | 1111          | 04769548  | 1156          | 05054861  |
| 1022          | 04220170  | 1067          | 04495422  | 1112          | 04775835  | 1157          | 05061257  |
| 1023          | 04226229  | 1068          | 04501598  | 1113          | 04782124  | 1158          | 05067655  |
| 1024          | 04232291  | 1069          | 04507777  | 1114          | 04788415  | 1159          | 05074056  |
| 1025          | 04238356  | 1070          | 04513958  | 1115          | 04794709  | 1160          | 05080459  |
| 1026          | 04244424  | 1071          | 04520141  | 1116          | 04801005  | 1161          | 05086865  |
| 1027          | 04250494  | 1072          | 04526327  | 1117          | 04807304  | 1162          | 05093273  |
| 1028          | 04256566  | 1073          | 04532516  | 1118          | 04813605  | 1163          | 05099684  |
| 1029          | 04262642  | 1074          | 04538707  | 1119          | 04819908  | 1164          | 05106097  |
| 1030          | 04268719  | 1075          | 04544901  | 1120          | 04826215  | 1165          | 05112512  |
| 1031          | 04274800  | 1076          | 04551097  | 1121          | 04832523  | 1166          | 05118930  |
| 1032          | 04280883  | 1077          | 04557296  | 1122          | 04838834  | 1167          | 05125350  |
| 1033          | 04286969  | 1078          | 04563497  | 1123          | 04845148  | 1168          | 05131772  |
| 1034          | 04293057  | 1079          | 04569701  | 1124          | 04851464  | 1169          | 05138197  |
| 1035          | 04299148  | 1080          | 04575907  | 1125          | 04857782  | 1170          | 05144624  |
| 1036          | 04305241  | 1081          | 04582116  | 1126          | 04864103  | 1171          | 05151054  |
| 1037          | 04311338  | 1082          | 04588327  | 1127          | 04870426  | 1172          | 05157486  |
| 1038          | 04317436  | 1083          | 04594541  | 1128          | 04876752  | 1173          | 05163920  |
| 1039          | 04323538  | 1084          | 04600758  | 1129          | 04883080  | 1174          | 05170357  |
| 1040          | 04329642  | 1085          | 04606977  | 1130          | 04889411  | 1175          | 05176796  |
| 1041          | 04335748  | 1086          | 04613198  | 1131          | 04895744  | 1176          | 05183237  |
| 1042          | 04341857  | 1087          | 04619422  | 1132          | 04902079  | 1177          | 05189681  |
| 1043          | 04347969  | 1088          | 04625649  | 1133          | 04908417  | 1178          | 05196127  |
| 1044          | 04354083  | 1089          | 04631878  | 1134          | 04914758  | 1179          | 05202576  |
| 1045          | 04360200  | 1090          | 04638109  | 1135          | 04921100  | 1180          | 05209027  |
| 1046          | 04366319  | 1091          | 04644343  | 1136          | 04927446  | 1181          | 05215480  |
| 1047          | 04372442  | 1092          | 04650580  | 1137          | 04933793  | 1182          | 05221936  |
| 1048          | 04378566  | 1093          | 04656819  | 1138          | 04940144  | 1183          | 05228394  |
| 1049          | 04384693  | 1094          | 04663060  | 1139          | 04946496  | 1184          | 05234855  |
| 1050          | 04390823  | 1095          | 04669304  | 1140          | 04952851  | 1185          | 05241317  |
| 1051          | 04396955  | 1096          | 04675551  | 1141          | 04959209  | 1186          | 05247783  |
| 1052          | 04403090  | 1097          | 04681800  | 1142          | 04965568  | 1187          | 05254250  |
| 1053          | 04409228  | 1098          | 04688052  | 1143          | 04971931  | 1188          | 05260720  |
| 1054          | 04415368  | 1099          | 04694306  | 1144          | 04978295  | 1189          | 05267192  |
| 1055          | 04421511  | 1100          | 04700562  | 1145          | 04984663  | 1190          | 05273667  |
| 1056          | 04427656  | 1101          | 04706821  | 1146          | 04991032  | 1191          | 05280144  |
| 1057          | 04433804  | 1102          | 04713083  | 1147          | 04997404  | 1192          | 05286623  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 1193          | 05293105  | 1238          | 05587188  | 1283          | 05885898  | 1328          | 06189110  |
| 1194          | 05299589  | 1239          | 05593776  | 1284          | 05892588  | 1329          | 06195898  |
| 1195          | 05306075  | 1240          | 05600367  | 1285          | 05899279  | 1330          | 06202688  |
| 1196          | 05312564  | 1241          | 05606960  | 1286          | 05905973  | 1331          | 06209481  |
| 1197          | 05319055  | 1242          | 05613555  | 1287          | 05912670  | 1332          | 06216276  |
| 1198          | 05325548  | 1243          | 05620152  | 1288          | 05919368  | 1333          | 06223073  |
| 1199          | 05332044  | 1244          | 05626752  | 1289          | 05926069  | 1334          | 06229872  |
| 1200          | 05338542  | 1245          | 05633353  | 1290          | 05932772  | 1335          | 06236673  |
| 1201          | 05345042  | 1246          | 05639958  | 1291          | 05939477  | 1336          | 06243476  |
| 1202          | 05351545  | 1247          | 05646564  | 1292          | 05946184  | 1337          | 06250282  |
| 1203          | 05358050  | 1248          | 05653173  | 1293          | 05952894  | 1338          | 06257090  |
| 1204          | 05364558  | 1249          | 05659784  | 1294          | 05959605  | 1339          | 06263899  |
| 1205          | 05371067  | 1250          | 05666397  | 1295          | 05966319  | 1340          | 06270711  |
| 1206          | 05377579  | 1251          | 05673012  | 1296          | 05973035  | 1341          | 06277525  |
| 1207          | 05384094  | 1252          | 05679730  | 1297          | 05979754  | 1342          | 06284342  |
| 1208          | 05390610  | 1253          | 05686250  | 1298          | 05986474  | 1343          | 06291160  |
| 1209          | 05397130  | 1254          | 05692873  | 1299          | 05993197  | 1344          | 06297981  |
| 1210          | 05403651  | 1255          | 05699497  | 1300          | 05999922  | 1345          | 06304803  |
| 1211          | 05410175  | 1256          | 05706124  | 1301          | 06006649  | 1346          | 06311628  |
| 1212          | 05416701  | 1257          | 05712753  | 1302          | 06013379  | 1347          | 06318455  |
| 1213          | 05423229  | 1258          | 05719384  | 1303          | 06020110  | 1348          | 06325284  |
| 1214          | 05429760  | 1259          | 05726018  | 1304          | 06026844  | 1349          | 06332116  |
| 1215          | 05436293  | 1260          | 05732654  | 1305          | 06033580  | 1350          | 06338949  |
| 1216          | 05442828  | 1261          | 05739292  | 1306          | 06040318  | 1351          | 06345785  |
| 1217          | 05449366  | 1262          | 05745932  | 1307          | 06047058  | 1352          | 06352622  |
| 1218          | 05455906  | 1263          | 05752575  | 1308          | 06053801  | 1353          | 06359462  |
| 1219          | 05462448  | 1264          | 05759220  | 1309          | 06060546  | 1354          | 06366304  |
| 1220          | 05468992  | 1265          | 05765867  | 1310          | 06067293  | 1355          | 06373148  |
| 1221          | 05475539  | 1266          | 05772516  | 1311          | 06074042  | 1356          | 06379994  |
| 1222          | 05482088  | 1267          | 05779168  | 1312          | 06080793  | 1357          | 06386843  |
| 1223          | 05488640  | 1268          | 05785822  | 1313          | 06087546  | 1358          | 06393693  |
| 1224          | 05495194  | 1269          | 05792478  | 1314          | 06094302  | 1359          | 06400546  |
| 1225          | 05501750  | 1270          | 05799136  | 1315          | 06101060  | 1360          | 06407400  |
| 1226          | 05508308  | 1271          | 05805797  | 1316          | 06107820  | 1361          | 06414257  |
| 1227          | 05514869  | 1272          | 05812460  | 1317          | 06114582  | 1362          | 06421116  |
| 1228          | 05521432  | 1273          | 05819125  | 1318          | 06121347  | 1363          | 06427977  |
| 1229          | 05527997  | 1274          | 05825792  | 1319          | 06128113  | 1364          | 06434840  |
| 1230          | 05534565  | 1275          | 05832461  | 1320          | 06134882  | 1365          | 06441706  |
| 1231          | 05541135  | 1276          | 05839133  | 1321          | 06141653  | 1366          | 06448573  |
| 1232          | 05547707  | 1277          | 05845807  | 1322          | 06148426  | 1367          | 06455443  |
| 1233          | 05554281  | 1278          | 05852483  | 1323          | 06155201  | 1368          | 06462314  |
| 1234          | 05560858  | 1279          | 05859162  | 1324          | 06161978  | 1369          | 06469188  |
| 1235          | 05567437  | 1280          | 05865843  | 1325          | 06168758  | 1370          | 06476064  |
| 1236          | 05574019  | 1281          | 05872525  | 1326          | 06175540  | 1371          | 06482942  |
| 1237          | 05580602  | 1282          | 05879211  | 1327          | 06182324  | 1372          | 06489822  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 1373          | 06496704  | 1418          | 06808567  | 1463          | 07124589  | 1508          | 07444665  |
| 1374          | 06503589  | 1419          | 06815545  | 1464          | 07131659  | 1509          | 07451823  |
| 1375          | 06510475  | 1420          | 06822525  | 1465          | 07138730  | 1510          | 07458983  |
| 1376          | 06517364  | 1421          | 06829507  | 1466          | 07145803  | 1511          | 07466145  |
| 1377          | 06524254  | 1422          | 06836491  | 1467          | 07152878  | 1512          | 07473309  |
| 1378          | 06531147  | 1423          | 06843478  | 1468          | 07159955  | 1513          | 07480475  |
| 1379          | 06538042  | 1424          | 06850466  | 1469          | 07167034  | 1514          | 07487643  |
| 1380          | 06544939  | 1425          | 06857456  | 1470          | 07174115  | 1515          | 07494812  |
| 1381          | 06551838  | 1426          | 06864448  | 1471          | 07181198  | 1516          | 07501984  |
| 1382          | 06558739  | 1427          | 06871443  | 1472          | 07188284  | 1517          | 07509158  |
| 1383          | 06565642  | 1428          | 06878439  | 1473          | 07195371  | 1518          | 07516333  |
| 1384          | 06572548  | 1429          | 06885437  | 1474          | 07202460  | 1519          | 07523511  |
| 1385          | 06579455  | 1430          | 06892438  | 1475          | 07209551  | 1520          | 07530690  |
| 1386          | 06586365  | 1431          | 06899440  | 1476          | 07216644  | 1521          | 07537872  |
| 1387          | 06593276  | 1432          | 06906445  | 1477          | 07223739  | 1522          | 07545055  |
| 1388          | 06600190  | 1433          | 06913451  | 1478          | 07230836  | 1523          | 07552240  |
| 1389          | 06607106  | 1434          | 06920460  | 1479          | 07237935  | 1524          | 07559427  |
| 1390          | 06614024  | 1435          | 06927470  | 1480          | 07245036  | 1525          | 07566616  |
| 1391          | 06620944  | 1436          | 06934483  | 1481          | 07252139  | 1526          | 07573808  |
| 1392          | 06627866  | 1437          | 06941498  | 1482          | 07259244  | 1527          | 07581001  |
| 1393          | 06634790  | 1438          | 06948515  | 1483          | 07266351  | 1528          | 07588195  |
| 1394          | 06641716  | 1439          | 06955533  | 1484          | 07273460  | 1529          | 07595392  |
| 1395          | 06648644  | 1440          | 06962554  | 1485          | 07280571  | 1530          | 07602591  |
| 1396          | 06655575  | 1441          | 06969577  | 1486          | 07287684  | 1531          | 07609792  |
| 1397          | 06662507  | 1442          | 06976602  | 1487          | 07294798  | 1532          | 07616994  |
| 1398          | 06669442  | 1443          | 06983629  | 1488          | 07301915  | 1533          | 07624199  |
| 1399          | 06676378  | 1444          | 06990657  | 1489          | 07309034  | 1534          | 07631406  |
| 1400          | 06683317  | 1445          | 06997688  | 1490          | 07316155  | 1535          | 07638614  |
| 1401          | 06690258  | 1446          | 07004721  | 1491          | 07323278  | 1536          | 07645824  |
| 1402          | 06697201  | 1447          | 07011756  | 1492          | 07330402  | 1537          | 07653037  |
| 1403          | 06704146  | 1448          | 07018793  | 1493          | 07337529  | 1538          | 07660251  |
| 1404          | 06711093  | 1449          | 07025832  | 1494          | 07344658  | 1539          | 07667467  |
| 1405          | 06718042  | 1450          | 07032873  | 1495          | 07351788  | 1540          | 07674685  |
| 1406          | 06724993  | 1451          | 07039916  | 1496          | 07358921  | 1541          | 07681905  |
| 1407          | 06731946  | 1452          | 07046961  | 1497          | 07366056  | 1542          | 07689127  |
| 1408          | 06738901  | 1453          | 07054008  | 1498          | 07373192  | 1543          | 07696350  |
| 1409          | 06745859  | 1454          | 07061057  | 1499          | 07380331  | 1544          | 07703576  |
| 1410          | 06752818  | 1455          | 07068108  | 1500          | 07387471  | 1545          | 07710804  |
| 1411          | 06759780  | 1456          | 07075162  | 1501          | 07394613  | 1546          | 07718033  |
| 1412          | 06766743  | 1457          | 07082217  | 1502          | 07401758  | 1547          | 07725265  |
| 1413          | 06773709  | 1458          | 07089274  | 1503          | 07408904  | 1548          | 07732498  |
| 1414          | 06780676  | 1459          | 07096333  | 1504          | 07416052  | 1549          | 07739733  |
| 1415          | 06787646  | 1460          | 07103394  | 1505          | 07423203  | 1550          | 07746970  |
| 1416          | 06794618  | 1461          | 07110457  | 1506          | 07430355  | 1551          | 07754209  |
| 1417          | 06801592  | 1462          | 07117522  | 1507          | 07437509  | 1552          | 07761450  |



| Veri.<br>fine | Seg. area | Veri.<br>fine | Seg. area | Veri.<br>fine | Seg. area | Veri.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 1553          | 07768693  | 1598          | 08096576  | 1643          | 08428219  | 1688          | 08763532  |
| 1554          | 07775938  | 1599          | 08103905  | 1644          | 08435631  | 1689          | 08771024  |
| 1555          | 07783185  | 1600          | 08111236  | 1645          | 08443044  | 1690          | 08778518  |
| 1556          | 07790433  | 1601          | 08118569  | 1646          | 08450460  | 1691          | 08786014  |
| 1557          | 07797684  | 1602          | 08125904  | 1647          | 08457877  | 1692          | 08793512  |
| 1558          | 07804936  | 1603          | 08133241  | 1648          | 08465296  | 1693          | 08801011  |
| 1559          | 07812190  | 1604          | 08140580  | 1649          | 08472717  | 1694          | 08808512  |
| 1560          | 07819446  | 1605          | 08147920  | 1650          | 08480140  | 1695          | 08816015  |
| 1561          | 07826704  | 1606          | 08155262  | 1651          | 08487564  | 1696          | 08823520  |
| 1562          | 07833964  | 1607          | 08162607  | 1652          | 08494991  | 1697          | 08831027  |
| 1563          | 07841226  | 1608          | 08169953  | 1653          | 08502419  | 1698          | 08838535  |
| 1564          | 07848490  | 1609          | 08177300  | 1654          | 08509849  | 1699          | 08846045  |
| 1565          | 07855756  | 1610          | 08184650  | 1655          | 08517281  | 1700          | 08853557  |
| 1566          | 07863023  | 1611          | 08192002  | 1656          | 08524714  | 1701          | 08861070  |
| 1567          | 07870292  | 1612          | 08199355  | 1657          | 08532149  | 1702          | 08868585  |
| 1568          | 07877564  | 1613          | 08206710  | 1658          | 08539587  | 1703          | 08876103  |
| 1569          | 07884837  | 1614          | 08214067  | 1659          | 08547025  | 1704          | 08883621  |
| 1570          | 07892112  | 1615          | 08221426  | 1660          | 08554466  | 1705          | 08891142  |
| 1571          | 07899389  | 1616          | 08228789  | 1661          | 08561909  | 1706          | 08898664  |
| 1572          | 07906668  | 1617          | 08236150  | 1662          | 08569353  | 1707          | 08906188  |
| 1573          | 07913949  | 1618          | 08243514  | 1663          | 08576799  | 1708          | 08913714  |
| 1574          | 07921231  | 1619          | 08250880  | 1664          | 08584247  | 1709          | 08921242  |
| 1575          | 07928516  | 1620          | 08258248  | 1665          | 08591697  | 1710          | 08928771  |
| 1576          | 07935802  | 1621          | 08265618  | 1666          | 08599148  | 1711          | 08936302  |
| 1577          | 07943090  | 1622          | 08272990  | 1667          | 08606601  | 1712          | 08943835  |
| 1578          | 07950380  | 1623          | 08280364  | 1668          | 08614056  | 1713          | 08951369  |
| 1579          | 07957672  | 1624          | 08287739  | 1669          | 08621513  | 1714          | 08958906  |
| 1580          | 07964966  | 1625          | 08295116  | 1670          | 08628972  | 1715          | 08966444  |
| 1581          | 07972262  | 1626          | 08302495  | 1671          | 08636432  | 1716          | 08973983  |
| 1582          | 07979560  | 1627          | 08309876  | 1672          | 08643894  | 1717          | 08981525  |
| 1583          | 07986859  | 1628          | 08317259  | 1673          | 08651358  | 1718          | 08989068  |
| 1584          | 07994161  | 1629          | 08324644  | 1674          | 08658824  | 1719          | 08996613  |
| 1585          | 08001464  | 1630          | 08332030  | 1675          | 08666292  | 1720          | 09004160  |
| 1586          | 08008769  | 1631          | 08339418  | 1676          | 08673761  | 1721          | 09011709  |
| 1587          | 08016076  | 1632          | 08346808  | 1677          | 08681232  | 1722          | 09019259  |
| 1588          | 08023385  | 1633          | 08354200  | 1678          | 08688705  | 1723          | 09026811  |
| 1589          | 08030696  | 1634          | 08361594  | 1679          | 08696180  | 1724          | 09034364  |
| 1590          | 08038008  | 1635          | 08368989  | 1680          | 08703656  | 1725          | 09041920  |
| 1591          | 08045323  | 1636          | 08376387  | 1681          | 08711134  | 1726          | 09049477  |
| 1592          | 08052639  | 1637          | 08383786  | 1682          | 08718614  | 1727          | 09057036  |
| 1593          | 08059957  | 1638          | 08391187  | 1683          | 08726096  | 1728          | 09064596  |
| 1594          | 08067277  | 1639          | 08398590  | 1684          | 08733580  | 1729          | 09072159  |
| 1595          | 08074599  | 1640          | 08405994  | 1685          | 08741065  | 1730          | 09079723  |
| 1596          | 08081923  | 1641          | 08413401  | 1686          | 08748552  | 1731          | 09087289  |
| 1597          | 08089248  | 1642          | 08420809  | 1687          | 08756041  | 1732          | 09094856  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 1733          | 09102425  | 1778          | 09444815  | 1823          | 09790618  | 1868          | 10139754  |
| 1734          | 09109996  | 1779          | 09452463  | 1824          | 09798341  | 1869          | 10147549  |
| 1735          | 09117569  | 1780          | 09460112  | 1825          | 09806065  | 1870          | 10155347  |
| 1736          | 09125143  | 1781          | 09467764  | 1826          | 09813791  | 1871          | 10163146  |
| 1737          | 09132720  | 1782          | 09475416  | 1827          | 09821519  | 1872          | 10170947  |
| 1738          | 09140298  | 1783          | 09483071  | 1828          | 09829248  | 1873          | 10178749  |
| 1739          | 09147877  | 1784          | 09490727  | 1829          | 09836979  | 1874          | 10186553  |
| 1740          | 09155458  | 1785          | 09498385  | 1830          | 09844711  | 1875          | 10194358  |
| 1741          | 09163042  | 1786          | 09506044  | 1831          | 09852445  | 1876          | 10202165  |
| 1742          | 09170626  | 1787          | 09513705  | 1832          | 09860181  | 1877          | 10209974  |
| 1743          | 09178213  | 1788          | 09521368  | 1833          | 09867919  | 1878          | 10217784  |
| 1744          | 09185801  | 1789          | 09529033  | 1834          | 09875658  | 1879          | 10225596  |
| 1745          | 09193391  | 1790          | 09536699  | 1835          | 09883398  | 1880          | 10233409  |
| 1746          | 09200983  | 1791          | 09544367  | 1836          | 09891141  | 1881          | 10241224  |
| 1747          | 09208576  | 1792          | 09552036  | 1837          | 09898885  | 1882          | 10249041  |
| 1748          | 09216171  | 1793          | 09559708  | 1838          | 09906630  | 1883          | 10256859  |
| 1749          | 09223768  | 1794          | 09567381  | 1839          | 09914377  | 1884          | 10264679  |
| 1750          | 09231366  | 1795          | 09575055  | 1840          | 09922126  | 1885          | 10272500  |
| 1751          | 09238966  | 1796          | 09582731  | 1841          | 09929877  | 1886          | 10280323  |
| 1752          | 09246568  | 1797          | 09590409  | 1842          | 09937629  | 1887          | 10288148  |
| 1753          | 09254172  | 1798          | 09598089  | 1843          | 09945383  | 1888          | 10295974  |
| 1754          | 09261777  | 1799          | 09605770  | 1844          | 09953138  | 1889          | 10303802  |
| 1755          | 09269384  | 1800          | 09613453  | 1845          | 09960895  | 1890          | 10311631  |
| 1756          | 09276993  | 1801          | 09621138  | 1846          | 09968654  | 1891          | 10319462  |
| 1757          | 09284603  | 1802          | 09628824  | 1847          | 09976414  | 1892          | 10327295  |
| 1758          | 09292216  | 1803          | 09636512  | 1848          | 09984176  | 1893          | 10335129  |
| 1759          | 09299829  | 1804          | 09644201  | 1849          | 09991939  | 1894          | 10342965  |
| 1760          | 09307445  | 1805          | 09651893  | 1850          | 09999704  | 1895          | 10350802  |
| 1761          | 09315062  | 1806          | 09659585  | 1851          | 10007471  | 1896          | 10358641  |
| 1762          | 09322681  | 1807          | 09667280  | 1852          | 10015240  | 1897          | 10366481  |
| 1763          | 09330302  | 1808          | 09674976  | 1853          | 10023010  | 1898          | 10374324  |
| 1764          | 09337924  | 1809          | 09682674  | 1854          | 10030781  | 1899          | 10382167  |
| 1765          | 09345548  | 1810          | 09690374  | 1855          | 10038555  | 1900          | 10390013  |
| 1766          | 09353174  | 1811          | 09698075  | 1856          | 10046330  | 1901          | 10397859  |
| 1767          | 09360802  | 1812          | 09705778  | 1857          | 10054106  | 1902          | 10405708  |
| 1768          | 09368431  | 1813          | 09713482  | 1858          | 10061884  | 1903          | 10413558  |
| 1769          | 09376062  | 1814          | 09721188  | 1859          | 10069664  | 1904          | 10421409  |
| 1770          | 09383694  | 1815          | 09728896  | 1860          | 10077445  | 1905          | 10429262  |
| 1771          | 09391328  | 1816          | 09736606  | 1861          | 10085228  | 1906          | 10437117  |
| 1772          | 09398964  | 1817          | 09744317  | 1862          | 10093012  | 1907          | 10444973  |
| 1773          | 09406602  | 1818          | 09752029  | 1863          | 10100799  | 1908          | 10452831  |
| 1774          | 09414241  | 1819          | 09759744  | 1864          | 10108586  | 1909          | 10460691  |
| 1775          | 09421882  | 1820          | 09767460  | 1865          | 10116376  | 1910          | 10468552  |
| 1776          | 09429525  | 1821          | 09775178  | 1866          | 10124167  | 1911          | 10476414  |
| 1777          | 09437169  | 1822          | 09782897  | 1867          | 10131959  | 1912          | 10484278  |



## CIRCULAR SEGMENTS.

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| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 1913          | 10492144  | 1958          | 10847713  | 2003          | 11206387  | 2048          | 11568094  |
| 1914          | 10500011  | 1959          | 10855650  | 2004          | 11214392  | 2049          | 11576166  |
| 1915          | 10507880  | 1960          | 10863589  | 2005          | 11222399  | 2050          | 11584239  |
| 1916          | 10515751  | 1961          | 10871529  | 2006          | 11230407  | 2051          | 11592314  |
| 1917          | 10523623  | 1962          | 10879471  | 2007          | 11238417  | 2052          | 11600390  |
| 1918          | 10531496  | 1963          | 10887414  | 2008          | 11246428  | 2053          | 11608468  |
| 1919          | 10539371  | 1964          | 10895359  | 2009          | 11254441  | 2054          | 11616547  |
| 1920          | 10547248  | 1965          | 10903305  | 2010          | 11262455  | 2055          | 11624628  |
| 1921          | 10555126  | 1966          | 10911253  | 2011          | 11270471  | 2056          | 11632710  |
| 1922          | 10563006  | 1967          | 10919202  | 2012          | 11278488  | 2057          | 11640793  |
| 1923          | 10570887  | 1968          | 10927153  | 2013          | 11286507  | 2058          | 11648878  |
| 1924          | 10578770  | 1969          | 10935105  | 2014          | 11294527  | 2059          | 11656965  |
| 1925          | 10586655  | 1970          | 10943059  | 2015          | 11302549  | 2060          | 11665053  |
| 1926          | 10594541  | 1971          | 10951014  | 2016          | 11310572  | 2061          | 11673142  |
| 1927          | 10602428  | 1972          | 10958971  | 2017          | 11318597  | 2062          | 11681233  |
| 1928          | 10610318  | 1973          | 10966930  | 2018          | 11326623  | 2063          | 11689325  |
| 1929          | 10618208  | 1974          | 10974890  | 2019          | 11334650  | 2064          | 11697419  |
| 1930          | 10626101  | 1975          | 10982851  | 2020          | 11342679  | 2065          | 11705514  |
| 1931          | 10633994  | 1976          | 10990814  | 2021          | 11350710  | 2066          | 11713610  |
| 1932          | 10641890  | 1977          | 10998779  | 2022          | 11358742  | 2067          | 11721708  |
| 1933          | 10649787  | 1978          | 11006745  | 2023          | 11366776  | 2068          | 11729808  |
| 1934          | 10657685  | 1979          | 11014713  | 2024          | 11374811  | 2069          | 11737909  |
| 1935          | 10665585  | 1980          | 11022682  | 2025          | 11382847  | 2070          | 11746011  |
| 1936          | 10673487  | 1981          | 11030652  | 2026          | 11390885  | 2071          | 11754115  |
| 1937          | 10681390  | 1982          | 11038624  | 2027          | 11398925  | 2072          | 11762220  |
| 1938          | 10689295  | 1983          | 11046598  | 2028          | 11406966  | 2073          | 11770327  |
| 1939          | 10697201  | 1984          | 11054573  | 2029          | 11415008  | 2074          | 11778435  |
| 1940          | 10705109  | 1985          | 11062550  | 2030          | 11423052  | 2075          | 11786545  |
| 1941          | 10713018  | 1986          | 11070528  | 2031          | 11431097  | 2076          | 11794656  |
| 1942          | 10720929  | 1987          | 11078507  | 2032          | 11439144  | 2077          | 11802768  |
| 1943          | 10728842  | 1988          | 11086489  | 2033          | 11447193  | 2078          | 11810882  |
| 1944          | 10736756  | 1989          | 11094471  | 2034          | 11455242  | 2079          | 11818998  |
| 1945          | 10744671  | 1990          | 11102456  | 2035          | 11463294  | 2080          | 11827114  |
| 1946          | 10752588  | 1991          | 11110441  | 2036          | 11471346  | 2081          | 11835233  |
| 1947          | 10760507  | 1992          | 11118429  | 2037          | 11479401  | 2082          | 11843352  |
| 1948          | 10768427  | 1993          | 11126417  | 2038          | 11487456  | 2083          | 11851474  |
| 1949          | 10776349  | 1994          | 11134407  | 2039          | 11495514  | 2084          | 11859596  |
| 1950          | 10784272  | 1995          | 11142399  | 2040          | 11503572  | 2085          | 11867720  |
| 1951          | 10792197  | 1996          | 11150392  | 2041          | 11511632  | 2086          | 11875846  |
| 1952          | 10800123  | 1997          | 11158387  | 2042          | 11519694  | 2087          | 11883972  |
| 1953          | 10808051  | 1998          | 11166383  | 2043          | 11527757  | 2088          | 11892101  |
| 1954          | 10815980  | 1999          | 11174381  | 2044          | 11535821  | 2089          | 11900230  |
| 1955          | 10823911  | 2000          | 11182380  | 2045          | 11543887  | 2090          | 11908362  |
| 1956          | 10831844  | 2001          | 11190381  | 2046          | 11551955  | 2091          | 11916494  |
| 1957          | 10839778  | 2002          | 11198383  | 2047          | 11560024  | 2092          | 11924628  |



| Ver.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|--------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 2093         | 11932764  | 2138          | 12300328  | 2183          | 12670719  | 2228          | 13043872  |
| 2094         | 11940901  | 2139          | 12308528  | 2184          | 12678981  | 2229          | 13052195  |
| 2095         | 11949039  | 2140          | 12316730  | 2185          | 12687245  | 2230          | 13060520  |
| 2096         | 11957179  | 2141          | 12324933  | 2186          | 12695511  | 2231          | 13068846  |
| 2097         | 11965320  | 2142          | 12333138  | 2187          | 12703777  | 2232          | 13077173  |
| 2098         | 11973462  | 2143          | 12341344  | 2188          | 12712045  | 2233          | 13085501  |
| 2099         | 11981606  | 2144          | 12349551  | 2189          | 12720314  | 2234          | 13093831  |
| 2100         | 11989752  | 2145          | 12357760  | 2190          | 12728585  | 2235          | 13102162  |
| 2101         | 11997899  | 2146          | 12365970  | 2191          | 12736857  | 2236          | 13110495  |
| 2102         | 12006047  | 2147          | 12374182  | 2192          | 12745131  | 2237          | 13118828  |
| 2103         | 12014197  | 2148          | 12382395  | 2193          | 12753405  | 2238          | 13127164  |
| 2104         | 12022348  | 2149          | 12390609  | 2194          | 12761682  | 2239          | 13135500  |
| 2105         | 12030501  | 2150          | 12398825  | 2195          | 12769959  | 2240          | 13143838  |
| 2106         | 12038655  | 2151          | 12407042  | 2196          | 12778238  | 2241          | 13152177  |
| 2107         | 12046810  | 2152          | 12415261  | 2197          | 12786518  | 2242          | 13160517  |
| 2108         | 12054967  | 2153          | 12423481  | 2198          | 12794800  | 2243          | 13168859  |
| 2109         | 12063125  | 2154          | 12431702  | 2199          | 12803082  | 2244          | 13177202  |
| 2110         | 12071285  | 2155          | 12439924  | 2200          | 12811367  | 2245          | 13185546  |
| 2111         | 12079446  | 2156          | 12448149  | 2201          | 12819652  | 2246          | 13193892  |
| 2112         | 12087608  | 2157          | 12456374  | 2202          | 12827939  | 2247          | 13202239  |
| 2113         | 12095772  | 2158          | 12464601  | 2203          | 12836228  | 2248          | 13210588  |
| 2114         | 12103938  | 2159          | 12472829  | 2204          | 12844517  | 2249          | 13218937  |
| 2115         | 12112104  | 2160          | 12481059  | 2205          | 12852808  | 2250          | 13227288  |
| 2116         | 12120272  | 2161          | 12489290  | 2206          | 12861101  | 2251          | 13235641  |
| 2117         | 12128442  | 2162          | 12497522  | 2207          | 12869394  | 2252          | 13243994  |
| 2118         | 12136613  | 2163          | 12505756  | 2208          | 12877689  | 2253          | 13252349  |
| 2119         | 12144785  | 2164          | 12513991  | 2209          | 12885986  | 2254          | 13260705  |
| 2120         | 12152959  | 2165          | 12522227  | 2210          | 12894283  | 2255          | 13269063  |
| 2121         | 12161134  | 2166          | 12530465  | 2211          | 12902583  | 2256          | 13277422  |
| 2122         | 12169311  | 2167          | 12538704  | 2212          | 12910883  | 2257          | 13285782  |
| 2123         | 12177489  | 2168          | 12546945  | 2213          | 12919185  | 2258          | 13294143  |
| 2124         | 12185668  | 2169          | 12555187  | 2214          | 12927488  | 2259          | 13302506  |
| 2125         | 12193849  | 2170          | 12563430  | 2215          | 12935792  | 2260          | 13310870  |
| 2126         | 12202031  | 2171          | 12571675  | 2216          | 12944098  | 2261          | 13319236  |
| 2127         | 12210215  | 2172          | 12579921  | 2217          | 12952405  | 2262          | 13327603  |
| 2128         | 12218400  | 2173          | 12588169  | 2218          | 12960714  | 2263          | 13335971  |
| 2129         | 12226587  | 2174          | 12596418  | 2219          | 12969024  | 2264          | 13344340  |
| 2130         | 12234774  | 2175          | 12604668  | 2220          | 12977335  | 2265          | 13352711  |
| 2131         | 12242964  | 2176          | 12612919  | 2221          | 12985647  | 2266          | 13361083  |
| 2132         | 12251154  | 2177          | 12621172  | 2222          | 12993961  | 2267          | 13369456  |
| 2133         | 12259346  | 2178          | 12629427  | 2223          | 13002276  | 2268          | 13377831  |
| 2134         | 12267540  | 2179          | 12637682  | 2224          | 13010593  | 2269          | 13386206  |
| 2135         | 12275735  | 2180          | 12645939  | 2225          | 13018911  | 2270          | 13394584  |
| 2136         | 12283931  | 2181          | 12654198  | 2226          | 13027230  | 2271          | 13402962  |
| 2137         | 12292129  | 2182          | 12662458  | 2227          | 13035550  | 2272          | 13411342  |



| Verl.<br>fine | Seg. area | Verl.<br>fine | Seg. area | Verl.<br>fine | Seg. area | Verl.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 2372          | 13419723  | 2318          | 13798210  | 2363          | 14179270  | 2408          | 14562845  |
| 2274          | 13428105  | 2319          | 13806650  | 2364          | 14187767  | 2409          | 14571397  |
| 2275          | 13436489  | 2320          | 13815091  | 2365          | 14196265  | 2410          | 14579950  |
| 2276          | 13444874  | 2321          | 13823534  | 2366          | 14204764  | 2411          | 14588505  |
| 2277          | 13453260  | 2322          | 13831978  | 2367          | 14213265  | 2412          | 14597060  |
| 2278          | 13461648  | 2323          | 13840424  | 2368          | 14221767  | 2413          | 14605617  |
| 2279          | 13470037  | 2324          | 13848870  | 2369          | 14230270  | 2414          | 14614175  |
| 2280          | 13478427  | 2325          | 13857318  | 2370          | 14238774  | 2415          | 14622735  |
| 2281          | 13486819  | 2326          | 13865767  | 2371          | 14247279  | 2416          | 14631295  |
| 2282          | 13495212  | 2327          | 13874218  | 2372          | 14255786  | 2417          | 14639857  |
| 2283          | 13503606  | 2328          | 13882669  | 2373          | 14264294  | 2418          | 14648419  |
| 2284          | 13512001  | 2329          | 13891123  | 2374          | 14272803  | 2419          | 14656984  |
| 2285          | 13520398  | 2330          | 13899577  | 2375          | 14281314  | 2420          | 14665549  |
| 2286          | 13528796  | 2331          | 13908032  | 2376          | 14289825  | 2421          | 14674115  |
| 2287          | 13537195  | 2332          | 14916489  | 2377          | 14298338  | 2422          | 14682683  |
| 2288          | 13545595  | 2333          | 14924947  | 2378          | 14306852  | 2423          | 14691252  |
| 2289          | 13553997  | 2334          | 14933406  | 2379          | 14315368  | 2424          | 14699822  |
| 2290          | 13562400  | 2335          | 14941867  | 2380          | 14323884  | 2425          | 14708393  |
| 2291          | 13570805  | 2336          | 14950328  | 2381          | 14332402  | 2426          | 14716966  |
| 2292          | 13579211  | 2337          | 14958791  | 2382          | 14340921  | 2427          | 14725540  |
| 2293          | 13587618  | 2338          | 14967256  | 2383          | 14349441  | 2428          | 14734114  |
| 2294          | 13596026  | 2339          | 14975721  | 2384          | 14357963  | 2429          | 14742691  |
| 2295          | 13604435  | 2340          | 14984188  | 2385          | 14366485  | 2430          | 14751268  |
| 2296          | 13612846  | 2341          | 14992656  | 2386          | 14375009  | 2431          | 14759846  |
| 2297          | 13621259  | 2342          | 14001126  | 2387          | 14383535  | 2432          | 14768426  |
| 2298          | 13629672  | 2343          | 14009596  | 2388          | 14392061  | 2433          | 14777007  |
| 2299          | 13638087  | 2344          | 14018068  | 2389          | 14400589  | 2434          | 14785589  |
| 2300          | 13646503  | 2345          | 14026541  | 2390          | 14409117  | 2435          | 14794172  |
| 2301          | 13654920  | 2346          | 14035015  | 2391          | 14417647  | 2436          | 14802757  |
| 2302          | 13663339  | 2347          | 14043491  | 2392          | 14426179  | 2437          | 14811342  |
| 2303          | 13671758  | 2348          | 14051968  | 2393          | 14434711  | 2438          | 14819929  |
| 2304          | 13680180  | 2349          | 14060446  | 2394          | 14443245  | 2439          | 14828517  |
| 2305          | 13688602  | 2350          | 14068925  | 2395          | 14451780  | 2440          | 14837107  |
| 2306          | 13697026  | 2351          | 14077406  | 2396          | 14460316  | 2441          | 14845697  |
| 2307          | 13705451  | 2352          | 14085888  | 2397          | 14468854  | 2442          | 14854289  |
| 2308          | 13713877  | 2353          | 14094371  | 2398          | 14477392  | 2443          | 14862882  |
| 2309          | 13722304  | 2354          | 14102855  | 2399          | 14485932  | 2444          | 14871476  |
| 2310          | 13730733  | 2355          | 14111341  | 2400          | 14494473  | 2445          | 14880071  |
| 2311          | 13739163  | 2356          | 14119828  | 2401          | 14503015  | 2446          | 14888667  |
| 2312          | 13747595  | 2357          | 14128316  | 2402          | 14511559  | 2447          | 14897265  |
| 2313          | 13756027  | 2358          | 14136805  | 2403          | 14520103  | 2448          | 14905864  |
| 2314          | 13764461  | 2359          | 14145296  | 2404          | 14528649  | 2449          | 14914464  |
| 2315          | 13772896  | 2360          | 14153787  | 2405          | 14537197  | 2450          | 14923065  |
| 2316          | 13781333  | 2361          | 14162281  | 2406          | 14545745  | 2451          | 14931667  |
| 2317          | 13789771  | 2362          | 14170775  | 2407          | 14554294  | 2452          | 14940271  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 2453          | 14948875  | 2498          | 15337303  | 2543          | 15728071  | 2588          | 16121125  |
| 2454          | 14957481  | 2499          | 15345961  | 2544          | 15736781  | 2589          | 16129885  |
| 2455          | 14966088  | 2500          | 15354621  | 2545          | 15745492  | 2590          | 16138646  |
| 2456          | 14974696  | 2501          | 15363282  | 2546          | 15754205  | 2591          | 16147409  |
| 2457          | 14983306  | 2502          | 15371944  | 2547          | 15762918  | 2592          | 16156172  |
| 2458          | 14991917  | 2503          | 15380607  | 2548          | 15771632  | 2593          | 16164937  |
| 2459          | 15000528  | 2504          | 15389271  | 2549          | 15780348  | 2594          | 16173702  |
| 2460          | 15009141  | 2505          | 15397937  | 2550          | 15789064  | 2595          | 16182469  |
| 2461          | 15017755  | 2506          | 15406603  | 2551          | 15797782  | 2596          | 16191237  |
| 2462          | 15026371  | 2507          | 15415271  | 2552          | 15806501  | 2597          | 16200005  |
| 2463          | 15034987  | 2508          | 15423940  | 2553          | 15815221  | 2598          | 16208775  |
| 2464          | 15043605  | 2509          | 15432610  | 2554          | 15823942  | 2599          | 16217546  |
| 2465          | 15052224  | 2510          | 15441281  | 2555          | 15832665  | 2600          | 16226319  |
| 2466          | 15060844  | 2511          | 15449954  | 2556          | 15841388  | 2601          | 16235092  |
| 2467          | 15069465  | 2512          | 15458627  | 2557          | 15850113  | 2602          | 16243866  |
| 2468          | 15078088  | 2513          | 15467302  | 2558          | 15858838  | 2603          | 16252641  |
| 2469          | 15086711  | 2514          | 15475978  | 2559          | 15867565  | 2604          | 16261418  |
| 2470          | 15095336  | 2515          | 15484655  | 2560          | 15876293  | 2605          | 16270196  |
| 2471          | 15103962  | 2516          | 15493333  | 2561          | 15885022  | 2606          | 16278974  |
| 2472          | 15112589  | 2517          | 15502012  | 2562          | 15893752  | 2607          | 16287754  |
| 2473          | 15121217  | 2518          | 15510692  | 2563          | 15902483  | 2608          | 16296535  |
| 2474          | 15129847  | 2519          | 15519374  | 2564          | 15911215  | 2609          | 16305317  |
| 2475          | 15138477  | 2520          | 15528056  | 2565          | 15919949  | 2610          | 16314100  |
| 2476          | 15147109  | 2521          | 15536740  | 2566          | 15928684  | 2611          | 16322884  |
| 2477          | 15155742  | 2522          | 15545425  | 2567          | 15937419  | 2612          | 16331669  |
| 2478          | 15164376  | 2523          | 15554111  | 2568          | 15946157  | 2613          | 16340456  |
| 2479          | 15173011  | 2524          | 15562798  | 2569          | 15954894  | 2614          | 16349243  |
| 2480          | 15181648  | 2525          | 15571487  | 2570          | 15963633  | 2615          | 16358032  |
| 2481          | 15190285  | 2526          | 15580176  | 2571          | 15972373  | 2616          | 16366821  |
| 2482          | 15198924  | 2527          | 15588867  | 2572          | 15981114  | 2617          | 16375612  |
| 2483          | 15207564  | 2528          | 15597559  | 2573          | 15989856  | 2618          | 16384404  |
| 2484          | 15216205  | 2529          | 15606252  | 2574          | 15998600  | 2619          | 16393196  |
| 2485          | 15224848  | 2530          | 15614946  | 2575          | 16007345  | 2620          | 16401990  |
| 2486          | 15233491  | 2531          | 15623641  | 2576          | 16016091  | 2621          | 16410785  |
| 2487          | 15242136  | 2532          | 15632337  | 2577          | 16024837  | 2622          | 16419581  |
| 2488          | 15250781  | 2533          | 15641035  | 2578          | 16033585  | 2623          | 16428379  |
| 2489          | 15259428  | 2534          | 15649733  | 2579          | 16042334  | 2624          | 16437177  |
| 2490          | 15268077  | 2535          | 15658433  | 2580          | 16051084  | 2625          | 16445976  |
| 2491          | 15276726  | 2536          | 15667134  | 2581          | 16059836  | 2626          | 16454777  |
| 2492          | 15285376  | 2537          | 15675836  | 2582          | 16068588  | 2627          | 16463578  |
| 2493          | 15294028  | 2538          | 15684539  | 2583          | 16077341  | 2628          | 16472381  |
| 2494          | 15302681  | 2539          | 15693243  | 2584          | 16086096  | 2629          | 16481184  |
| 2495          | 15311334  | 2540          | 15701949  | 2585          | 16094852  | 2630          | 16489989  |
| 2496          | 15319989  | 2541          | 15710655  | 2586          | 16103608  | 2631          | 16498795  |
| 2497          | 15328646  | 2542          | 15719363  | 2587          | 16112366  | 2632          | 16507602  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 2633          | 16516410  | 2678          | 16913870  | 2723          | 17313455  | 2768          | 17715111  |
| 2634          | 16525219  | 2679          | 16922727  | 2724          | 17322358  | 2769          | 17724060  |
| 2635          | 16534029  | 2680          | 16931585  | 2725          | 17331263  | 2770          | 17733010  |
| 2636          | 16542840  | 2681          | 16940444  | 2726          | 17340168  | 2771          | 17741961  |
| 2637          | 16551652  | 2682          | 16949304  | 2727          | 17349075  | 2772          | 17750913  |
| 2638          | 16560465  | 2683          | 16958165  | 2728          | 17357982  | 2773          | 17759865  |
| 2639          | 16569280  | 2684          | 16967027  | 2729          | 17366891  | 2774          | 17768819  |
| 2640          | 16578095  | 2685          | 16975890  | 2730          | 17375800  | 2775          | 17777774  |
| 2641          | 16586912  | 2686          | 16984754  | 2731          | 17384711  | 2776          | 17786730  |
| 2642          | 16595729  | 2687          | 16993619  | 2732          | 17393622  | 2777          | 17795687  |
| 2643          | 16604548  | 2688          | 17002485  | 2733          | 17402535  | 2778          | 17804644  |
| 2644          | 16613368  | 2689          | 17011352  | 2734          | 17411448  | 2779          | 17813603  |
| 2645          | 16622188  | 2690          | 17020221  | 2735          | 17420363  | 2780          | 17822563  |
| 2646          | 16631010  | 2691          | 17029090  | 2736          | 17429278  | 2781          | 17831524  |
| 2647          | 16639833  | 2692          | 17037960  | 2737          | 17438195  | 2782          | 17840486  |
| 2648          | 16648657  | 2693          | 17046832  | 2738          | 17447113  | 2783          | 17849448  |
| 2649          | 16657482  | 2694          | 17055704  | 2739          | 17456031  | 2784          | 17858412  |
| 2650          | 16666308  | 2695          | 17064578  | 2740          | 17464951  | 2785          | 17867377  |
| 2651          | 16675136  | 2696          | 17073452  | 2741          | 17473872  | 2786          | 17876342  |
| 2652          | 16683964  | 2697          | 17082328  | 2742          | 17482793  | 2787          | 17885309  |
| 2653          | 16692793  | 2698          | 17091204  | 2743          | 17491716  | 2788          | 17894277  |
| 2654          | 16701624  | 2699          | 17100082  | 2744          | 17500640  | 2789          | 17903245  |
| 2655          | 16710455  | 2700          | 17108961  | 2745          | 17509565  | 2790          | 17912215  |
| 2656          | 16719287  | 2701          | 17117840  | 2746          | 17518490  | 2791          | 17921186  |
| 2657          | 16728121  | 2702          | 17126721  | 2747          | 17527417  | 2792          | 17930157  |
| 2658          | 16736956  | 2703          | 17135603  | 2748          | 17536345  | 2793          | 17939130  |
| 2659          | 16745791  | 2704          | 17144486  | 2749          | 17545274  | 2794          | 17948104  |
| 2660          | 16754628  | 2705          | 17153370  | 2750          | 17554203  | 2795          | 17957079  |
| 2661          | 16763466  | 2706          | 17162254  | 2751          | 17563134  | 2796          | 17966054  |
| 2662          | 16772305  | 2707          | 17171140  | 2752          | 17572066  | 2797          | 17975030  |
| 2663          | 16781145  | 2708          | 17180027  | 2753          | 17580999  | 2798          | 17984008  |
| 2664          | 16789986  | 2709          | 17188915  | 2754          | 17589933  | 2799          | 17992986  |
| 2665          | 16798828  | 2710          | 17197804  | 2755          | 17598867  | 2800          | 18001966  |
| 2666          | 16807671  | 2711          | 17206694  | 2756          | 17607803  | 2801          | 18010946  |
| 2667          | 16816515  | 2712          | 17215585  | 2757          | 17616740  | 2802          | 18019928  |
| 2668          | 16825360  | 2713          | 17224477  | 2758          | 17625678  | 2803          | 18028910  |
| 2669          | 16834207  | 2714          | 17233371  | 2759          | 17634617  | 2804          | 18037894  |
| 2670          | 16843054  | 2715          | 17242265  | 2760          | 17643556  | 2805          | 18046878  |
| 2671          | 16851902  | 2716          | 17251160  | 2761          | 17652497  | 2806          | 18055863  |
| 2672          | 16860752  | 2717          | 17260056  | 2762          | 17661439  | 2807          | 18064850  |
| 2673          | 16869602  | 2718          | 17268953  | 2763          | 17670382  | 2808          | 18073837  |
| 2674          | 16878454  | 2719          | 17277852  | 2764          | 17679326  | 2809          | 18082825  |
| 2675          | 16887306  | 2720          | 17286751  | 2765          | 17688271  | 2810          | 18091815  |
| 2676          | 16896160  | 2721          | 17295651  | 2766          | 17697217  | 2811          | 18100805  |
| 2677          | 16905015  | 2722          | 17304553  | 2767          | 17706163  | 2812          | 18109796  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 2813          | 18118788  | 2858          | 18524435  | 2903          | 18932002  | 2948          | 19341440  |
| 2814          | 18127781  | 2859          | 18533471  | 2904          | 18941080  | 2949          | 19350560  |
| 2815          | 18136775  | 2860          | 18542509  | 2905          | 18950160  | 2950          | 19359680  |
| 2816          | 18145771  | 2861          | 18551547  | 2906          | 18959240  | 2951          | 19368801  |
| 2817          | 18154767  | 2862          | 18560586  | 2907          | 18968321  | 2952          | 19377924  |
| 2818          | 18163764  | 2863          | 18569626  | 2908          | 18977404  | 2953          | 19387047  |
| 2819          | 18172762  | 2864          | 18578667  | 2909          | 18986487  | 2954          | 19396171  |
| 2820          | 18181761  | 2865          | 18587709  | 2910          | 18995571  | 2955          | 19405296  |
| 2821          | 18190761  | 2866          | 18596752  | 2911          | 19004656  | 2956          | 19414421  |
| 2822          | 18199762  | 2867          | 18605796  | 2912          | 19013741  | 2957          | 19423548  |
| 2823          | 18208763  | 2868          | 18614841  | 2913          | 19022828  | 2958          | 19432676  |
| 2824          | 18217766  | 2869          | 18623887  | 2914          | 19031916  | 2959          | 19441804  |
| 2825          | 18226770  | 2870          | 18632934  | 2915          | 19041005  | 2960          | 19450933  |
| 2826          | 18235775  | 2871          | 18641981  | 2916          | 19050094  | 2961          | 19460064  |
| 2827          | 18244781  | 2872          | 18651030  | 2917          | 19059184  | 2962          | 19469195  |
| 2828          | 18253787  | 2873          | 18660080  | 2918          | 19068276  | 2963          | 19478327  |
| 2829          | 18262795  | 2874          | 18669130  | 2919          | 19077368  | 2964          | 19487460  |
| 2830          | 18271804  | 2875          | 18678182  | 2920          | 19086461  | 2965          | 19496594  |
| 2831          | 18280813  | 2876          | 18687234  | 2921          | 19095555  | 2966          | 19505728  |
| 2832          | 18289824  | 2877          | 18696287  | 2922          | 19104650  | 2967          | 19514864  |
| 2833          | 18298835  | 2878          | 18705342  | 2923          | 19113746  | 2968          | 19524001  |
| 2834          | 18307848  | 2879          | 18714397  | 2924          | 19122843  | 2969          | 19533138  |
| 2835          | 18316861  | 2880          | 18723453  | 2925          | 19131941  | 2970          | 19542276  |
| 2836          | 18325876  | 2881          | 18732510  | 2926          | 19141040  | 2971          | 19551415  |
| 2837          | 18334891  | 2882          | 18741568  | 2927          | 19150139  | 2972          | 19560556  |
| 2838          | 18343907  | 2883          | 18750627  | 2928          | 19159240  | 2973          | 19569696  |
| 2839          | 18352925  | 2884          | 18759687  | 2929          | 19168341  | 2974          | 19578838  |
| 2840          | 18361943  | 2885          | 18768748  | 2930          | 19177443  | 2975          | 19587981  |
| 2841          | 18370962  | 2886          | 18777810  | 2931          | 19186547  | 2976          | 19597125  |
| 2842          | 18379982  | 2887          | 18786872  | 2932          | 19195651  | 2977          | 19606269  |
| 2843          | 18389004  | 2888          | 18795936  | 2933          | 19204756  | 2978          | 19615415  |
| 2844          | 18398026  | 2889          | 18805001  | 2934          | 19213862  | 2979          | 19624561  |
| 2845          | 18407049  | 2890          | 18814066  | 2935          | 19222969  | 2980          | 19633708  |
| 2846          | 18416073  | 2891          | 18823132  | 2936          | 19232076  | 2981          | 19642856  |
| 2847          | 18425098  | 2892          | 18832200  | 2937          | 19241185  | 2982          | 19652005  |
| 2848          | 18434124  | 2893          | 18841268  | 2938          | 19250295  | 2983          | 19661155  |
| 2849          | 18443150  | 2894          | 18850337  | 2939          | 19259405  | 2984          | 19670305  |
| 2850          | 18452178  | 2895          | 18859407  | 2940          | 19268517  | 2985          | 19679457  |
| 2851          | 18461207  | 2896          | 18868479  | 2941          | 19277629  | 2986          | 19688609  |
| 2852          | 18470237  | 2897          | 18877551  | 2942          | 19286742  | 2987          | 19697763  |
| 2853          | 18479267  | 2898          | 18886623  | 2943          | 19295856  | 2988          | 19706917  |
| 2854          | 18488299  | 2899          | 18895697  | 2944          | 19304971  | 2989          | 19716072  |
| 2855          | 18497332  | 2900          | 18904772  | 2945          | 19314087  | 2990          | 19725228  |
| 2856          | 18506365  | 2901          | 18913848  | 2946          | 19323204  | 2991          | 19734385  |
| 2857          | 18515400  | 2902          | 18922924  | 2947          | 19332321  | 2992          | 19743542  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 2993          | 19752701  | 3038          | 20165737  | 3083          | 20580501  | 3128          | 20996946  |
| 2994          | 19761861  | 3039          | 20174935  | 3084          | 20589737  | 3129          | 21006219  |
| 2995          | 19771021  | 3040          | 20184134  | 3085          | 20598974  | 3130          | 21015493  |
| 2996          | 19780182  | 3041          | 20193335  | 3086          | 20608212  | 3131          | 21024767  |
| 2997          | 19789344  | 3042          | 20202535  | 3087          | 20617451  | 3132          | 21034043  |
| 2998          | 19798507  | 3043          | 20211737  | 3088          | 20626690  | 3133          | 21043319  |
| 2999          | 19807671  | 3044          | 20220940  | 3089          | 20635930  | 3134          | 21052596  |
| 3000          | 19816836  | 3045          | 20230143  | 3090          | 20645172  | 3135          | 21061874  |
| 3001          | 19826001  | 3046          | 20239348  | 3091          | 20654414  | 3136          | 21071153  |
| 3002          | 19835168  | 3047          | 20248553  | 3092          | 20663657  | 3137          | 21080432  |
| 3003          | 19844335  | 3048          | 20257759  | 3093          | 20672900  | 3138          | 21089713  |
| 3004          | 19853503  | 3049          | 20266966  | 3094          | 20682145  | 3139          | 21098994  |
| 3005          | 19862672  | 3050          | 20276173  | 3095          | 20691390  | 3140          | 21108276  |
| 3006          | 19871842  | 3051          | 20285382  | 3096          | 20700636  | 3141          | 21117558  |
| 3007          | 19881013  | 3052          | 20294591  | 3097          | 20709883  | 3142          | 21126842  |
| 3008          | 19890185  | 3053          | 20303802  | 3098          | 20719131  | 3143          | 21136126  |
| 3009          | 19899357  | 3054          | 20313013  | 3099          | 20728380  | 3144          | 21145411  |
| 3010          | 19908531  | 3055          | 20322225  | 3100          | 20737629  | 3145          | 21154697  |
| 3011          | 19917705  | 3056          | 20331438  | 3101          | 20746879  | 3146          | 21163984  |
| 3012          | 19926880  | 3057          | 20340651  | 3102          | 20756131  | 3147          | 21173272  |
| 3013          | 19936056  | 3058          | 20349866  | 3103          | 20765382  | 3148          | 21182560  |
| 3014          | 19945233  | 3059          | 20359081  | 3104          | 20774635  | 3149          | 21191849  |
| 3015          | 19954411  | 3060          | 20368297  | 3105          | 20783889  | 3150          | 21201139  |
| 3016          | 19963589  | 3061          | 20377514  | 3106          | 20793143  | 3151          | 21210430  |
| 3017          | 19972769  | 3062          | 20386732  | 3107          | 20802398  | 3152          | 21219721  |
| 3018          | 19981949  | 3063          | 20395951  | 3108          | 20811654  | 3153          | 21229013  |
| 3019          | 19991130  | 3064          | 20405170  | 3109          | 20820911  | 3154          | 21238307  |
| 3020          | 20000313  | 3065          | 20414391  | 3110          | 20830169  | 3155          | 21247600  |
| 3021          | 20009495  | 3066          | 20423612  | 3111          | 20839427  | 3156          | 21256895  |
| 3022          | 20018679  | 3067          | 20432834  | 3112          | 20848687  | 3157          | 21266191  |
| 3023          | 20027864  | 3068          | 20442057  | 3113          | 20857947  | 3158          | 21275487  |
| 3024          | 20037049  | 3069          | 20451281  | 3114          | 20867208  | 3159          | 21284784  |
| 3025          | 20046236  | 3070          | 20460505  | 3115          | 20876469  | 3160          | 21294082  |
| 3026          | 20055423  | 3071          | 20469731  | 3116          | 20885732  | 3161          | 21303380  |
| 3027          | 20064611  | 3072          | 20478957  | 3117          | 20894995  | 3162          | 21312680  |
| 3028          | 20073800  | 3073          | 20488184  | 3118          | 20904259  | 3163          | 21321980  |
| 3029          | 20082990  | 3074          | 20497412  | 3119          | 20913524  | 3164          | 21331281  |
| 3030          | 20092181  | 3075          | 20506641  | 3120          | 20922790  | 3165          | 21340583  |
| 3031          | 20101372  | 3076          | 20515870  | 3121          | 20932057  | 3166          | 21349886  |
| 3032          | 20110565  | 3077          | 20525101  | 3122          | 20941324  | 3167          | 21359189  |
| 3033          | 20119758  | 3078          | 20534332  | 3123          | 20950592  | 3168          | 21368493  |
| 3034          | 20128952  | 3079          | 20543564  | 3124          | 20959861  | 3169          | 21377798  |
| 3035          | 20138147  | 3080          | 20552797  | 3125          | 20969131  | 3170          | 21387104  |
| 3036          | 20147343  | 3081          | 20562031  | 3126          | 20978402  | 3171          | 21396410  |
| 3037          | 20156539  | 3082          | 20571265  | 3127          | 20987673  | 3172          | 21405718  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 3173          | 21415026  | 3218          | 21834696  | 3263          | 22255911  | 3308          | 22678627  |
| 3174          | 21424335  | 3219          | 21844040  | 3264          | 22265289  | 3309          | 22688037  |
| 3175          | 21433644  | 3220          | 21853384  | 3265          | 22274667  | 3310          | 22697448  |
| 3176          | 21442955  | 3221          | 21862729  | 3266          | 22284046  | 3311          | 22706860  |
| 3177          | 21452266  | 3222          | 21872076  | 3267          | 22293426  | 3312          | 22716273  |
| 3178          | 21461578  | 3223          | 21881422  | 3268          | 22302806  | 3313          | 22725686  |
| 3179          | 21470891  | 3224          | 21890770  | 3269          | 22312187  | 3314          | 22735100  |
| 3180          | 21480205  | 3225          | 21900118  | 3270          | 22321569  | 3315          | 22744514  |
| 3181          | 21489519  | 3226          | 21909467  | 3271          | 22330952  | 3316          | 22753930  |
| 3182          | 21498834  | 3227          | 21918817  | 3272          | 22340336  | 3317          | 22763346  |
| 3183          | 21508150  | 3228          | 21928168  | 3273          | 22349720  | 3318          | 22772763  |
| 3184          | 21517467  | 3229          | 21937519  | 3274          | 22359105  | 3319          | 22782180  |
| 3185          | 21526784  | 3230          | 21946871  | 3275          | 22368490  | 3320          | 22791599  |
| 3186          | 21536103  | 3231          | 21956224  | 3276          | 22377877  | 3321          | 22801018  |
| 3187          | 21545422  | 3232          | 21965577  | 3277          | 22387264  | 3322          | 22810437  |
| 3188          | 21554742  | 3233          | 21974932  | 3278          | 22396652  | 3323          | 22819858  |
| 3189          | 21564062  | 3234          | 21984287  | 3279          | 22406040  | 3324          | 22829279  |
| 3190          | 21573384  | 3235          | 21993643  | 3280          | 22415430  | 3325          | 22838701  |
| 3191          | 21582706  | 3236          | 22002999  | 3281          | 22424820  | 3326          | 22848123  |
| 3192          | 21592029  | 3237          | 22012357  | 3282          | 22434211  | 3327          | 22857546  |
| 3193          | 21601352  | 3238          | 22021715  | 3283          | 22443602  | 3328          | 22866970  |
| 3194          | 21610677  | 3239          | 22031074  | 3284          | 22452994  | 3329          | 22876395  |
| 3195          | 21620002  | 3240          | 22040433  | 3285          | 22462387  | 3330          | 22885820  |
| 3196          | 21629328  | 3241          | 22049794  | 3286          | 22471781  | 3331          | 22895246  |
| 3197          | 21638655  | 3242          | 22059155  | 3287          | 22481175  | 3332          | 22904673  |
| 3198          | 21647983  | 3243          | 22068517  | 3288          | 22490571  | 3333          | 22914101  |
| 3199          | 21657311  | 3244          | 22077879  | 3289          | 22499966  | 3334          | 22923529  |
| 3200          | 21666640  | 3245          | 22087243  | 3290          | 22509363  | 3335          | 22932958  |
| 3201          | 21675970  | 3246          | 22096607  | 3291          | 22518761  | 3336          | 22942388  |
| 3202          | 21685301  | 3247          | 22105972  | 3292          | 22528159  | 3337          | 22951818  |
| 3203          | 21694632  | 3248          | 22115337  | 3293          | 22537557  | 3338          | 22961249  |
| 3204          | 21703964  | 3249          | 22124704  | 3294          | 22546957  | 3339          | 22970681  |
| 3205          | 21713297  | 3250          | 22134071  | 3295          | 22556357  | 3340          | 22980113  |
| 3206          | 21722631  | 3251          | 22143439  | 3296          | 22565758  | 3341          | 22989546  |
| 3207          | 21731966  | 3252          | 22152807  | 3297          | 22575160  | 3342          | 22998980  |
| 3208          | 21741301  | 3253          | 22162177  | 3298          | 22584562  | 3343          | 23008415  |
| 3209          | 21750637  | 3254          | 22171547  | 3299          | 22593966  | 3344          | 23017850  |
| 3210          | 21759974  | 3255          | 22180918  | 3300          | 22603370  | 3345          | 23027286  |
| 3211          | 21769311  | 3256          | 22190289  | 3301          | 22612774  | 3346          | 23036722  |
| 3212          | 21778650  | 3257          | 22199661  | 3302          | 22622179  | 3347          | 23046160  |
| 3213          | 21787989  | 3258          | 22209035  | 3303          | 22631586  | 3348          | 23055598  |
| 3214          | 21797329  | 3259          | 22218408  | 3304          | 22640992  | 3349          | 23065037  |
| 3215          | 21806669  | 3260          | 22227783  | 3305          | 22650400  | 3350          | 23074476  |
| 3216          | 21816011  | 3261          | 22237158  | 3306          | 22659808  | 3351          | 23083916  |
| 3217          | 21825353  | 3262          | 22246534  | 3307          | 22669217  | 3352          | 23093357  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 3353          | 23102799  | 3398          | 23528384  | 3443          | 23955339  | 3488          | 24383621  |
| 3354          | 23112241  | 3399          | 23537857  | 3444          | 23964842  | 3489          | 24393153  |
| 3355          | 23121684  | 3400          | 23547331  | 3445          | 23974345  | 3490          | 24402685  |
| 3356          | 23131128  | 3401          | 23556805  | 3446          | 23983850  | 3491          | 24412219  |
| 3357          | 23140572  | 3402          | 23566280  | 3447          | 23993355  | 3492          | 24421753  |
| 3358          | 23150017  | 3403          | 23575756  | 3448          | 24002861  | 3493          | 24431288  |
| 3359          | 23159463  | 3404          | 23585233  | 3449          | 24012367  | 3494          | 24440823  |
| 3360          | 23168909  | 3405          | 23594710  | 3450          | 24021874  | 3495          | 24450359  |
| 3361          | 23178356  | 3406          | 23604188  | 3451          | 24031382  | 3496          | 24459895  |
| 3362          | 23187804  | 3407          | 23613666  | 3452          | 24040890  | 3497          | 24469432  |
| 3363          | 23197253  | 3408          | 23623146  | 3453          | 24050399  | 3498          | 24478970  |
| 3364          | 23206702  | 3409          | 23632626  | 3454          | 24059909  | 3499          | 24488509  |
| 3365          | 23216152  | 3410          | 23642106  | 3455          | 24069419  | 3500          | 24498048  |
| 3366          | 23225602  | 3411          | 23651587  | 3456          | 24078930  | 3501          | 24507588  |
| 3367          | 23235054  | 3412          | 23661069  | 3457          | 24088442  | 3502          | 24517128  |
| 3368          | 23244505  | 3413          | 23670552  | 3458          | 24097954  | 3503          | 24526669  |
| 3369          | 23253958  | 3414          | 23680035  | 3459          | 24107467  | 3504          | 24536210  |
| 3370          | 23263411  | 3415          | 23689519  | 3460          | 24116980  | 3505          | 24545753  |
| 3371          | 23272866  | 3416          | 23699004  | 3461          | 24126494  | 3506          | 24555295  |
| 3372          | 23282320  | 3417          | 23708489  | 3462          | 24136009  | 3507          | 24564839  |
| 3373          | 23291776  | 3418          | 23717975  | 3463          | 24145525  | 3508          | 24574383  |
| 3374          | 23301232  | 3419          | 23727461  | 3464          | 24155041  | 3509          | 24583928  |
| 3375          | 23310689  | 3420          | 23736949  | 3465          | 24164558  | 3510          | 24593473  |
| 3376          | 23320146  | 3421          | 23746437  | 3466          | 24174075  | 3511          | 24603019  |
| 3377          | 23329604  | 3422          | 23755925  | 3467          | 24183593  | 3512          | 24612566  |
| 3378          | 23339063  | 3423          | 23765414  | 3468          | 24193112  | 3513          | 24622113  |
| 3379          | 23348523  | 3424          | 23774904  | 3469          | 24202631  | 3514          | 24631661  |
| 3380          | 23357983  | 3425          | 23784395  | 3470          | 24212151  | 3515          | 24641209  |
| 3381          | 23367444  | 3426          | 23793886  | 3471          | 24221672  | 3516          | 24650758  |
| 3382          | 23376905  | 3427          | 23803378  | 3472          | 24231193  | 3517          | 24660308  |
| 3383          | 23386368  | 3428          | 23812871  | 3473          | 24240715  | 3518          | 24669858  |
| 3384          | 23395831  | 3429          | 23822364  | 3474          | 24250238  | 3519          | 24679409  |
| 3385          | 23405294  | 3430          | 23831858  | 3475          | 24259761  | 3520          | 24688961  |
| 3386          | 23414759  | 3431          | 23841352  | 3476          | 24269285  | 3521          | 24698513  |
| 3387          | 23424224  | 3432          | 23850848  | 3477          | 24278809  | 3522          | 24708066  |
| 3388          | 23433689  | 3433          | 23860343  | 3478          | 24288334  | 3523          | 24717619  |
| 3389          | 23443156  | 3434          | 23869840  | 3479          | 24297860  | 3524          | 24727173  |
| 3390          | 23452623  | 3435          | 23879337  | 3480          | 24307386  | 3525          | 24736728  |
| 3391          | 23462090  | 3436          | 23888835  | 3481          | 24316913  | 3526          | 24746283  |
| 3392          | 23471559  | 3437          | 23898334  | 3482          | 24326441  | 3527          | 24755839  |
| 3393          | 23481028  | 3438          | 23907833  | 3483          | 24335969  | 3528          | 24765396  |
| 3394          | 23490498  | 3439          | 23917333  | 3484          | 24345498  | 3529          | 24774953  |
| 3395          | 23499968  | 3440          | 23926833  | 3485          | 24355028  | 3530          | 24784511  |
| 3396          | 23509439  | 3441          | 23936334  | 3486          | 24364558  | 3531          | 24794069  |
| 3397          | 23518911  | 3442          | 23945836  | 3487          | 24374089  | 3532          | 24803628  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 3533          | 24813187  | 3578          | 25243997  | 3623          | 25676009  | 3668          | 26109180  |
| 3534          | 24822748  | 3579          | 25253585  | 3624          | 25685622  | 3669          | 26118819  |
| 3535          | 24832308  | 3580          | 25263173  | 3625          | 25695237  | 3670          | 26128459  |
| 3536          | 24841870  | 3581          | 25272761  | 3626          | 25704851  | 3671          | 26138099  |
| 3537          | 24851432  | 3582          | 25282350  | 3627          | 25714467  | 3672          | 26147739  |
| 3538          | 24860995  | 3583          | 25291940  | 3628          | 25724082  | 3673          | 26157381  |
| 3539          | 24870558  | 3584          | 25301530  | 3629          | 25733699  | 3674          | 26167022  |
| 3540          | 24880122  | 3585          | 25311121  | 3630          | 25743316  | 3675          | 26176664  |
| 3541          | 24889686  | 3586          | 25320713  | 3631          | 25752933  | 3676          | 26186307  |
| 3542          | 24899251  | 3587          | 25330305  | 3632          | 25762552  | 3677          | 26195950  |
| 3543          | 24908817  | 3588          | 25339898  | 3633          | 25772170  | 3678          | 26205594  |
| 3544          | 24918383  | 3589          | 25349491  | 3634          | 25781790  | 3679          | 26215239  |
| 3545          | 24927950  | 3590          | 25359085  | 3635          | 25791410  | 3680          | 26224884  |
| 3546          | 24937518  | 3591          | 25368679  | 3636          | 25801030  | 3681          | 26234529  |
| 3547          | 24947086  | 3592          | 25378274  | 3637          | 25810651  | 3682          | 26244175  |
| 3548          | 24956655  | 3593          | 25387870  | 3638          | 25820272  | 3683          | 26253822  |
| 3549          | 24966224  | 3594          | 25397466  | 3639          | 25829895  | 3684          | 26263469  |
| 3550          | 24975794  | 3595          | 25407063  | 3640          | 25839517  | 3685          | 26273117  |
| 3551          | 24985365  | 3596          | 25416660  | 3641          | 25849141  | 3686          | 26282765  |
| 3552          | 24994936  | 3597          | 25426258  | 3642          | 25858764  | 3687          | 26292414  |
| 3553          | 25004508  | 3598          | 25435857  | 3643          | 25868389  | 3688          | 26302063  |
| 3554          | 25014080  | 3599          | 25445456  | 3644          | 25878014  | 3689          | 26311713  |
| 3555          | 25023653  | 3600          | 25455055  | 3645          | 25887639  | 3690          | 26321363  |
| 3556          | 25033227  | 3601          | 25464656  | 3646          | 25897265  | 3691          | 26331014  |
| 3557          | 25042801  | 3602          | 25474257  | 3647          | 25906892  | 3692          | 26340666  |
| 3558          | 25052376  | 3603          | 25483858  | 3648          | 25916519  | 3693          | 26350318  |
| 3559          | 25061951  | 3604          | 25493460  | 3649          | 25926147  | 3694          | 26359970  |
| 3560          | 25071527  | 3605          | 25503063  | 3650          | 25935775  | 3695          | 26369623  |
| 3561          | 25081104  | 3606          | 25512666  | 3651          | 25945404  | 3696          | 26379277  |
| 3562          | 25090681  | 3607          | 25522270  | 3652          | 25955034  | 3697          | 26388931  |
| 3563          | 25100259  | 3608          | 25531874  | 3653          | 25964664  | 3698          | 26398586  |
| 3564          | 25109837  | 3609          | 25541479  | 3654          | 25974294  | 3699          | 26408241  |
| 3565          | 25119416  | 3610          | 25551085  | 3655          | 25983925  | 3700          | 26417897  |
| 3566          | 25128996  | 3611          | 25560691  | 3656          | 25993557  | 3701          | 26427553  |
| 3567          | 25138576  | 3612          | 25570297  | 3657          | 26003189  | 3702          | 26437210  |
| 3568          | 25148157  | 3613          | 25579905  | 3658          | 26012822  | 3703          | 26446868  |
| 3569          | 25157738  | 3614          | 25589512  | 3659          | 26022455  | 3704          | 26456526  |
| 3570          | 25167320  | 3615          | 25599121  | 3660          | 26032089  | 3705          | 26466184  |
| 3571          | 25176903  | 3616          | 25608730  | 3661          | 26041724  | 3706          | 26475843  |
| 3572          | 25186486  | 3617          | 25618339  | 3662          | 26051359  | 3707          | 26485503  |
| 3573          | 25196070  | 3618          | 25627950  | 3663          | 26060994  | 3708          | 26495163  |
| 3574          | 25205654  | 3619          | 25637560  | 3664          | 26070630  | 3709          | 26504824  |
| 3575          | 25215239  | 3620          | 25647172  | 3665          | 26080267  | 3710          | 26514485  |
| 3576          | 25224824  | 3621          | 25656783  | 3666          | 26089904  | 3711          | 26524147  |
| 3577          | 25234411  | 3622          | 25666396  | 3667          | 26099542  | 3712          | 26533809  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 3713          | 26543472  | 3758          | 26978841  | 3803          | 27415259  | 3848          | 27852657  |
| 3714          | 26553135  | 3759          | 26988528  | 3804          | 27424959  | 3849          | 27862388  |
| 3715          | 26562799  | 3760          | 26998216  | 3805          | 27434669  | 3850          | 27872120  |
| 3716          | 26572463  | 3761          | 27007903  | 3806          | 27444380  | 3851          | 27881852  |
| 3717          | 26582128  | 3762          | 27017592  | 3807          | 27454091  | 3852          | 27891585  |
| 3718          | 26591793  | 3763          | 27027281  | 3808          | 27463802  | 3853          | 27901318  |
| 3719          | 26601459  | 3764          | 27036970  | 3809          | 27473514  | 3854          | 27911051  |
| 3720          | 26611125  | 3765          | 27046660  | 3810          | 27483226  | 3855          | 27920785  |
| 3721          | 26620792  | 3766          | 27056350  | 3811          | 27492939  | 3856          | 27930520  |
| 3722          | 26630460  | 3767          | 27066041  | 3812          | 27502653  | 3857          | 27940255  |
| 3723          | 26640128  | 3768          | 27075733  | 3813          | 27512367  | 3858          | 27949990  |
| 3724          | 26649796  | 3769          | 27085425  | 3814          | 27522081  | 3859          | 27959726  |
| 3725          | 26659465  | 3770          | 27095117  | 3815          | 27531796  | 3860          | 27969463  |
| 3726          | 26669135  | 3771          | 27104810  | 3816          | 27541511  | 3861          | 27979199  |
| 3727          | 26678805  | 3772          | 27114503  | 3817          | 27551227  | 3862          | 27988937  |
| 3728          | 26688476  | 3773          | 27124198  | 3818          | 27560943  | 3863          | 27998675  |
| 3729          | 26698147  | 3774          | 27133892  | 3819          | 27570660  | 3864          | 28008413  |
| 3730          | 26707818  | 3775          | 27143587  | 3820          | 27580377  | 3865          | 28018152  |
| 3731          | 26717491  | 3776          | 27153282  | 3821          | 27590095  | 3866          | 28027891  |
| 3732          | 26727164  | 3777          | 27162978  | 3822          | 27599813  | 3867          | 28037630  |
| 3733          | 26736838  | 3778          | 27172675  | 3823          | 27609532  | 3868          | 28047370  |
| 3734          | 26746511  | 3779          | 27182372  | 3824          | 27619251  | 3869          | 28057111  |
| 3735          | 26756186  | 3780          | 27192069  | 3825          | 27628971  | 3870          | 28066852  |
| 3736          | 26765861  | 3781          | 27201767  | 3826          | 27638691  | 3871          | 28076594  |
| 3737          | 26775536  | 3782          | 27211466  | 3827          | 27648412  | 3872          | 28086336  |
| 3738          | 26785212  | 3783          | 27221165  | 3828          | 27658133  | 3873          | 28096078  |
| 3739          | 26794889  | 3784          | 27230864  | 3829          | 27667855  | 3874          | 28105821  |
| 3740          | 26804566  | 3785          | 27240565  | 3830          | 27677577  | 3875          | 28115564  |
| 3741          | 26814243  | 3786          | 27250265  | 3831          | 27687300  | 3876          | 28125308  |
| 3742          | 26823921  | 3787          | 27259966  | 3832          | 27697023  | 3877          | 28135052  |
| 3743          | 26833601  | 3788          | 27269668  | 3833          | 27706746  | 3878          | 28144797  |
| 3744          | 26843280  | 3789          | 27279370  | 3834          | 27716470  | 3879          | 28154542  |
| 3745          | 26852960  | 3790          | 27289072  | 3835          | 27726195  | 3880          | 28164288  |
| 3746          | 26862640  | 3791          | 27298775  | 3836          | 27735920  | 3881          | 28174034  |
| 3747          | 26872320  | 3792          | 27308479  | 3837          | 27745645  | 3882          | 28183781  |
| 3748          | 26882001  | 3793          | 27318183  | 3838          | 27755371  | 3883          | 28193528  |
| 3749          | 26891683  | 3794          | 27327887  | 3839          | 27765098  | 3884          | 28203275  |
| 3750          | 26901365  | 3795          | 27337592  | 3840          | 27774825  | 3885          | 28213023  |
| 3751          | 26911048  | 3796          | 27347298  | 3841          | 27784552  | 3886          | 28222772  |
| 3752          | 26920731  | 3797          | 27357004  | 3842          | 27794280  | 3887          | 28232520  |
| 3753          | 26930415  | 3798          | 27366710  | 3843          | 27804008  | 3888          | 28242270  |
| 3754          | 26940099  | 3799          | 27376417  | 3844          | 27813737  | 3889          | 28252019  |
| 3755          | 26949784  | 3800          | 27386125  | 3845          | 27823466  | 3890          | 28261770  |
| 3756          | 26959469  | 3801          | 27395833  | 3846          | 27833196  | 3891          | 28271520  |
| 3757          | 26969155  | 3802          | 27405541  | 3847          | 27842926  | 3892          | 28281272  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 3893          | 28291023  | 3938          | 28730309  | 3983          | 29170475  | 4028          | 29611482  |
| 3894          | 28300775  | 3939          | 28740081  | 3984          | 29180266  | 4029          | 29621291  |
| 3895          | 28310528  | 3940          | 28749853  | 3985          | 29190057  | 4030          | 29631101  |
| 3896          | 28320281  | 3941          | 28759626  | 3986          | 29199849  | 4031          | 29640911  |
| 3897          | 28330034  | 3942          | 28769400  | 3987          | 29209642  | 4032          | 29650722  |
| 3898          | 28339788  | 3943          | 28779173  | 3988          | 29219435  | 4033          | 29660533  |
| 3899          | 28349342  | 3944          | 28788948  | 3989          | 29229228  | 4034          | 29670344  |
| 3900          | 28359297  | 3945          | 28798722  | 3990          | 29239022  | 4035          | 29680156  |
| 3901          | 28369052  | 3946          | 28808497  | 3991          | 29248816  | 4036          | 29689968  |
| 3902          | 28378808  | 3947          | 28818273  | 3992          | 29258610  | 4037          | 29699781  |
| 3903          | 28388564  | 3948          | 28828049  | 3993          | 29268405  | 4038          | 29709594  |
| 3904          | 28398321  | 3949          | 28837825  | 3994          | 29278200  | 4039          | 29719407  |
| 3905          | 28408078  | 3950          | 28847602  | 3995          | 29287996  | 4040          | 29729221  |
| 3906          | 28417835  | 3951          | 28857379  | 3996          | 29297792  | 4041          | 29739035  |
| 3907          | 28427593  | 3952          | 28867157  | 3997          | 29307589  | 4042          | 29748850  |
| 3908          | 28437351  | 3953          | 28876935  | 3998          | 29317386  | 4043          | 29758665  |
| 3909          | 28447110  | 3954          | 28886713  | 3999          | 29327183  | 4044          | 29768480  |
| 3910          | 28456870  | 3955          | 28896492  | 4000          | 29336981  | 4045          | 29778296  |
| 3911          | 28466629  | 3956          | 28906272  | 4001          | 29346779  | 4046          | 29788112  |
| 3912          | 28476389  | 3957          | 28916051  | 4002          | 29356577  | 4047          | 29797928  |
| 3913          | 28486150  | 3958          | 28925832  | 4003          | 29366376  | 4048          | 29807745  |
| 3914          | 28495911  | 3959          | 28935612  | 4004          | 29376176  | 4049          | 29817562  |
| 3915          | 28505673  | 3960          | 28945393  | 4005          | 29385976  | 4050          | 29827380  |
| 3916          | 28515435  | 3961          | 28955175  | 4006          | 29395776  | 4051          | 29837198  |
| 3917          | 28525197  | 3962          | 28964957  | 4007          | 29405576  | 4052          | 29847017  |
| 3918          | 28534960  | 3963          | 28974739  | 4008          | 29415378  | 4053          | 29856835  |
| 3919          | 28544723  | 3964          | 28984522  | 4009          | 29425179  | 4054          | 29866655  |
| 3920          | 28554487  | 3965          | 28994305  | 4010          | 29434981  | 4055          | 29876474  |
| 3921          | 28564251  | 3966          | 29004089  | 4011          | 29444783  | 4056          | 29886294  |
| 3922          | 28574015  | 3967          | 29013873  | 4012          | 29454585  | 4057          | 29896114  |
| 3923          | 28583781  | 3968          | 29023657  | 4013          | 29464388  | 4058          | 29905935  |
| 3924          | 28593546  | 3969          | 29033442  | 4014          | 29474192  | 4059          | 29915756  |
| 3925          | 28603312  | 3970          | 29043228  | 4015          | 29483996  | 4060          | 29925578  |
| 3926          | 28613078  | 3971          | 29053013  | 4016          | 29493800  | 4061          | 29935400  |
| 3927          | 28622845  | 3972          | 29062799  | 4017          | 29503605  | 4062          | 29945222  |
| 3928          | 28632612  | 3973          | 29072586  | 4018          | 29513410  | 4063          | 29955045  |
| 3929          | 28642380  | 3974          | 29082373  | 4019          | 29523215  | 4064          | 29964868  |
| 3930          | 28652148  | 3975          | 29092160  | 4020          | 29533021  | 4065          | 29974691  |
| 3931          | 28661917  | 3976          | 29101948  | 4021          | 29542827  | 4066          | 29984515  |
| 3932          | 28671686  | 3977          | 29111736  | 4022          | 29552634  | 4067          | 29994339  |
| 3933          | 28681455  | 3978          | 29121525  | 4023          | 29562441  | 4068          | 30004164  |
| 3934          | 28691225  | 3979          | 29131314  | 4024          | 29572248  | 4069          | 30013988  |
| 3935          | 28700995  | 3980          | 29141104  | 4025          | 29582056  | 4070          | 30023814  |
| 3936          | 28710766  | 3981          | 29150894  | 4026          | 29591864  | 4071          | 30033639  |
| 3937          | 28720537  | 3982          | 29160684  | 4027          | 29601673  | 4072          | 30043466  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 4073          | 30053292  | 4118          | 30495866  | 4163          | 30939166  | 4208          | 31383153  |
| 4074          | 30063119  | 4119          | 30505709  | 4164          | 30949025  | 4209          | 31393027  |
| 4075          | 30072946  | 4120          | 30515553  | 4165          | 30958884  | 4210          | 31402902  |
| 4076          | 30082774  | 4121          | 30525397  | 4166          | 30968744  | 4211          | 31412776  |
| 4077          | 30092602  | 4122          | 30535242  | 4167          | 30978604  | 4212          | 31422651  |
| 4078          | 30102430  | 4123          | 30545086  | 4168          | 30988465  | 4213          | 31432526  |
| 4079          | 30112259  | 4124          | 30554932  | 4169          | 30998325  | 4214          | 31442402  |
| 4080          | 30122088  | 4125          | 30564777  | 4170          | 31008186  | 4215          | 31452278  |
| 4081          | 30131917  | 4126          | 30574623  | 4171          | 31018048  | 4216          | 31462154  |
| 4082          | 30141747  | 4127          | 30584469  | 4172          | 31027910  | 4217          | 31472030  |
| 4083          | 30151577  | 4128          | 30594316  | 4173          | 31037772  | 4218          | 31481907  |
| 4084          | 30161408  | 4129          | 30604163  | 4174          | 31047634  | 4219          | 31491784  |
| 4085          | 30171239  | 4130          | 30614010  | 4175          | 31057497  | 4220          | 31501661  |
| 4086          | 30181070  | 4131          | 30623858  | 4176          | 31067360  | 4221          | 31511539  |
| 4087          | 30190902  | 4132          | 30633706  | 4177          | 31077223  | 4222          | 31521417  |
| 4088          | 30200734  | 4133          | 30643554  | 4178          | 31087087  | 4223          | 31531296  |
| 4089          | 30210566  | 4134          | 30653403  | 4179          | 31096951  | 4224          | 31541174  |
| 4090          | 30220399  | 4135          | 30663252  | 4180          | 31106816  | 4225          | 31551053  |
| 4091          | 30230232  | 4136          | 30673101  | 4181          | 31116681  | 4226          | 31560933  |
| 4092          | 30240066  | 4137          | 30682951  | 4182          | 31126546  | 4227          | 31570812  |
| 4093          | 30249899  | 4138          | 30692801  | 4183          | 31136411  | 4228          | 31580692  |
| 4094          | 30259734  | 4139          | 30702651  | 4184          | 31146277  | 4229          | 31590572  |
| 4095          | 30269568  | 4140          | 30712502  | 4185          | 31156143  | 4230          | 31600453  |
| 4096          | 30279403  | 4141          | 30722353  | 4186          | 31166009  | 4231          | 31610334  |
| 4097          | 30289239  | 4142          | 30732205  | 4187          | 31175876  | 4232          | 31620215  |
| 4098          | 30299075  | 4143          | 30742057  | 4188          | 31185743  | 4233          | 31630096  |
| 4099          | 30308911  | 4144          | 30751909  | 4189          | 31195611  | 4234          | 31639978  |
| 4100          | 30318747  | 4145          | 30761761  | 4190          | 31205478  | 4235          | 31649860  |
| 4101          | 30328584  | 4146          | 30771614  | 4191          | 31215346  | 4236          | 31659743  |
| 4102          | 30338421  | 4147          | 30781467  | 4192          | 31225215  | 4237          | 31669625  |
| 4103          | 30348259  | 4148          | 30791321  | 4193          | 31235084  | 4238          | 31679509  |
| 4104          | 30358097  | 4149          | 30801175  | 4194          | 31244953  | 4239          | 31689392  |
| 4105          | 30367935  | 4150          | 30811029  | 4195          | 31254822  | 4240          | 31699276  |
| 4106          | 30377774  | 4151          | 30820884  | 4196          | 31264692  | 4241          | 31709159  |
| 4107          | 30387613  | 4152          | 30830739  | 4197          | 31274562  | 4242          | 31719044  |
| 4108          | 30397452  | 4153          | 30840594  | 4198          | 31284432  | 4243          | 31728928  |
| 4109          | 30407292  | 4154          | 30850450  | 4199          | 31294303  | 4244          | 31738813  |
| 4110          | 30417132  | 4155          | 30860306  | 4200          | 31304174  | 4245          | 31748698  |
| 4111          | 30426973  | 4156          | 30870162  | 4201          | 31314045  | 4246          | 31758584  |
| 4112          | 30436813  | 4157          | 30880019  | 4202          | 31323917  | 4247          | 31768470  |
| 4113          | 30446655  | 4158          | 30889876  | 4203          | 31333789  | 4248          | 31778356  |
| 4114          | 30456496  | 4159          | 30899733  | 4204          | 31343661  | 4249          | 31788242  |
| 4115          | 30466338  | 4160          | 30909591  | 4205          | 31353534  | 4250          | 31798129  |
| 4116          | 30476180  | 4161          | 30919449  | 4206          | 31363407  | 4251          | 31808016  |
| 4117          | 30486023  | 4162          | 30929307  | 4207          | 31373280  | 4252          | 31817903  |



## A TABLE of

| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 4253          | 31827791  | 4298          | 32273040  | 4343          | 32718864  | 4388          | 33165224  |
| 4254          | 31837679  | 4299          | 32282941  | 4344          | 32728777  | 4389          | 33175149  |
| 4255          | 31847567  | 4300          | 32292843  | 4345          | 32738691  | 4390          | 33185074  |
| 4256          | 31857455  | 4301          | 32302744  | 4346          | 32748605  | 4391          | 33195000  |
| 4257          | 31867344  | 4302          | 32312646  | 4347          | 32758529  | 4392          | 33204925  |
| 4258          | 31877233  | 4303          | 32322548  | 4348          | 32768434  | 4393          | 33214851  |
| 4259          | 31887123  | 4304          | 32332451  | 4349          | 32778348  | 4394          | 33224777  |
| 4260          | 31897013  | 4305          | 32342354  | 4350          | 32788263  | 4395          | 33234704  |
| 4261          | 31906903  | 4306          | 32352257  | 4351          | 32798179  | 4396          | 33244630  |
| 4262          | 31916793  | 4307          | 32362160  | 4352          | 32808094  | 4397          | 33254557  |
| 4263          | 31926684  | 4308          | 32372064  | 4353          | 32818010  | 4398          | 33264484  |
| 4264          | 31936574  | 4309          | 32381968  | 4354          | 32827926  | 4399          | 33274412  |
| 4265          | 31946466  | 4310          | 32391872  | 4355          | 32837842  | 4400          | 33284339  |
| 4266          | 31956357  | 4311          | 32401776  | 4356          | 32847759  | 4401          | 33294267  |
| 4267          | 31966249  | 4312          | 32411681  | 4357          | 32857676  | 4402          | 33304195  |
| 4268          | 31976141  | 4313          | 32421586  | 4358          | 32867593  | 4403          | 33314124  |
| 4269          | 31986033  | 4314          | 32431491  | 4359          | 32877510  | 4404          | 33324052  |
| 4270          | 31995926  | 4315          | 32441397  | 4360          | 32887428  | 4405          | 33333981  |
| 4271          | 32005819  | 4316          | 32451303  | 4361          | 32897346  | 4406          | 33343910  |
| 4272          | 32015712  | 4317          | 32461209  | 4362          | 32907264  | 4407          | 33353840  |
| 4273          | 32025606  | 4318          | 32471115  | 4363          | 32917182  | 4408          | 33363769  |
| 4274          | 32035500  | 4319          | 32481022  | 4364          | 32927101  | 4409          | 33373699  |
| 4275          | 32045394  | 4320          | 32490929  | 4365          | 32937020  | 4410          | 33383629  |
| 4276          | 32055289  | 4321          | 32500836  | 4366          | 32946939  | 4411          | 33393559  |
| 4277          | 32065183  | 4322          | 32510744  | 4367          | 32956858  | 4412          | 33403490  |
| 4278          | 32075078  | 4323          | 32520651  | 4368          | 32966778  | 4413          | 33413420  |
| 4279          | 32084974  | 4324          | 32530559  | 4369          | 32976698  | 4414          | 33423351  |
| 4280          | 32094869  | 4325          | 32540468  | 4370          | 32986618  | 4415          | 33433282  |
| 4281          | 32104765  | 4326          | 32550376  | 4371          | 32996538  | 4416          | 33443214  |
| 4282          | 32114661  | 4327          | 32560285  | 4372          | 33006459  | 4417          | 33453146  |
| 4283          | 32124558  | 4328          | 32570194  | 4373          | 33016380  | 4418          | 33463078  |
| 4284          | 32134455  | 4329          | 32580104  | 4374          | 33026301  | 4419          | 33473010  |
| 4285          | 32144352  | 4330          | 32590013  | 4375          | 33036223  | 4420          | 33482942  |
| 4286          | 32154249  | 4331          | 32599923  | 4376          | 33046144  | 4421          | 33492875  |
| 4287          | 32164147  | 4332          | 32609834  | 4377          | 33056066  | 4422          | 33502807  |
| 4288          | 32174045  | 4333          | 32619744  | 4378          | 33065988  | 4423          | 33512741  |
| 4289          | 32183943  | 4334          | 32629655  | 4379          | 33075911  | 4424          | 33522674  |
| 4290          | 32193842  | 4335          | 32639566  | 4380          | 33085834  | 4425          | 33532607  |
| 4291          | 32203740  | 4336          | 32649477  | 4381          | 33095757  | 4426          | 33542541  |
| 4292          | 32213640  | 4337          | 32659389  | 4382          | 33105680  | 4427          | 33552475  |
| 4293          | 32223539  | 4338          | 32669301  | 4383          | 33115603  | 4428          | 33562409  |
| 4294          | 32233439  | 4339          | 32679213  | 4384          | 33125527  | 4429          | 33572344  |
| 4295          | 32243339  | 4340          | 32689125  | 4385          | 33135451  | 4430          | 33582279  |
| 4296          | 32253239  | 4341          | 32699038  | 4386          | 33145375  | 4431          | 33592213  |
| 4297          | 32263139  | 4342          | 32708951  | 4387          | 33155299  | 4432          | 33602149  |



## CIRCULAR SEGMENTS.

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| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 4433          | 33612084  | 4478          | 34059406  | 4523          | 34507154  | 4568          | 34955289  |
| 4434          | 33622020  | 4479          | 34069352  | 4524          | 34517108  | 4569          | 34965252  |
| 4435          | 33631955  | 4480          | 34079297  | 4525          | 34527063  | 4570          | 34975215  |
| 4436          | 33641892  | 4481          | 34089243  | 4526          | 34537018  | 4571          | 34985178  |
| 4437          | 33651828  | 4482          | 34099189  | 4527          | 34546973  | 4572          | 34995141  |
| 4438          | 33661764  | 4483          | 34109136  | 4528          | 34556928  | 4573          | 35005104  |
| 4439          | 33671701  | 4484          | 34119082  | 4529          | 34566883  | 4574          | 35015068  |
| 4440          | 33681638  | 4485          | 34129029  | 4530          | 34576839  | 4575          | 35025031  |
| 4441          | 33691575  | 4486          | 34138976  | 4531          | 34586795  | 4576          | 35034995  |
| 4442          | 33701513  | 4487          | 34148923  | 4532          | 34596751  | 4577          | 35044959  |
| 4443          | 33711450  | 4488          | 34158870  | 4533          | 34606707  | 4578          | 35054924  |
| 4444          | 33721388  | 4489          | 34168818  | 4534          | 34616663  | 4579          | 35064888  |
| 4445          | 33731326  | 4490          | 34178765  | 4535          | 34626620  | 4580          | 35074853  |
| 4446          | 33741265  | 4491          | 34188713  | 4536          | 34636577  | 4581          | 35084817  |
| 4447          | 33751203  | 4492          | 34198662  | 4537          | 34646534  | 4582          | 35094782  |
| 4448          | 33761142  | 4493          | 34208610  | 4538          | 34656491  | 4583          | 35104747  |
| 4449          | 33771081  | 4494          | 34218558  | 4539          | 34666448  | 4584          | 35114713  |
| 4450          | 33781020  | 4495          | 34228507  | 4540          | 34676406  | 4585          | 35124678  |
| 4451          | 33790959  | 4496          | 34238456  | 4541          | 34686363  | 4586          | 35134644  |
| 4452          | 33800899  | 4497          | 34248405  | 4542          | 34696321  | 4587          | 35144609  |
| 4453          | 33810839  | 4498          | 34258355  | 4543          | 34706279  | 4588          | 35154575  |
| 4454          | 33820779  | 4499          | 34268304  | 4544          | 34716237  | 4589          | 35164541  |
| 4455          | 33830719  | 4500          | 34278254  | 4545          | 34726196  | 4590          | 35174508  |
| 4456          | 33840660  | 4501          | 34288204  | 4546          | 34736154  | 4591          | 35184474  |
| 4457          | 33850601  | 4502          | 34298154  | 4547          | 34746113  | 4592          | 35194441  |
| 4458          | 33860542  | 4503          | 34308105  | 4548          | 34756072  | 4593          | 35204407  |
| 4459          | 33870483  | 4504          | 34318055  | 4549          | 34766031  | 4594          | 35214374  |
| 4460          | 33880424  | 4505          | 34328006  | 4550          | 34775991  | 4595          | 35224341  |
| 4461          | 33890366  | 4506          | 34337957  | 4551          | 34785950  | 4596          | 35234308  |
| 4462          | 33900308  | 4507          | 34347908  | 4552          | 34795910  | 4597          | 35244276  |
| 4463          | 33910250  | 4508          | 34357859  | 4553          | 34805870  | 4598          | 35254243  |
| 4464          | 33920192  | 4509          | 34367811  | 4554          | 34815830  | 4599          | 35264211  |
| 4465          | 33930134  | 4510          | 34377763  | 4555          | 34825790  | 4600          | 35274179  |
| 4466          | 33940077  | 4511          | 34387715  | 4556          | 34835750  | 4601          | 35284147  |
| 4467          | 33950020  | 4512          | 34397667  | 4557          | 34845711  | 4602          | 35294115  |
| 4468          | 33959963  | 4513          | 34407619  | 4558          | 34855672  | 4603          | 35304084  |
| 4469          | 33969907  | 4514          | 34417572  | 4559          | 34865633  | 4604          | 35314052  |
| 4470          | 33979850  | 4515          | 34427525  | 4560          | 34875594  | 4605          | 35324021  |
| 4471          | 33989794  | 4516          | 34437478  | 4561          | 34885555  | 4606          | 35333990  |
| 4472          | 33999738  | 4517          | 34447431  | 4562          | 34895516  | 4607          | 35343958  |
| 4473          | 34009682  | 4518          | 34457384  | 4563          | 34905478  | 4608          | 35353928  |
| 4474          | 34019626  | 4519          | 34467338  | 4564          | 34915440  | 4609          | 35363897  |
| 4475          | 34029571  | 4520          | 34477291  | 4565          | 34925402  | 4610          | 35373866  |
| 4476          | 34039516  | 4521          | 34487245  | 4566          | 34935364  | 4611          | 35383836  |
| 4477          | 34049461  | 4522          | 34497199  | 4567          | 34945326  | 4612          | 35393806  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 4613          | 35403776  | 4658          | 35852577  | 4703          | 36301656  | 4748          | 36750976  |
| 4614          | 35413746  | 4659          | 35862553  | 4704          | 36311638  | 4749          | 36760963  |
| 4615          | 35423716  | 4660          | 35872530  | 4705          | 36321621  | 4750          | 36770950  |
| 4616          | 35433686  | 4661          | 35882507  | 4706          | 36331603  | 4751          | 36780938  |
| 4617          | 35443657  | 4662          | 35892484  | 4707          | 36341586  | 4752          | 36790925  |
| 4618          | 35453628  | 4663          | 35902461  | 4708          | 36351569  | 4753          | 36800913  |
| 4619          | 35463598  | 4664          | 35912439  | 4709          | 36361552  | 4754          | 36810901  |
| 4620          | 35473569  | 4665          | 35922416  | 4710          | 36371535  | 4755          | 36820889  |
| 4621          | 35483541  | 4666          | 35932394  | 4711          | 36381518  | 4756          | 36830877  |
| 4622          | 35493512  | 4667          | 35942372  | 4712          | 36391501  | 4757          | 36840865  |
| 4623          | 35503483  | 4668          | 35952349  | 4713          | 36401485  | 4758          | 36850853  |
| 4624          | 35513455  | 4669          | 35962327  | 4714          | 36411469  | 4759          | 36860842  |
| 4625          | 35523427  | 4670          | 35972306  | 4715          | 36421452  | 4760          | 36870830  |
| 4626          | 35533399  | 4671          | 35982284  | 4716          | 36431436  | 4761          | 36880819  |
| 4627          | 35543371  | 4672          | 35992262  | 4717          | 36441420  | 4762          | 36890807  |
| 4628          | 35553343  | 4673          | 36002241  | 4718          | 36451404  | 4763          | 36900796  |
| 4629          | 35563315  | 4674          | 36012219  | 4719          | 36461388  | 4764          | 36910785  |
| 4630          | 35573288  | 4675          | 36022198  | 4720          | 36471372  | 4765          | 36920774  |
| 4631          | 35583260  | 4676          | 36032177  | 4721          | 36481357  | 4766          | 36930763  |
| 4632          | 35593233  | 4677          | 36042156  | 4722          | 36491341  | 4767          | 36940752  |
| 4633          | 35603206  | 4678          | 36052135  | 4723          | 36501326  | 4768          | 36950741  |
| 4634          | 35613179  | 4679          | 36062115  | 4724          | 36511310  | 4769          | 36960730  |
| 4635          | 35623153  | 4680          | 36072094  | 4725          | 36521295  | 4770          | 36970720  |
| 4636          | 35633126  | 4681          | 36082074  | 4726          | 36531280  | 4771          | 36980709  |
| 4637          | 35643099  | 4682          | 36092053  | 4727          | 36541265  | 4772          | 36990699  |
| 4638          | 35653073  | 4683          | 36102033  | 4728          | 36551250  | 4773          | 37000688  |
| 4639          | 35663047  | 4684          | 36112013  | 4729          | 36561236  | 4774          | 37010678  |
| 4640          | 35673021  | 4685          | 36121993  | 4730          | 36571221  | 4775          | 37020668  |
| 4641          | 35682995  | 4686          | 36131973  | 4731          | 36581206  | 4776          | 37030658  |
| 4642          | 35692969  | 4687          | 36141954  | 4732          | 36591192  | 4777          | 37040648  |
| 4643          | 35702944  | 4688          | 36151934  | 4733          | 36601178  | 4778          | 37050638  |
| 4644          | 35712918  | 4689          | 36161915  | 4734          | 36611163  | 4779          | 37060628  |
| 4645          | 35722893  | 4690          | 36171895  | 4735          | 36621149  | 4780          | 37070618  |
| 4646          | 35732868  | 4691          | 36181876  | 4736          | 36631135  | 4781          | 37080609  |
| 4647          | 35742843  | 4692          | 36191857  | 4737          | 36641121  | 4782          | 37090599  |
| 4648          | 35752818  | 4693          | 36201838  | 4738          | 36651108  | 4783          | 37100590  |
| 4649          | 35762793  | 4694          | 36211820  | 4739          | 36661094  | 4784          | 37110580  |
| 4650          | 35772769  | 4695          | 36221801  | 4740          | 36671080  | 4785          | 37120571  |
| 4651          | 35782744  | 4696          | 36231783  | 4741          | 36681067  | 4786          | 37130562  |
| 4652          | 35792720  | 4697          | 36241764  | 4742          | 36691053  | 4787          | 37140553  |
| 4653          | 35802696  | 4698          | 36251746  | 4743          | 36701040  | 4788          | 37150544  |
| 4654          | 35812672  | 4699          | 36261728  | 4744          | 36711027  | 4789          | 37160535  |
| 4655          | 35822648  | 4700          | 36271710  | 4745          | 36721014  | 4790          | 37170526  |
| 4656          | 35832624  | 4701          | 36281691  | 4746          | 36731001  | 4791          | 37180517  |
| 4657          | 35842600  | 4702          | 36291673  | 4747          | 36740988  | 4792          | 37190508  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 4793          | 37200500  | 4838          | 37650192  | 4883          | 38100015  | 4928          | 38549933  |
| 4794          | 37210491  | 4839          | 37660186  | 4884          | 38110012  | 4929          | 38559932  |
| 4795          | 37220483  | 4840          | 37670181  | 4885          | 38120010  | 4930          | 38569931  |
| 4796          | 37230474  | 4841          | 37680176  | 4886          | 38130007  | 4931          | 38579930  |
| 4797          | 37240466  | 4842          | 37690171  | 4887          | 38140004  | 4932          | 38589929  |
| 4798          | 37250458  | 4843          | 37700166  | 4888          | 38150002  | 4933          | 38599928  |
| 4799          | 37260450  | 4844          | 37710161  | 4889          | 38159999  | 4934          | 38609927  |
| 4800          | 37270442  | 4845          | 37720156  | 4890          | 38169997  | 4935          | 38619926  |
| 4801          | 37280434  | 4846          | 37730152  | 4891          | 38179994  | 4936          | 38629926  |
| 4802          | 37290426  | 4847          | 37740147  | 4892          | 38189992  | 4937          | 38639925  |
| 4803          | 37300418  | 4848          | 37750142  | 4893          | 38199990  | 4938          | 38649924  |
| 4804          | 37310410  | 4849          | 37760138  | 4894          | 38209988  | 4939          | 38659923  |
| 4805          | 37320403  | 4850          | 37770133  | 4895          | 38219985  | 4940          | 38669923  |
| 4806          | 37330395  | 4851          | 37780129  | 4896          | 38229983  | 4941          | 38679922  |
| 4807          | 37340387  | 4852          | 37790124  | 4897          | 38239981  | 4942          | 38689921  |
| 4808          | 37350380  | 4853          | 37800120  | 4898          | 38249979  | 4943          | 38699920  |
| 4809          | 37360373  | 4854          | 37810116  | 4899          | 38259977  | 4944          | 38709920  |
| 4810          | 37370365  | 4855          | 37820111  | 4900          | 38269975  | 4945          | 38719919  |
| 4811          | 37380358  | 4856          | 37830107  | 4901          | 38279973  | 4946          | 38729919  |
| 4812          | 37390351  | 4857          | 37840103  | 4902          | 38289971  | 4947          | 38739918  |
| 4813          | 37400344  | 4858          | 37850099  | 4903          | 38299969  | 4948          | 38749918  |
| 4814          | 37410337  | 4859          | 37860095  | 4904          | 38309967  | 4949          | 38759917  |
| 4815          | 37420330  | 4860          | 37870091  | 4905          | 38319965  | 4950          | 38769917  |
| 4816          | 37430324  | 4861          | 37880087  | 4906          | 38329963  | 4951          | 38779916  |
| 4817          | 37440317  | 4862          | 37890083  | 4907          | 38339961  | 4952          | 38789916  |
| 4818          | 37450310  | 4863          | 37900080  | 4908          | 38349960  | 4953          | 38799915  |
| 4819          | 37460304  | 4864          | 37910076  | 4909          | 38359958  | 4954          | 38809915  |
| 4820          | 37470297  | 4865          | 37920072  | 4910          | 38369956  | 4955          | 38819914  |
| 4821          | 37480291  | 4866          | 37930069  | 4911          | 38379955  | 4956          | 38829914  |
| 4822          | 37490284  | 4867          | 37940065  | 4912          | 38389953  | 4957          | 38839913  |
| 4823          | 37500278  | 4868          | 37950062  | 4913          | 38399952  | 4958          | 38849913  |
| 4824          | 37510272  | 4869          | 37960058  | 4914          | 38409950  | 4959          | 38859913  |
| 4825          | 37520266  | 4870          | 37970055  | 4915          | 38419949  | 4960          | 38869912  |
| 4826          | 37530259  | 4871          | 37980051  | 4916          | 38429947  | 4961          | 38879912  |
| 4827          | 37540253  | 4872          | 37990048  | 4917          | 38439946  | 4962          | 38889912  |
| 4828          | 37550247  | 4873          | 38000045  | 4918          | 38449945  | 4963          | 38899912  |
| 4829          | 37560242  | 4874          | 38010042  | 4919          | 38459943  | 4964          | 38909911  |
| 4830          | 37570236  | 4875          | 38020038  | 4920          | 38469942  | 4965          | 38919911  |
| 4831          | 37580230  | 4876          | 38030035  | 4921          | 38479941  | 4966          | 38929911  |
| 4832          | 37590224  | 4877          | 38040032  | 4922          | 38489939  | 4967          | 38939911  |
| 4833          | 37600219  | 4878          | 38050029  | 4923          | 38499938  | 4968          | 38949910  |
| 4834          | 37610213  | 4879          | 38060026  | 4924          | 38509937  | 4969          | 38959910  |
| 4835          | 37620208  | 4880          | 38070023  | 4925          | 38519936  | 4970          | 38969910  |
| 4836          | 37630202  | 4881          | 38080020  | 4926          | 38529935  | 4971          | 38979910  |
| 4837          | 37640197  | 4882          | 38090018  | 4927          | 38539934  | 4972          | 38989910  |



| Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area | Verf.<br>fine | Seg. area |
|---------------|-----------|---------------|-----------|---------------|-----------|---------------|-----------|
| 4973          | 38999909  | 4980          | 39069909  | 4987          | 39139908  | 4994          | 39209908  |
| 4974          | 39009909  | 4981          | 39079909  | 4988          | 39149908  | 4995          | 39219908  |
| 4975          | 39019909  | 4982          | 39089908  | 4989          | 39159908  | 4996          | 39229908  |
| 4976          | 39029909  | 4983          | 39099908  | 4990          | 39169908  | 4997          | 39239908  |
| 4977          | 39039909  | 4984          | 39109908  | 4991          | 39179908  | 4998          | 39249908  |
| 4978          | 39049909  | 4985          | 39119908  | 4992          | 39189908  | 4999          | 39259908  |
| 4979          | 39059909  | 4986          | 39129908  | 4993          | 39199908  | 5000          | 39269908  |

In the preceding table, each number in the column of segments is the area of the circular segment whose height or versed sine of its half arc is the number immediately on the left of it, the diameter of the circle being 1, and the whole area  $\cdot 78539816$ . This table is here extended to ten times the length which it had usually been, the diameter being divided into ten thousand equal parts instead of one thousand: And this size of it also allowed me to carry each number out to eight places of figures instead of six; for when the diameter is divided into only one thousand parts, the proportional part for the figures in the versed sine beyond the third will not give the area true beyond the 6th figure; but in this table it is always found true in the 8th place.

Rule the 8th to the circular segment in page 108 to add there, when a segment greater than a semicircle is plains the use of it; to which it may only be necessary to be found, subtract the quotient of its versed sine divided by its diameter from 1, then subtract the tabular segment which corresponds to the remainder from  $\cdot 78539816$  the whole tabular circle, and the remainder will be the tabular segment corresponding to the segment required, and which must then be multiplied by the square of the diameter.

F I N I S.



# E R R A T A.

| Page | Line         | For                                 | Read                                                                                                                   |
|------|--------------|-------------------------------------|------------------------------------------------------------------------------------------------------------------------|
| 1    | 7            | stroke                              | trace of a pen &c.                                                                                                     |
| 17   | last         | $DF : EG$                           | $EG : DF$                                                                                                              |
| 28   | last         | prob. 7                             | prob. 6                                                                                                                |
| 30   | 5b           | $\frac{aa - cc}{b} +$               | $\frac{aa - cc}{b} =$                                                                                                  |
| 31   | 13           | $c^3$                               | $3c^3$                                                                                                                 |
| 32   | 7b           | $\frac{aa}{aa}$                     | $\frac{aa}{rr}$                                                                                                        |
| 36   | 10           | eye                                 | center of the quadrant                                                                                                 |
| 37   | 1            | 2.7473774                           | 2.7474774                                                                                                              |
| 37   | 2            | 219.790192                          | 219.798192                                                                                                             |
| 39   | 3            | 2.0845048                           | 2.0845333, which will make the last line of that ex. become<br>$62.623 = AD$                                           |
| 41   | 4b           | $LDCB$                              | $LDBC$                                                                                                                 |
| 44   | 7            | 8.04                                | 8.40                                                                                                                   |
| 47   | 5            | $53^\circ 20'$                      | $53^\circ 30'$ , which being corrected throughout the example, will make the last line of it become<br>$HM = 479.7933$ |
| 51   | 12           | 8.6513363                           | 8.6513357                                                                                                              |
| 61   | 12b          | 245 Acres                           | 24.5 Acres                                                                                                             |
| 67   | 3            | 61 ac. 3 r. 36.8064 p.              | 61 ac. 1 r. 39.68 p.                                                                                                   |
| 70   | 12           | greater sum from the less           | less sum from the greater                                                                                              |
| 74   | 9b           | $\frac{AB + CD}{2}$                 | $\frac{AB + CD}{2} \times AP$                                                                                          |
| 75   |              | In the 2d ex. say see fig. in p. 69 |                                                                                                                        |
| 98   | 10 }<br>16 } | radius                              | diameter                                                                                                               |
| 103  | 3b           | $n = \frac{1}{1.1.3} + \&c.$        | $n = 2 \times \frac{1}{1.1.3} + \&c.$                                                                                  |
| 112  | last         | he                                  | the                                                                                                                    |
| 115  |              | In the triangle T, supply the T     |                                                                                                                        |
| 120  | 3            | ar                                  | lar                                                                                                                    |
|      | ib.          | $\sqrt{BC^2 - CD^2}$                | $\sqrt{BC^2 - CD^2}$                                                                                                   |
| 126  | 17           | 16.944305                           | 16.944302                                                                                                              |
| 133  | 11b          | $KE =$                              | $KE = 13$                                                                                                              |
| 138  | 7            | extentions                          | extensions                                                                                                             |
| 140  | 6            | rowing                              | rolling                                                                                                                |
| 144  | 12           | feet to the lb.                     | lb. to the foot                                                                                                        |
| 154  | 14           | 100                                 | 110                                                                                                                    |
| 160  | 6            | $5 \times 15$                       | $15 \times 15$                                                                                                         |
| 164  | 2            | diffimular                          | diffimilar                                                                                                             |
|      | ib.          | at the end add the word and         |                                                                                                                        |
| 174  | 9            | corollary 2                         | corollary 1                                                                                                            |
| 175  | 13           | $20 - 30 - 19\frac{1}{2}$           | $20 - 30 - 19\frac{1}{2}$                                                                                              |



# E R R A T A.

| Page | Line        | For                                                                                                       | Read                                                                                                                                                                                                                                                          |
|------|-------------|-----------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 175  | 5 <i>b</i>  | CP being                                                                                                  | (CP being                                                                                                                                                                                                                                                     |
| 178  | 10 <i>b</i> | 1606 <sup>c</sup> 41934                                                                                   | 1606 <sup>c</sup> 41834                                                                                                                                                                                                                                       |
| 200  | 8           | will                                                                                                      | will                                                                                                                                                                                                                                                          |
| 201  | 9           | 263857624944                                                                                              | 263858149120                                                                                                                                                                                                                                                  |
| 217  | 11          | parabolid                                                                                                 | paraboloid                                                                                                                                                                                                                                                    |
| 236  | 7           | $x \sqrt{\frac{aa - \frac{1}{2}dxx}{aa - \frac{1}{2}xx}} \times$                                          | $x \sqrt{\frac{aa - \frac{1}{2}dxx}{aa - \frac{1}{2}xx}}$ is $= x \times$                                                                                                                                                                                     |
| 405  | 11          | solidity                                                                                                  | superficies                                                                                                                                                                                                                                                   |
| 414  | 7 <i>b</i>  | 18 + 2 = 20 = diameter                                                                                    | 18 + 4 = 22 = diameter, which<br>will make the whole expence<br>become 3 <i>l</i> . 12 <i>s</i> . 4½ <i>d</i> .<br>This will also alter the contents<br>in the example in the following<br>page, the number of pounds<br>being 240 and the price 4 <i>l</i> . |
| 428  | 5           | 128 × 6 <sup>3</sup>                                                                                      | 128 × 6                                                                                                                                                                                                                                                       |
| 443  | 11 <i>b</i> | b - x                                                                                                     | b + x                                                                                                                                                                                                                                                         |
| ib.  | 6 <i>b</i>  | $\frac{5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8} -$                                                           | $\frac{5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8} +$                                                                                                                                                                                                               |
| ib.  | ib.         | $\frac{1 + \sqrt{2}}{3\sqrt{2}} =$                                                                        | $\frac{1 + \sqrt{2}}{3\sqrt{2}} = \frac{2 + \sqrt{2}}{6} =$                                                                                                                                                                                                   |
| 452  | 14          | duadrable                                                                                                 | quadrable                                                                                                                                                                                                                                                     |
| 459  | 9           | or (EG)                                                                                                   | (or EG)                                                                                                                                                                                                                                                       |
| 469  | 8           | 25 C                                                                                                      | 50 C                                                                                                                                                                                                                                                          |
| ib.  | 14          | second                                                                                                    | third                                                                                                                                                                                                                                                         |
| 489  | 3           | nothings                                                                                                  | northings                                                                                                                                                                                                                                                     |
| ib.  | ib.         | ib.                                                                                                       |                                                                                                                                                                                                                                                               |
| 492  |             | At the bottom of the fig. draw<br>a line through all the points<br>a i b c h e d g f.                     |                                                                                                                                                                                                                                                               |
| 496  | 21          | 78                                                                                                        | 102                                                                                                                                                                                                                                                           |
| ib.  | 4 <i>b</i>  | ib.                                                                                                       |                                                                                                                                                                                                                                                               |
| 524  | 9 <i>b</i>  | paraboidal                                                                                                | paraboloidal                                                                                                                                                                                                                                                  |
| 526  | 1           | After the words then if the<br>sum be, add the words mul-<br>tiplied by the length, and<br>the product be |                                                                                                                                                                                                                                                               |
| 531  | 1 <i>b</i>  | dele the words excepting the                                                                              |                                                                                                                                                                                                                                                               |
|      | 2 <i>b</i>  | allowance, &c.                                                                                            |                                                                                                                                                                                                                                                               |
| 539  | 22          | Briton                                                                                                    | Britain                                                                                                                                                                                                                                                       |
| 549  | 8           | Square-roots                                                                                              | squares                                                                                                                                                                                                                                                       |
| 600  | 15          | thick                                                                                                     | broad                                                                                                                                                                                                                                                         |

N. B. *b* at the line figures denotes them to be counted from the bottom.



















Hutton, Charles,

A treatise on mensuration both in theory and practice

Newcastle-upon-Tine 1770

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